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Ground state of an arbitrary triangle with a Calogero–Sutherland–Moser potential

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Abstract. We construct the expression for the ground state eigenfunction of the Schrödinger equation for a particle inside an arbitrary planar triangle under the influence of a potential.

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Let us consider the multispecies Calogero–Sutherland–Moser (CSM) system for N particles on a circle of perimeter L with Hamiltonian [1]:

$$H = \sum_i \frac{p_i^2}{2m_i} + \frac{\pi^2\hbar^2}{L^2}\sum_{i<j} g_{ij}(g_{ij}-1)\frac{(m_i+m_j)}{2m_im_j}\frac{1}{\sin^2\frac{\pi}{L}(x_i-x_j)}. \qquad (1)$$

There are 'a' species, each with N_a particles such that $N=\sum_a N_a$. Because H is symmetric in g_{ij} and $g_{ij}-1$, the allowed wave functions satisfy the boundary conditions

$$\psi \sim |x_i - x_j|^{g_{ij}}, \quad x_i \to x_j. \qquad (2)$$

Corresponding to different ordering of particles, there are different sectors distinguished by coincident points where the wave functions vanish. Working in the bosonic basis, if all particles belong to the same species, we can expect

$$\psi(0, x_1, x_2, ..., x_{N-1}) = \psi(x_1, x_2, ..., x_{N-1}, L); \qquad (3)$$

$(0 < x_1 < x_2 < ... < x_{N-1} < L)$.

For $N = 2$, ground state as well as excited states can be found. However, for $N = 3$, ground state can only be found if the condition

$$g_{ij} = \alpha m_i m_j \qquad (4)$$

is satisfied, a condition suggested by asymptotic Bethe ansatz. Remarkably, the ground state is [1]

$$\Psi_0 = \prod_{i<j} \left| \sin \frac{\pi}{L}(x_i - x_j) \right|^{g_{ij}} \tag{5}$$

with energy

$$E_0 = \frac{\pi^2 \hbar^2 \alpha}{6L^2} \left[\left(\sum_a m_a N_a \right)^3 - \sum_a m_a^3 N_a \right]. \tag{6}$$

Here we consider the special case of three particles with masses, m_1, m_2, m_3 on a circle. We now map this system to a 'particle' in a triangle with angles depending on masses and ratios of their combinations. In the work, we solve the Schrödinger equation for a particle inside a triangular shaped box with a CSM potential. Due to the nature of the potential, the equation is solved with Dirichlet boundary conditions. The details of the triangle are obtained by the map that we describe below. This remarkable map was discovered by Rabouw and Ruijgrok [2] (much later, re-discovered by Glashow and Mittag [3]). At the expense of repetition, we have to outline the map to present our final result.

Observe that the total momentum can be set to zero:

$$P = \sum_k m_k v_k = 0. \tag{7}$$

Let x_k be the arclength between the other two particles via the route avoiding the particle, k; $\sum_k x_k = L$. Kinetic energy is given by

$$T = \frac{\Pi}{M} \sum_k \frac{v_k^2}{m_k}, \tag{8}$$

where $\Pi = m_1 m_2 m_3$, $M = m_1 + m_2 + m_3$ and v_k is the speed of the kth particle. To establish a connection with a triangle billiard, we need three pairs of basis vectors, each having two components – one parallel to a side of the triangle and the other parallel to the corresponding altitude. These pairs will be related to each other by a rotation by an angle of the triangle.

Define

$$U_1 = (v_2 - v_3)\sqrt{\frac{m_2 m_3}{(m_2 + m_3)M}};$$
$$V_1 = v_1 \sqrt{\frac{m_1}{m_2 + m_3}}, \tag{9}$$

and similar cyclic relations for U_2, V_2 and U_3, V_3. When particles 2 and 3 collide, $U_1 \to -U_1$ and V_1 remains unchanged. The kinetic energy can be written as

$$T = \frac{M}{2} \sum_k (U_k^2 + V_k^2). \tag{10}$$

We note that (U_k, V_k) form three pairs, each defining a vector

$$\vec{W} = U_k \hat{e}_k + V_k \hat{f}_k, \tag{11}$$

the norm of which is given by

$$W^2 = \frac{\Pi}{M^2} \sum_k \frac{v_k^2}{m_k}. \tag{12}$$

As desired, $(\hat{e}_2, \hat{f}_2)$ and $(\hat{e}_1, \hat{f}_1)$ are related to each other by rotation:

$$\begin{bmatrix} \hat{e}_2 \\ \hat{f}_2 \end{bmatrix} = \begin{bmatrix} -\cos\theta_3 & -\sin\theta_3 \\ \sin\theta_3 & -\cos\theta_3 \end{bmatrix} \begin{bmatrix} \hat{e}_1 \\ \hat{f}_1 \end{bmatrix}, \tag{13}$$

where θ_k is given by

$$m_k \cot\theta_k = \sqrt{\frac{\Pi}{M}} = M \cot\theta_1 \cot\theta_2 \cot\theta_3. \tag{14}$$

With these transformations, motion of three particles on a circle is mapped to a uniform motion in a triangle billiard. Collision of two particles is equivalent to a particle striking a side of the triangle in accordance with Snell's law.

The billiard is an acute triangle with interior angles θ_k, sides ℓ_k parallel to $\hat{f}_k$, and altitudes a_k parallel to $\hat{e}_k$. An interior point is given by trilinear coordinates, the distances d_k from each side [3] which are related to x_k by the relations

$$x_1 = d_1 \sqrt{\frac{(m_2 + m_3)M}{m_2 m_3}} \quad \text{and similar relations for } x_2 \text{ and } x_3. \tag{15}$$

For obtuse triangles, we have to work with negative masses, but everything works well as far as the map is concerned. For particles 1 and 3, if the masses are $-m_1$ and $-m_3$ and $m_2 - m_1 - m_3 > 0$. The norm of $\vec{W}$ will be negative, so it seems unphysical. But the connection still works with angles of the triangle given by

$$\tan\theta_k = (-1)^{k+1} m_k \sqrt{\frac{M}{\Pi}}. \tag{16}$$

Here $\theta_2 > \pi/2$.

Now, combining (5) and (15), we obtain the ground state eigenfunction of an arbitrary acute triangle with CSM potential. Figure 1 shows the contour plot for angles obtained by choosing masses 1, 2 and 3, α equal to 1; it is an irrational triangle billiard. The CSM potential remains finite and smooth throughout the triangle and rises to infinity, thus providing a barrier for the particle to escape.

In figure 2, one can see the level curves of the potential and the contour plot of the eigenfunction together, just to facilitate a more visual impression of the result.

The above result is only valid for acute triangles and not for obtuse ones. Thus, as observed in [2], obtuse triangles are more complicated somehow than the acute triangles.

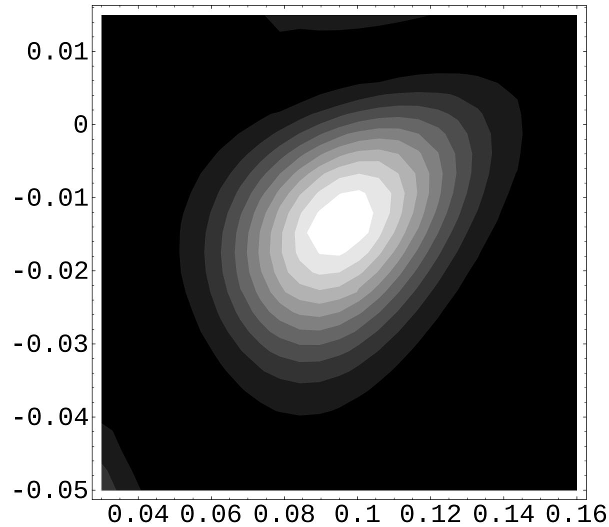

Figure 1. Contour plot of the ground state of an irrational triangle billiard with a CSM potential in the d_1–d_2 plane. Two of the interior angles are $\cot^{-1}\frac{1}{2}$ ($\sim$ 63.4095...degrees) and $\cot^{-1}\frac{1}{3}$ ($\sim$ 71.5365...degrees).

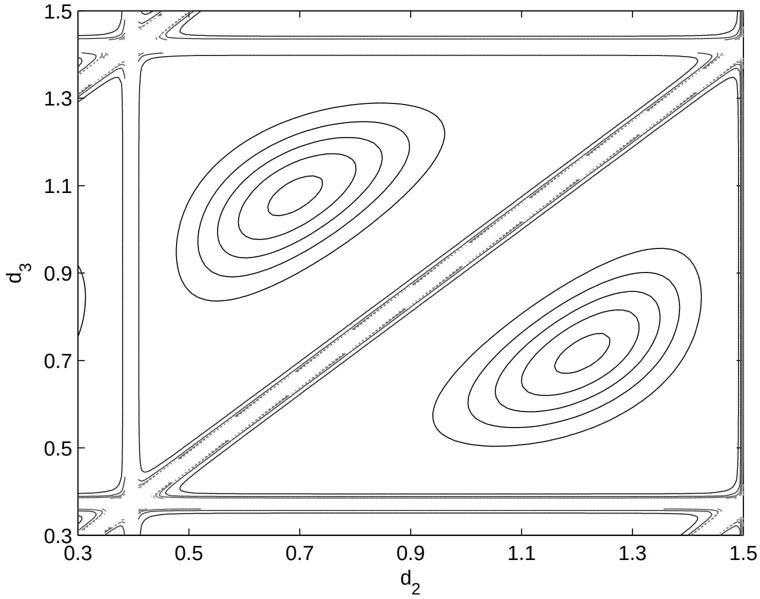

Figure 2. Level curves of the potential and the contours of the eigenfunction are shown together in d_2–d_3 plane.

The Rabouw–Ruijgrok map [2] reduces to that found by Onsager [4] connecting the dynamics of two particles of masses m_1 and m_2 on a line segment to a right triangle billiard.

We note that there have been studies on the ground state of polygonal systems in the past. Ground state energy of a particle in an n-sided regular polygon was estimated as an expansion in $1/n$ [5]. For the same system, a sharper result was obtained [6] by means of conformal mapping from the circle, which provided with an expansion parameter for an approximate evaluation of the lowest eigenvalue and eigenvector. In this paper, the result is exact, and it is for an arbitrary acute triangle. It has recently been shown that the solution of Helmholtz equation can be found for an N-simplex by relating it to $(N+1)$-particle hard point boson problem on a circle [7]. The result presented here leaves us with a hope that the solution of three dissimilar point particles on a circle will lead us to the solution for an arbitrary triangle billiard. This is an outstanding open problem, in addition to finding analytically exact, complete solutions for a fully chaotic system. It would be good to recall that, surprisingly, first few energy levels and eigenfunctions were found analytically in closed form for a class of chaotic systems some years ago [8].

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