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Weak gravitational field and Coriolis potential

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Abstract. In mechanics of the mass point passage from one frame of reference to another moving with velocity $\vec{u}$ consists in subtracting this vector from the velocity of the particle. In general case the vector $\vec{u}$ is not constant, as, for example, when passing through a rotating frame, this operation creates inertial forces. Analysis of this phenomenon from the point of view of Lagrangian and Hamiltonian mechanics is interesting from the general relativistic point of view due to Einstein's principle of equivalence. We show that the vector $\vec{u}$ plays the role of vector potential which, however, essentially differs from vector potential known in classical electrodynamics. Comparative analysis of the two kinds of vector potentials is completed.

Keywords. Lagrangian and Hamiltonian formalism; gravimagnetic field; vector potential; principle of equivalence.

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1. Introduction

The source of gravitational field in Kerr space-time was identified with rotating mass due to the form of the component $g_{t\varphi}$ in Boyer–Lindquist coordinates which asymptotically coincides with that of the φ-component of the vector potential produced by magnetic dipole [1]

$$g_{t\varphi} = \frac{a(r^2+a^2)\sin^2\theta}{r^2+a^2\cos^2\varphi} \to \frac{a\sin^2\theta}{r^2}. \tag{1}$$

Analysis of the time-like geodesics shows that this metric component acts like vector potential, though explicit form of Hamilton–Jacobi equation differs from Hamilton–Jacobi equation for charged particle in the field of magnetic dipole. As is well-known, Hamilton–Jacobi equation for a test particle in Kerr space-time separates whereas that for a charged particle in the field of magnetic dipole, does not. Indeed, components of momentum of a particle with charge e, mass m and velocity $\dot{x}^i$ in vector potential A_i are $g_{ij}m\dot{x}^j + eA_i$. So, non-relativistic Hamilton–Jacobi equation for this case has the form [3]

$$\frac{1}{2m} g^{ij} \left(\frac{\partial S}{\partial x^i} + eA_i \right) \left(\frac{\partial S}{\partial x^j} + eA_j \right) = E \tag{2}$$

that contains the square of vector potential. Unlike this, Hamilton–Jacobi equation for a test particle in Kerr space-time does not contain the square of $g_{t\varphi}$. This difference reveals when one tries to solve the equation and finds that it is the square of the vector potential which makes variables no-separable. The question arises, as to why one and the same vector potential yields two distinct forms of Hamilton–Jacobi and which one is to be used.

The goal of the present work is to show that two distinct forms of Hamilton–Jacobi equation which yield two distinct forms of dynamics correspond to two distinct objects both of which are called 'vector potential'. In fact only one of them is indeed a vector while the other is 1-form. The one appeared in classical electrodynamics is 1-form and should be called 'co-vector potential' whereas genuine vector potential appears when passing from one frame of reference say, inertial one, to another which is uniformly rotating. Therefore, we suggest to call the genuine vector potential concerned with non-inertial frames 'Coriolis potential' as it was done by Einstein [2]. Below we discuss the appearance of vector potential under this operation and restore Lagrangian and Hamiltonian formalism of dynamics of mass point. Existence of two kinds of potentials explains the difference between motion of test particle in Kerr space-time and that of the charged particle in the magnetic field of a dipole.

2. Coriolis potential

The action principle remains in force in any frame. Therefore, Lagrangian of mass point in a non-inertial frame can be obtained from that in an inertial one simply by substituting $\vec{v} - \vec{u}$ for the mass point velocity $\vec{v}$:

$$L(\vec{v}) = \frac{m}{2}(\vec{v} - \vec{u})^2 = \frac{m}{2}\vec{v}^2 - m\vec{u} \cdot \vec{v} + \frac{m}{2}\vec{u}^2. \tag{3}$$

Applying the definition of generalized momentum, one obtains

$$\vec{p} \equiv \frac{\partial L}{\partial \vec{v}} = m(\vec{v} - \vec{u}). \tag{4}$$

This expression is similar to the expression for the momentum of a charged particle in vector potential. Indeed, Lagrangian of a particle with unit charge in a vector potential of magnetostatic field $\vec{A}$ has the form [3]

$$L = \frac{m\vec{v}^2}{2} - \vec{A} \cdot \vec{v} \tag{5}$$

and generalized momentum

$$\vec{p} = m\vec{v} - \vec{A}. \tag{6}$$

Therefore, it seems that there is no difference between magnetic field and non-inertial character of the frame of reference. Let us show that it is not so and equation of motion in the magnetic field and in the non-inertial frame are different.

To see this, construct Hamiltonian of the particle for both Lagrangians (3) and (5). Substituting the pairs of Lagrangian and corresponding momentum into the definition of Hamiltonian

$$H(\vec{p}, x) = \vec{p} \cdot \dot{\vec{x}} - L \tag{7}$$

yields two different forms of this function:

$$H = \vec{p} \cdot \vec{v} - L = \vec{p} \cdot \vec{v} - \frac{m(\vec{v} - \vec{u})^2}{2} = \frac{\vec{p}^2}{2m} + \vec{p} \cdot \vec{u}, \tag{8}$$

and the well-known Hamiltonian of charged particle in the magnetic field

$$H = \frac{1}{m}\vec{p} \cdot (\vec{p} + \vec{A}) - L = \frac{1}{2m}(\vec{p}^2 + 2\vec{A} \cdot \vec{p} + \vec{A}^2). \tag{9}$$

It is seen that Hamiltonian for non-inertial frame does not contain square of the velocity of the frame whereas Hamiltonian of charged particle contains the term $\vec{A}^2/2m$. Therefore, these two kinds of vector potentials are really different and there is a reason to make difference between electromagnetic 'vector' potential which in fact is co-vector and Coriolis potential which is a genuine vector.

In fact, genuine vector potential is $\vec{C}$ added to velocity, whose square appears in Lagrangian and does not appear in Hamiltonian, whereas potential of the magnetic field is the co-vector added to momentum and its square appears in Hamiltonian. This difference is valid in both relativistic and non-relativistic mechanics. In this work we start with relativistic mechanics of mass point and then pass to non-relativistic limit to show the role of Coriolis potential in passages from one rotating frame to another both in relativistic and non-relativistic mechanics of mass point. To this end we use a coordinate system with one of the coordinates being azimuthal angle φ such that ∂_φ is the Killing vector. Then passage to a frame rotating with angular velocity ω produces vector Coriolis potential with ω as its only non-zero φ-component:

$$\vec{C} = \omega \partial_\varphi. \tag{10}$$

3. Weak gravitational field and coordinate transformations

Consider a space-time in which gravitational field is weak. Then it is possible to introduce a function t whose gradient has square $\langle \mathrm{d}t\,, \mathrm{d}t \rangle$ close enough to one. Hereafter this function plays the role of time coordinate. Note that this function alone determines both space and time because its surfaces of level $t = \text{const.}$ specify space at a given moment of time. Other coordinates also are functions of the space-time which can be chosen arbitrarily. Together they form a coordinate system $\{t, x^i\}$. If the space-time has time-like Killing vector, it is convenient to chose the time coordinate such that the Killing vector is exactly ∂_t.

By spatial metric we mean Riemannian metric of the surfaces $t = \text{const.}$ which is

$$\langle \mathrm{d}x^i, \mathrm{d}x^j \rangle|_{t=\text{const.}} \equiv g^{ij}. \tag{11}$$

Since functions x^i on the space-time are quite arbitrary, genuine gradients $\mathrm{d}x^i$ are not orthogonal to $\mathrm{d}t$. Denote scalar products of 1-forms $\mathrm{d}t$ and $\mathrm{d}x^1$ as components of Coriolis potential $\left\langle \mathrm{d}t\,, \mathrm{d}x^i \right\rangle \equiv C^i/c$ where c is the speed of light and assume that C^i's are small compared to c. Then, though genuine scalar products $\langle \mathrm{d}x^i, \mathrm{d}x^j \rangle$ differ from g^{ij}, the difference is of order c^{-2} and these coordinates can always be chosen in such a way that this difference can be ignored. By the result we have space-time metric in the form

$$\langle \mathrm{d}t, \mathrm{d}t \rangle = 1 - \frac{2\Phi}{c^2}, \quad \left\langle \mathrm{d}t, \mathrm{d}x^i \right\rangle = \frac{C^i}{c}, \quad \left\langle \mathrm{d}x^i, \mathrm{d}x^j \right\rangle = -g^{ij} - \frac{C^i C^j}{c^2}. \tag{12}$$

It will be shown below that the terms Φ/c^2 in g^{tt} and $g^{t\varphi}$ which is of order c^{-1}, cannot be neglected, while difference between spatial metric and that of the surface $t = \text{const.}$ is negligibly small.

Though 3 + 1-splitting of the space-time is specified by certain choice of what will be called space and time, frame of reference is not fixed ultimately. The frame can be changed without changing the space and time as they are specified by the function t. Therefore, afterwards we make coordinate transformations which leave the coordinate t unchanged. It must be noted that t is only a coordinate, not proper time of an observer because length of its gradient $\mathrm{d}t$ is not exactly one.

Intersection of three coordinate surfaces $x^i = \text{const.}$ is the coordinate line, a time-like curve which specifies the time axis through a given point of the space. In general, the time axis and the space $t = \text{const.}$ are not orthogonal to each other because, on one hand, this curve is specified only by coordinate surfaces $x^i = \text{const.}$ and, on the other hand, these surfaces are introduced regardless of choice of the surfaces $t = \text{const.}$ Non-orthogonality of space and time reveals in the components g^{ti} of the metric (12).

Invariance of the space means that the transformations do not touch the coordinate t. Thus, they are of the form

$$t \to t, \quad x^i \to y^a(x^i, t). \tag{13}$$

These transformations are of two different kinds. Ordinarily coordinate transformations

$$t \to t,\ x^i \to y^a(x^i) \tag{14}$$

leave the time axis immobile everywhere and saves the frame of reference. Another kind is given by changes of frame of reference. Its simplified form is

$$x^i \to y^i = x^i + f^i(t). \tag{15}$$

General transformations can be obtained as superposition of transformations of these two kinds. Hereafter we employ only simplified form (15) of this transformation. Differentiation of this equation gives

$$\mathrm{d}y^i = \mathrm{d}x^i + f'^i \mathrm{d}t. \tag{16}$$

The corresponding transformation of the metric (12) reduces to that of the coefficients C^i:

$$C^i \to C^i + cf'^i \left(1 - \frac{2\Phi}{c^2}\right). \tag{17}$$

Hereafter we transform the vector potential $\vec{C}$ this way leaving the form of metric invariant. Remarkably, in Minkowski space-time one can pass to uniformly accelerated frame another way, without creating any vector potential [4]. Passage to rotating frame is transformation (15) which in axially-symmetric coordinates has the form $\varphi \to \varphi + \omega t$. This transformation produces Coriolis potential (10).

4. Mechanics of mass point

Equations of motion of a mass point in a gravitational field can be solved analytically if the field admits known integrals of motion, particularly, energy. Total energy is an integral in stationary space-times. Therefore, hereafter we assume that the space-time under consideration is stationary, and hence, possesses time-like Killing vector. Hereafter we assume that such a Killing vector exists and the time coordinate t is chosen in such a way that the Killing vector is exactly ∂_t. Hamilton–Jacobi equation for time-like geodesics has the form

$$\left(1 - \frac{2\Phi}{c^2}\right)\left(\frac{\partial S}{\partial t}\right)^2 + \frac{2C^i}{c}\frac{\partial S}{\partial t}\frac{\partial S}{\partial x^i} - \langle \nabla S\,, \nabla S \rangle = m^2c^2, \tag{18}$$

where ∇S stands for the spatial part of the 1-form $\mathrm{d}S$. Since the coordinate t is chosen in a special way there exists a function W satisfying the equations

$$2m\frac{\partial W}{\partial t} = \left(\frac{\partial S}{\partial t}\right)^2 - m^2c^2, \quad \nabla S = \nabla W \tag{19}$$

because S_t can always be assumed to depend only on t. Then eq. (18) can be rewritten as follows:

$$\left(1 - \frac{2\Phi}{c^2}\right)\left(m^2c^2 + 2m\frac{\partial W}{\partial t}\right) + \frac{2C^i}{c}\sqrt{m^2c^2 + 2m\frac{\partial W}{\partial t}} - \langle \nabla W, \nabla W \rangle = m^2c^2.$$

Removing the parentheses and omitting negligibly small terms yields

$$\frac{1}{2m}\langle \nabla W, \nabla W \rangle - C^i\frac{\partial W}{\partial x^i} + m\Phi = \frac{\partial W}{\partial t} \tag{20}$$

that almost coincides with the well-known form of non-relativistic Hamilton–Jacobi equation for a particle moving in gravitational potential Φ and vector potential $\vec{C}$. The only difference is that this equation does not contain the term proportional to $|\vec{C}|^2$ which is obligatory in mechanics of electric charge in magnetic field where 1-form of potential $A \equiv A_i \mathrm{d}x^i$ enters only in combination with $\nabla W - A$.

Reduction of the Hamilton–Jacobi equation for isotropic geodesics is similar and leads to that for geodesics on the 3-space with metric g^{ij}. Indeed, in this case $m = 0$ and all terms containing negative powers of c can be neglected. Consequently, world lines of massless particles are just geodesics of the space which should be in a non-relativistic theory. As for strictly space-like geodesics ($m^2 < 0$), they have no physical meaning in the case.

In stationary axially symmetric fields, passage to rotating frame does not create any problems with variable separation in the Hamilton–Jacobi equation. Indeed, since square of $\vec{C}$ which has the form $(\omega\rho)^2$ where ρ is the distance from the axis of symmetry, does not enter the equation the only new term appearing is ωS_φ (10). In this case the derivative S_φ is the integral of motion, and consequently this term is constant. This example shows the importance of the difference between electrodynamical co-vector and Coriolis potentials. In uniformly rotating frame and in uniform magnetic field they have the same form, but Hamilton–Jacobi equation for unit charge in magnetic field and for mass point in rotating frame are different:

$$\frac{1}{2m}\left[g_{ij}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^j} + \left(\frac{\partial S}{\partial \varphi} - A\right)^2\right] = E,$$
$$\frac{1}{2m}\left[g_{ij}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^j} + \left(\frac{\partial S}{\partial \varphi}\right)^2\right] - \omega\frac{\partial S}{\partial \varphi} = E, \tag{21}$$

It is seen that the second equation separates in all axially-symmetric coordinate systems which separate it in inertial frame because we can put $\frac{\partial S}{\partial \varphi} = L$ that is constant, whereas the first one does only in cylindric coordinates $\{z, \rho, \varphi\}$. For example, in oblate spheroidal coordinates uniform magnetic field has potential only with component $A_\varphi = \Omega\rho^2/2$ where $\rho = a\cosh u \cos v$ and square of this potential equals $\Omega\rho^2 = a^2\Omega\cosh^2 u\cos^2 v$. Appearance of a term of this form in the Hamilton–Jacobi equation makes it impossible to separate variables in the first equation of eq. (21) whereas the second equation separates because this term does not appear. The same happens in the case of more complicated potential $A_\varphi = \cos^2 v/(\cosh^2 u - \cos^2 v)$ which corresponds to magnetic field of some particular distribution of electric current on disk or Coriolis field produced by rotating massive disk with corresponding surface mass density. This potential separates the second equation but does not separate the first one. Under large values of the coordinate u this potential coincides with the potential of point-like dipole and finally we have separability of Hamilton–Jacobi equation for test particle in Kerr space-time and non-separability of the equation for charged particle in the field of magnetic dipole.

5. Conclusion

Non-inertial frames play important roles in various areas of physics, especially in astrophysics in which there exists the problem of equilibrium of rotating matter in its own gravitational field. Sometimes the matter has no fixed angular velocity of rotation and different layers rest in different rotating frames. To describe objects of this sort one needs to pass from one rotating frame to another. Our analysis shows that it is more convenient to consider passages of this sort from general relativistic point of view. Explicit forms of transformation from one rotating frame to another provided above are made for mechanics of mass point. It turns out that passage to uniformly rotating frames leaves Hamilton–Jacobi equation separated in all axially-symmetric coordinate systems which separate it in inertial frame. The same approach can be applied to other covariant equations of mathematical physics.

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