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An integral transform of Green's function, off-shell Jost solution and T-matrix for Coulomb–Yamaguchi potential in coordinate representation

U LAHA
Department of Physics, National Institute of Technology, Jamshedpur 831 014, India
E-mail: ujjwal.laha@gmail.com

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Abstract. By exploiting the theory of ordinary differential equations together with certain properties of higher transcendental functions, a useful analytical expression for the integral transform of the Green's function for motion in Coulomb–Yamaguchi potential is derived via the r-space approach. This integral transform is applied to construct an analytical expression for off-shell Jost solution in its 'maximal reduced form' involving confluent and Gaussian hypergeometric functions. Corresponding Jost functions automatically follow from this solution. Finally, as another application of the off-shell Jost solution, the off-shell T-matrix is calculated by using a modified relation between off-shell physical wave function and T-matrix which does not involve the potential explicitly, thereby avoiding certain difficult integrals, and expressed it in terms of rational functions and simple hypergeometric functions which is in exact agreement with the results given previously by other authors.

Keywords. Integral transform; off-shell Jost solution; off-shell T-matrix.

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1. Introduction

Experiments which involve scattering by additive interactions are analysed using Gell–Mann–Goldberger scattering by two-potential theorem [1]. Applicability of the Gell–Mann–Goldburger theorem is directly related to the existence/completeness of the wave operators of the scattering system [2]. The wave operators exist for short-range interactions but not for long-range interactions. To deal with Coulomb or Coulomb-like potentials the wave operators need modification. This has been discussed by Chandler [3], van Haeringen [4] and van Haeringen and van Wageningen [5] on several occasions to deal with Coulomb-modified nuclear scattering. Calculations in physics involving the two-nucleon interaction generally use off-energy-shell elements of the two-nucleon transition matrix T. Any nuclear process depends on these elements in one way or another. Examples of such processes are knockout reactions and bremsstrahlung.

Recently, in [6], this author has derived the expression for off-shell Jost solution and T-matrix for Coulomb-like potentials based on a differential equation approach of van Leeuwen and Reiner [7]. As it is of importance to have in the literature an alternative mathematical prescription a relatively different approach is adopted here. In this work it will be shown that with certain modification of the differential equation approach Green's function technique can be used to find off-shell Jost solution and T-matrix related to Coulomb-distorted nuclear processes. The modification amounts to obtaining the required integral transform $\tilde{G}^{(+)}_{\ell\mathrm{CS}}(r,q)$ of the Green's function for motion in Coulomb plus separable potential (also known as Coulomb-distorted Green's function) based on the theory of ordinary differential equations together with certain properties of higher transcendental functions for use in the calculation of the corresponding off-shell quantities without the explicit use of two-potential theorem and an expression for T-matrix which does not involve the potential explicitly, thereby avoiding certain difficult integrals because of the presence of Coulomb potential. The method proposed will be applicable for Coulomb–separable potentials of arbitrary rank. The result for $\tilde{G}^{(+)}_{\ell\mathrm{CS}}(r,q)$ is obtained by integral transforming the differential equation for $G^{(+)}_{\ell\mathrm{CS}}(r,r')$ in r' to obtain a second-order ordinary differential equation for $\tilde{G}^{(+)}_{\ell\mathrm{CS}}(r,q)$ which is solved with appropriate boundary conditions.

The results for the integral transforms $\tilde{G}^{(+)}_{\ell\mathrm{CS}}(r,q)$ and $\tilde{G}^{(+)}_{\ell\mathrm{CS}}(r,-q)$ will be used to write an expression for off-shell physical/outgoing wave solution $\psi^{(+)}_{\ell}(k,q,r)$. The off-shell physical solution can be expressed in terms of half-shell T-matrix $T_\ell(k,q,k^2)$ and off-shell Jost solution $f_\ell(k,q,r)$. Also by exploiting the relation that exists between the off-shell physical solution and T-matrix $T_\ell(p,q,k^2)$, one can express $T_\ell(p,q,k^2)$ in closed form. If this procedure is correct, sizeable off- and half-shell effects will be obtained. The plan of the present paper is as follows.

In §2 a differential equation approach to find an integral transform of the outgoing wave Green's function $G^{(+)}_{\mathrm{CY}}(r,r')$ for Coulomb–Yamaguchi potential which plays a crucial role in the study of quantum mechanical scattering by additive interactions is developed to obtain $\psi^{(+)}_{\mathrm{CY}}(k,q,r)$. Section 3 is devoted to construct an exact analytical expression for off-shell Jost solution $f_{\mathrm{CY}}(k,q,r)$ for motion in Coulomb–Yamaguchi potential. The result for Coulomb–Yamaguchi Jost function $f_{\mathrm{CY}}(k,q)$ has appeared in a number of publications [8–10], while the result for $f_{\mathrm{CY}}(k,q,r)$ has only been discussed in the literature in the recent past [6]. The off-shell Jost function $f_{\mathrm{CY}}(k,q)$ automatically follows from $f_{\mathrm{CY}}(k,q,r)$. In §4 off-shell T-matrix $T_{\mathrm{CY}}(p,q,k^2)$ is expressed in closed form which is in agreement with that of Zachary [11], Bajzer [2] and van Haeringen and van Wageningen [5]. Earlier authors derived this result in the momentum representation, while the coordinate representation is followed here. Finally, §5 contains related numerical results, discussion and conclusion.

2. Differential equation approach for $\tilde{G}_{\mathrm{CY}}^{(+)}(r,q)$

Yamaguchi [12] has introduced a one-term separable potential

$$V(r,s)=\lambda \mathrm{e}^{-\beta(r+s)} \tag{1}$$

to describe the nucleon–nucleon scattering. At a centre of mass energy $E=k^2+i\varepsilon$ the outgoing wave Coulomb Green's function $G_{\mathrm{C}}^{(+)}(r,r')$ satisfies the inhomogeneous differential equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+k^2-\frac{2k\eta}{r}\right]G_{\mathrm{C}}^{(+)}(r,r')=\delta(r-r') \tag{2}$$

whereas the same $G_{\mathrm{CY}}^{(+)}(r,r')$ for Coulomb–Yamaguchi potential obeys the integro-differential equation

$$\begin{aligned}&\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+k^2-\frac{2k\eta}{r}\right]G_{\mathrm{CY}}^{(+)}(r,r')\\&\quad=\lambda \mathrm{e}^{-\beta r'}\int_0^\infty \mathrm{d}s\,\mathrm{e}^{-\beta s}G_{\mathrm{CY}}^{(+)}(s,r')+\delta(r-r').\end{aligned} \tag{3}$$

Let the function $F(r,r')$ be related to $G_{\mathrm{CY}}^{(+)}(r,r')$ by

$$G_{\mathrm{CY}}^{(+)}(r,r')=r\mathrm{e}^{ikr}F(r,r'). \tag{4}$$

Then the integral transform $\tilde{F}(r,q)=\int_0^\infty \mathrm{d}r\mathrm{e}^{iqr}F(r,r')=[F(r,r');r'\to q]$ is related to $\tilde{G}_{\mathrm{CY}}(r,q)=[G_{\mathrm{CY}}(r,r');r'\to q]$ by

$$\begin{aligned}\tilde{G}_{\mathrm{CY}}^{(+)}(r,q)&=r\mathrm{e}^{ikr}\int_0^\infty \mathrm{d}r'\mathrm{e}^{iqr'}G(r,r')=r\mathrm{e}^{ikr}\int_0^\infty \mathrm{d}r'\mathrm{e}^{iqr'}F(r,r')\\&=r\mathrm{e}^{ikr}\tilde{F}(r,q).\end{aligned} \tag{5}$$

To construct a closed form expression for $\tilde{F}(r,q)$ a differential equation approach to the problem is adopted here and it is observed that it satisfies a non-homogeneous confluent hypergeometric equation.

Equation (4) is substituted in eq. (3) to get

$$\begin{aligned}&\left[r\frac{\mathrm{d}^2}{\mathrm{d}r^2}+(2-2ik)\frac{\mathrm{d}}{\mathrm{d}r}+(2ik-2k\eta)\right]F(r,r')\\&\quad=\mathrm{e}^{-(\beta+ik)r}d(k,\beta,r')+\mathrm{e}^{-ikr}\delta(r-r')\end{aligned} \tag{6}$$

where

$$d(k,\beta,r')=\lambda\int_0^\infty \mathrm{d}s\,\mathrm{e}^{-\beta s}G_{\mathrm{CY}}^{(+)}(s,r'). \tag{7}$$

Taking the integral transform $[F(r,r');r'\to q]$ and substituting $z=-2ikr$, eq. (6) is obtained as

$$\left[z\frac{\mathrm{d}^2}{\mathrm{d}z^2}+(c-z)\frac{\mathrm{d}}{\mathrm{d}z}-a\right]\tilde{F}(z,q)=-\frac{1}{2ik}[d(k,\beta,q)\mathrm{e}^{\rho z}+\mathrm{e}^{\gamma z}] \tag{8}$$

with

$$a=(1+i\eta),\quad c=2,\quad \rho=\left(\frac{(\beta+ik)}{2ik}\right)\quad\text{and}\quad \gamma=\left(\frac{(k-q)}{2k}\right). \tag{9}$$

Equation (8) represents a non-homogeneous confluent hypergeometric equation of which the two independent solutions [13] of its homogeneous part are given by

$$\Phi(a,c;z)=\frac{\Gamma(c)}{\Gamma(a)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)z^n}{\Gamma(c+n)n!} \tag{10}$$

and

$$\bar{\Phi}(a,c;z)=z^{1-c}\Phi(a-c+1,2-c;z). \tag{11}$$

According to Babister [14] the particular solution of the non-homogeneous confluent hypergeometric equation

$$\left[z\frac{\mathrm{d}^2}{\mathrm{d}z^2}+(c-z)\frac{\mathrm{d}}{\mathrm{d}z}-a\right]y=\mathrm{e}^{\rho z}z^{\sigma-1}, \tag{12}$$

where a,c,ρ and σ are constants, is written as

$$y_P=\sum_{n=0}^{\infty}\frac{\theta_{\sigma+n}(a,c;z)\rho^n}{n!}. \tag{13}$$

Here $\theta_\sigma(a,c;z)$ is the particular solution of the non-homogeneous confluent hypergeometric equation [14]

$$\left[z\frac{\mathrm{d}^2}{\mathrm{d}z^2}+(c-z)\frac{\mathrm{d}}{\mathrm{d}z}-a\right]y=z^{\sigma-1}. \tag{14}$$

In the light of eq. (12), the particular solution of eq. (8) is expressed as

$$\tilde{F}(z,q)\Big|_P=-\frac{1}{2ik}\left[d(k,\beta,q)\Lambda_{\rho,1}(a,c;z)+\Lambda_{\gamma,1}(a,c;z)\right]. \tag{15}$$

On expanding $\mathrm{e}^{\rho z}$ as a power series in z, the particular integral [14] of eq. (12) can also be expressed in terms of the function $\Lambda_{\rho,\sigma}(a,c;z)$, where

$$\begin{aligned}\Lambda_{\rho,\sigma}(a,c;z)&=\sum_{n=0}^{\infty}\frac{\rho^n}{n!}\theta_{\sigma+n}(a,c;z)\\&=\sum_{n=0}^{\infty}\frac{\Gamma(\sigma+a+n)\Gamma(\sigma)\Gamma(\sigma+c-1)}{\Gamma(\sigma+a)\Gamma(\sigma+n+1)\Gamma(\sigma+c+n)}\\&\quad\times F_{(n+1)}(\sigma,\sigma+c-1;\sigma+a;\rho)z^{\sigma+n},\end{aligned} \tag{16}$$

where $F_{(n+1)}$ stands for the first $(n+1)$ terms of the Gaussian hypergeometric series with the given parameters. In view of eqs (10)–(16) the complete primitive of eq. (8) is written as

$$\tilde{F}(z,q) = A\Phi(1+i\eta,2;z) + B\bar{\Phi}(1+i\eta,2;z) - \frac{1}{2ik}[d(k,\beta,q)\Lambda_{\rho,1}(1+i\eta,2;z) + \Lambda_{\gamma,1}(1+i\eta,2;z)] \tag{17}$$

where A and B are two arbitrary constants. From eqs (5), (17) and $z=-2ikr$, $\tilde{G}^{(+)}_{\mathrm{CY}}(r,q)$ is obtained as

$$\tilde{G}^{(+)}_{\mathrm{CY}}(r,q) = r\mathrm{e}^{ikr}[A\Phi(1+i\eta,2;-2ikr) + B\bar{\Phi}(1+i\eta,2;-2ikr) - \frac{1}{2ik}[d(k,\beta,q)\Lambda_{\rho,1}(1+i\eta,2;-2ikr) + \Lambda_{\gamma,1}(1+i\eta,2;-2ikr)]]. \tag{18}$$

To determine the constants A and B boundary conditions will be applied judiciously at $r=0$ and ∞. As all the quantities in eq. (18) except $\bar{\Phi}(1+i\eta,2;-2ikr)$ are equal to zero at $r=0$, the above equation yields $B=0$ and thus $\tilde{G}^{(+)}_{\mathrm{CY}}(r,q)$ takes the form

$$\tilde{G}^{(+)}_{\mathrm{CY}}(r,q) = r\mathrm{e}^{ikr}\Bigg[A\Phi(1+i\eta,2;-2ikr) - \frac{1}{2ik}\{d(k,\beta,q)\Lambda_{\rho,1}(1+i\eta,2;-2ikr) + \Lambda_{\gamma,1}(1+i\eta,2;-2ikr)\}\Bigg]. \tag{19}$$

To take the limit $r\to\infty$ in the above equation is rather tricky and the procedure is as follows.

The Coulomb and Coulomb–Yamaguchi Green's functions are expressed as [15]

$$G^{(+)}_{\mathrm{C}}(r,r') = -k^{-1}\psi^{(+)}_{\mathrm{C}}(k,r_<)f_{\mathrm{C}}(k,r_>) \tag{20}$$

and

$$G^{(+)}_{\mathrm{CY}}(r,r') = -k^{-1}\psi^{(+)}_{\mathrm{CY}}(k,r_<)f_{\mathrm{CY}}(k,r_>). \tag{21}$$

Here $\psi^{(+)}_{\mathrm{C}}(k,r)$, $f_{\mathrm{C}}(k,r)$, $\psi^{(+)}_{\mathrm{CY}}(k,r)$ and $f_{\mathrm{CY}}(k,r)$ stand for the on-shell physical and Jost solutions for Coulomb and Coulomb–Yamaguchi potentials. The functions $\psi^{(+)}_{\mathrm{CY}}(k,r)$ and $f_{\mathrm{CY}}(k,r)$ are written as [6]

$$\psi^{(+)}_{\mathrm{CY}}(k,r) = \psi^{(+)}_{\mathrm{C}}(k,r) - R(\beta,k)X(\beta,k,r) \tag{22}$$

and

$$f_{\mathrm{CY}}(k,r) = f_{\mathrm{C}}(k,r) - W(\beta,k)Y(\beta,k,r), \tag{23}$$

where

$$R(\beta,k)=\lambda\left[\frac{\mathrm{e}^{i\pi/2}\Gamma(1+i\eta)\mathrm{e}^{-\pi\eta/2}}{2(\beta^2+k^2)D^{(+)}(k)}\right]\left(\frac{\beta-ik}{\beta+ik}\right)^{i\eta}, \tag{24}$$

$$X(\beta,k,r)=r\mathrm{e}^{ikr}\left[-\frac{(2ik)}{(1+i\eta)(\beta-ik)}{}_2F_1\left(1,i\eta;2+i\eta;\frac{\beta+ik}{\beta-ik}\right)\right.$$
$$\left.\times\Phi(1+i\eta,2;-2ikr)-\sum_{n=0}^{\infty}\frac{\rho^n}{n!}\theta_{n+1}(1+i\eta,2;-2ikr)\right], \tag{25}$$

$$W(\beta,k)=\lambda\left[\frac{\mathrm{e}^{\pi\eta/2}}{D(k)\Gamma(2+i\eta)(\beta-ik)}\right]{}_2F_1\left(1,i\eta;2+i\eta;\frac{\beta+ik}{\beta-ik}\right), \tag{26}$$

$$Y(\beta,k,r)=r\mathrm{e}^{ikr}\left[\frac{1}{(1+i\eta)(\beta-ik)^2}\ {}_2F_1\left(1,i\eta;2+i\eta;\frac{\beta+ik}{\beta-ik}\right)\right.$$
$$\times\Phi(1+i\eta,2;-2ikr)+\frac{2ik\Gamma(1+i\eta)}{(\beta^2+k^2)}\left(\frac{\beta-ik}{\beta+ik}\right)^{i\eta}$$
$$\left.\times\Psi(1+i\eta,2;-2ikr)-\sum_{n=0}^{\infty}\frac{\rho^n}{n!}\theta_{n+1}(1+i\eta,2;-2ikr)\right], \tag{27}$$

with the Fredholm determinants $D(k)$ and $D^{(+)}(k)$ for regular, irregular and physical boundary conditions [16].

Also, the last two terms in eq. (19) can be expressed directly in terms of regular Coulomb Green's function as [6]

$$-\frac{1}{2ik}r\mathrm{e}^{ikr}d(k,\beta,q)\Lambda_{\rho,1}(1+i\eta,2;-2ikr)=\int_0^r G_{\mathrm{C}}^{(\mathrm{R})}(r,s)\mathrm{e}^{-\beta s}\mathrm{d}s \tag{28}$$

and

$$-\frac{1}{2ik}r\mathrm{e}^{ikr}\Lambda_{\gamma,1}(1+i\eta,2;-2ikr)=\int_0^r G_{\mathrm{C}}^{(\mathrm{R})}(r,s)\mathrm{e}^{iqs}\mathrm{d}s, \tag{29}$$

where $G_{\mathrm{C}}^{(\mathrm{R})}(r,r')$ is the regular Coulomb Green's function [15]. Substitution of eqs (20)–(23) together with eqs (28) and (29) in eq. (19) leads to

$$-k^{-1}\left[f_{\mathrm{C}}(k,r)\int_0^r\mathrm{d}r'\mathrm{e}^{iqr'}\psi_{\mathrm{C}}^{(+)}(k,r')+\psi_{\mathrm{C}}^{(+)}(k,r)\int_r^\infty\mathrm{d}r'\mathrm{e}^{iqr'}f_{\mathrm{C}}(k,r')\right]$$
$$+W(\beta,k)k^{-1}\left[Y(\beta,k,r)\int_0^r\mathrm{d}r'\mathrm{e}^{iqr'}\psi_{\mathrm{C}}^{(+)}(k,r')+\psi_{\mathrm{C}}^{(+)}(k,r)\right.$$
$$\left.\times\int_r^\infty\mathrm{d}r'\mathrm{e}^{iqr'}Y(\beta,k,r')\right]+R(\beta,k)k^{-1}$$

$$\times\left[f_{\rm C}(k,r)\int_0^r {\rm d}r'{\rm e}^{iqr'}X(\beta,k,r')+X(\beta,k,r)\int_r^\infty {\rm d}r'{\rm e}^{iqr'}f_{\rm C}(k,r')\right]$$
$$-W(\beta,k)R(\beta,k)k^{-1}\left[Y(\beta,k,r)\int_0^r {\rm d}r'{\rm e}^{iqr'}X(\beta,k,r')\right.$$
$$\left.+X(\beta,k,r)\int_r^\infty {\rm d}r'{\rm e}^{iqr'}Y(\beta,k,r')\right]$$
$$= Ar\,{\rm e}^{ikr}\Phi(1+i\eta,2;-2ikr)+d(k,q,\beta)$$
$$\times\int_0^r {\rm d}s\,{\rm e}^{-\beta s}G_{\rm C}^{({\rm R})}(r,s)+\int_0^r {\rm d}s\,{\rm e}^{iqs}G_{\rm C}^{({\rm R})}(r,s). \tag{30}$$

Substitution of eqs (24)–(27) in the above equation, implementation of limit $r\to\infty$ and certain algebraic manipulations leads to

$$\lim_{r\to\infty}\left\{r{\rm e}^{ikr}\Phi(1+i\eta,2;-2ikr)\right.$$
$$\times\left[-\frac{{\rm e}^{i\pi/2}}{(1+i\eta)(k+q)}\,{}_2F_1\left(1,i\eta;2+i\eta;\frac{q-k}{q+k}\right)-A\right.$$
$$\left.\left.-\frac{d(k,\beta,q)}{(1+i\eta)(\beta-ik)}\,{}_2F_1\left(1,i\eta;2+i\eta;\frac{\beta+ik}{\beta-ik}\right)\right]\right\}$$
$$=\lim_{r\to\infty}\left\{r{\rm e}^{ikr}\Psi(1+i\eta,2;-2ikr)\left[\frac{2ik\Gamma(1+i\eta)}{(\beta^2+k^2)}\left(\frac{\beta-ik}{\beta+ik}\right)^{i\eta}\right.\right.$$
$$\times\left\{d(k,\beta,q)+\lambda\frac{{\rm e}^{i\pi/2}}{D^{(+)}(k)(q+k)(1+i\eta)(\beta-ik)}\right.$$
$$\left.\left.\left.\times\,{}_2F_1\left(1,i\eta;2+i\eta;\frac{(q-k)(\beta+ik)}{(q+k)(\beta-ik)}\right)\right\}\right]\right\}. \tag{31}$$

In evaluating eq. (31) the following standard integrals [13,14]

$$\int_0^\infty {\rm e}^{-ax}x^{s-1}\Psi(b,d;\mu x){\rm d}x=\frac{\Gamma(1+s-d)\Gamma(s)}{a^s\Gamma(1+b+s-d)}$$
$$\times F(b,s;1+b+s-d;1-\mu/a), \tag{32}$$

$$\int_0^\infty {\rm e}^{-\lambda z}z^\nu\Phi(a,c;pz){\rm d}z=\frac{\Gamma(\nu+1)}{\lambda^{\nu+1}}F\left(a,\nu+1;c;\frac{p}{\lambda}\right) \tag{33}$$

and relation between Gaussian hypergeometric functions [13]

$$F(a,b;c;z)=(1-z)^{c-a-b}F(c-a,c-b;c;z) \tag{34}$$

have been used. In the pure Coulomb case [17] it has been observed that the coefficient related with $r{\rm e}^{ikr}\Psi(1+i\eta,2;-2ikr)$ in eq. (31) led to zero. Thus in analogy with Coulomb case it is assumed here that the coefficient of $r{\rm e}^{ikr}\Psi(1+i\eta,2;-2ikr)$ in eq. (31) must go to zero to have

$$d(k,\beta,q) = -\lambda \frac{e^{i\pi/2}}{D^{(+)}(k)(1+i\eta)(q+k)(\beta-ik)(\beta-iq)} \times {}_2F_1\left(1, i\eta; 2+i\eta; \frac{(q-k)(\beta+ik)}{(q+k)(\beta-ik)}\right). \tag{35}$$

Substituting eq. (35) in eq. (31) the desired value of constant A is obtained as

$$\begin{aligned} A = -\frac{e^{i\pi/2}}{(1+i\eta)(q+k)} \Bigg[{}_2F_1\left(1, i\eta; 2+i\eta; \frac{(q-k)}{(q+k)}\right) \\ +\lambda \frac{e^{i\pi/2}}{D^{(+)}(k)(1+i\eta)(\beta-ik)^2(\beta-iq)} {}_2F_1\left(1, i\eta; 2+i\eta; \frac{(\beta+ik)}{(\beta-ik)}\right) \\ \times {}_2F_1\left(1, i\eta; 2+i\eta; \frac{(q-k)(\beta+ik)}{(q+k)(\beta-ik)}\right)\Bigg]. \end{aligned} \tag{36}$$

Having the value of the constant A and $d(k,\beta,q)$, eq. (19) yields

$$\begin{aligned} \tilde{G}^{(+)}_{\rm CY}(r,q) = r e^{ikr} \Bigg[\frac{e^{-i\pi/2}}{(1+i\eta)(q+k)} {}_2F_1\left(1, i\eta; 2+i\eta; \frac{(q-k)}{(q+k)}\right) \\ \times \Phi(1+i\eta, 2; -2ikr) - \frac{1}{2ik}\Lambda_{\gamma,1}(1+i\eta, 2; -2ikr) \\ +\lambda \frac{e^{i\pi/2}}{D^{(+)}(k)(1+i\eta)(\beta-ik)^2(\beta-iq)} \\ \times {}_2F_1\left(1, i\eta; 2+i\eta; \frac{(q-k)(\beta+ik)}{(q+k)(\beta-ik)}\right) \\ \times \Bigg\{ \frac{1}{(1+i\eta)(\beta-ik)} {}_2F_1\left(1, i\eta; 2+i\eta; \frac{(\beta+ik)}{(\beta-ik)}\right) \\ \times \Phi(1+i\eta, 2; -2ikr) + \frac{1}{2ik}\Lambda_{\rho,1}(1+i\eta, 2; -2ikr)\Bigg\}. \end{aligned} \tag{37}$$

The expression for $\tilde{G}^{(+)}_{\rm CY}(r,-q)$ can be written by replacing q by $-q$ in eq. (37). Therefore, having the expressions for $\tilde{G}^{(+)}_{\rm CY}(r,q)$ and $\tilde{G}^{(+)}_{\rm CY}(r,-q)$ one will be in a position to write the off-shell physical/outgoing wave function for Coulomb–Yamaguchi potential by exploiting the relation [17]

$$\psi^{(+)}_{\rm CY}(k,q,r) = \frac{(k^2-q^2)}{2i}\left[\tilde{G}^{(+)}_{\rm CY}(r,q) - \tilde{G}^{(+)}_{\rm CY}(r,-q)\right]. \tag{38}$$

3. Coulomb–Yamaguchi off-shell Jost solution

The integral transforms of the physical/outgoing wave Coulomb Green's function can be exploited to construct exact analytical expressions for off-shell physical and Jost solutions. There exists a relation between off-shell physical and Jost solutions written as

$$\psi_{\ell\mathrm{CS}}^{(+)}(k,q,r) = -\frac{1}{2}\pi q T_{\ell\mathrm{CS}}(k,q,k^2)\mathrm{e}^{-i\ell\pi/2} f_{\ell\mathrm{CS}}(k,r) + \frac{1}{2i}\left[\mathrm{e}^{-i\ell\pi/2} f_{\ell\mathrm{CS}}(k,q,r) - \mathrm{e}^{i\ell\pi/2} f_{\ell\mathrm{CS}}(k,-q,r)\right], \tag{39}$$

where the half-off-shell T-matrix

$$T_{\ell\mathrm{CS}}(k,q,k^2) = \left(\frac{k}{q}\right)^{\ell} \left\{\frac{f_{\ell\mathrm{CS}}(k,q) - f_{\ell\mathrm{CS}}(k,-q)}{i\pi q f_{\ell\mathrm{CS}}(k)}\right\}. \tag{40}$$

Here the subscript CS stands for Coulomb plus separable potential. The quantities $f_{\ell\mathrm{CS}}(k,r)$, $f_{\ell\mathrm{CS}}(k,q,r)$ and $f_{\ell\mathrm{CS}}(k,q)$, $f_{\ell\mathrm{CS}}(k)$ in eqs (39) and (40) represent the on- and off-shell Jost solutions and Jost functions [6,8,10] respectively.

The basic aim here is to express eq. (38) in the form of eq. (39). To that end substitution of eqs (37) in eq. (38), some rearrangements and manipulations lead to

$$\begin{aligned}
\psi_{\mathrm{CY}}^{(+)}(k,q,r) = &-\frac{1}{2}\pi q T(k,q,k^2) f_{\mathrm{CY}}(k,r) \\
&+\frac{1}{2i}\left\{ f_{\mathrm{C}}(k,q,r) - \lambda \frac{f_{\mathrm{C}}(k,q) r \mathrm{e}^{ikr}}{D(k)(1+i\eta)(\beta-ik)} \right. \\
&\times \left[{}_2F_1\left(1, i\eta; 2+i\eta; \frac{(\beta+ik)}{(\beta-ik)}\right) + \frac{\mathrm{e}^{i\pi/2}(q-k)}{(\beta-iq) f^{C}(k,q)} \right. \\
&\left. \times F\left(1, i\eta; 2+i\eta; \frac{(q-k)(\beta+ik)}{(q+k)(\beta-ik)}\right)\right] \\
&\times \left\{ \frac{1}{(1+i\eta)(\beta-ik)} {}_2F_1\left(1, i\eta; 2+i\eta; \frac{(\beta+ik)}{(\beta-ik)}\right) \right. \\
&\times \Phi(1+i\eta, 2; -2ikr) + \frac{2ik\Gamma(1+i\eta)}{(\beta^2+k^2)}\left(\frac{\beta-ik}{\beta+ik}\right)^{i\eta} \\
&\left. \times \Psi(1+i\eta, 2; -2ikr) + \frac{1}{2ik}\Lambda_{\rho,1}(1+i\eta, 2; -2ikr) \right\} \\
&- f_{\mathrm{C}}(k,-q,r) + \lambda \frac{f_{\mathrm{C}}(k,-q) r \mathrm{e}^{ikr}}{D(k)(1+i\eta)(\beta-ik)} \\
&\times \left[{}_2F_1\left(1, i\eta; 2+i\eta; \frac{(\beta+ik)}{(\beta-ik)}\right) - \frac{\mathrm{e}^{i\pi/2}(q+k)}{(\beta+iq) f^{C}(k,-q)} \right. \\
&\left. \times F\left(1, i\eta; 2+i\eta; \frac{(q+k)(\beta+ik)}{(q-k)(\beta-ik)}\right)\right] \\
&\times \left\{ \frac{1}{(1+i\eta)(\beta-ik)^2} \; {}_2F_1\left(1, i\eta; 2+i\eta; \frac{(\beta+ik)}{(\beta-ik)}\right) \right. \\
&\times \Phi(1+i\eta, 2; -2ikr) + \frac{2ik\Gamma(1+i\eta)}{(\beta^2+k^2)}\left(\frac{\beta-ik}{\beta+ik}\right)^{i\eta} \\
&\left.\left. \times \Psi(1+i\eta, 2; -2ikr) + \frac{1}{2ik}\Lambda_{\rho,1}(1+i\eta, 2; -2ikr) \right\}\right\}.
\end{aligned} \tag{41}$$

On comparing the above equation with s-wave version of eq. (39) the Coulomb–Yamaguchi Jost solution $f_{\rm CY}(k,q,r)$ is identified as

$$\begin{aligned} f_{\rm CY}(k,q,r) &= f_{\rm C}(k,q,r) - \lambda \frac{f_{\rm C}(k,q) r {\rm e}^{ikr}}{D(k)(1+i\eta)(\beta - ik)} \\ &\times \Bigg[{}_2F_1 \left(1, i\eta; 2+i\eta; \frac{(\beta+ik)}{(\beta-ik)}\right) \\ &+ \frac{{\rm e}^{i\pi/2}(q-k)}{(\beta-iq) f^C(k,q)} F\left(1, i\eta; 2+i\eta; \frac{(q-k)(\beta+ik)}{(q+k)(\beta-ik)}\right)\Bigg] \\ &\times \Bigg\{ \frac{1}{(1+i\eta)(\beta-ik)} {}_2F_1\left(1, i\eta; 2+i\eta; \frac{(\beta+ik)}{(\beta-ik)}\right) \\ &\times \Phi(1+i\eta, 2; -2ikr) + \frac{2ik\Gamma(1+i\eta)}{(\beta^2+k^2)} \left(\frac{\beta-ik}{\beta+ik}\right)^{i\eta} \\ &\times \Psi(1+i\eta, 2; -2ikr) + \frac{1}{2ik}\Lambda_{\rho,1}(1+i\eta, 2; -2ikr)\Bigg\}. \end{aligned} \quad (42)$$

The above result is in exact agreement with the expression in ref. [6] derived via a different approach to the problem. This proves the correctness of the expression for $\psi^{(+)}_{\rm CY}(k,q,r)$. A couple of useful checks is made on the expression for Coulomb–Yamaguchi off-shell Jost solution with particular emphasis on their limiting behaviour and on-shell discontinuity. For example, in the absence of Yamaguchi potential, i.e. $\lambda = 0$, $f_{\rm CY}(k,q,r)$ goes to pure Coulomb Jost solution [17,18]. Secondly, in the limit of no Coulomb field, $\eta = 0$, Yamaguchi Jost solution is obtained [19]. When both λ and η go to zero, $f_{\rm CY}(k,q,r) = {\rm e}^{ikr}$. Other useful checks consist in showing that $f_{\rm CY}(k,q,r)_{r\to 0} \to f_{\rm CY}(k,q)$ and

$$f_{\rm CY}(k,r) = \lim_{q\to k} [{\rm e}^{\pi\eta/2}/\Gamma(1+i\eta)](q+k)/(q+k)]^{i\eta} f_{\rm CY}(k,q,r).$$

It has been checked that the above results are in order and are in agreement with that of van Haeringen [20] and Talukdar *et al* [21].

4. Off-shell T-matrix

Knowledge of off-shell physical solution is indispensable to construct an analytical expression for off-shell T-matrix. Thus, by exploiting the following relation that exists between off-shell physical solution and T-matrix $T(p,q,k^2)$ [22,23]

$$\begin{aligned} T_{\ell {\rm CS}}(p,q,k^2) &= \frac{2}{\pi pq} \Bigg[\int_0^\infty {\rm d}r \hat{j}_\ell(pr) V_{\rm C}(r) \psi^{(+)}_{\ell{\rm CS}}(k,q,r) \\ &+ \sum_{i=1}^N \lambda_{\ell i} \int_0^\infty {\rm d}r \hat{j}_\ell(pr) g_{\ell i}(\beta_{\ell i}, r) \\ &\times \int_0^\infty {\rm d}s g_{\ell i}(\beta_{\ell i}, s) \psi^{(+)}_{\ell{\rm CS}}(k,q,s)\Bigg] \end{aligned} \quad (43)$$

with $\hat{j}_\ell^{(+)}(x)$, spherical Bessel function and $g_{\ell i}(\beta_{\ell i}, r)$s, the form factors of the separable potential the desired quantity $T_{\rm CY}(p, q, k^2)$ is obtained as [6]

$$\begin{aligned} T_{\rm CY}(p,q,k^2) &= T_{\rm C}(p,q,k^2) + \frac{2}{\pi pq} K_{\rm CY}(\beta,q,k^2) \\ &\times \Bigg[-\frac{p}{(\beta^2+p^2)} + \frac{k}{(\beta^2+k^2)} \\ &\times \Bigg\{ {}_2F_1\left(1, i\eta; 1+i\eta; \frac{(p-k)(\beta+ik)}{(p+k)(\beta-ik)}\right) \\ &- {}_2F_1\left(1, i\eta; 1+i\eta; \frac{(p+k)(\beta+ik)}{(p-k)(\beta-ik)}\right) \Bigg\} \Bigg], \end{aligned} \tag{44}$$

with the Coulomb off-shell T-matrix

$$\begin{aligned} T_{\rm C}(p,q,k^2) = \frac{{\rm e}^{i\pi/2}k}{\pi pq} \Bigg[&F\left(1, i\eta; 1+i\eta; \frac{(q-k)(p-k)}{(q+k)(p+k)}\right) \\ &- F\left(1, i\eta; 1+i\eta; \frac{(q+k)(p-k)}{(q-k)(p+k)}\right) \\ &- F\left(1, i\eta; 1+i\eta; \frac{(q-k)(p+k)}{(q+k)(p-k)}\right) \\ &+ F\left(1, i\eta; 1+i\eta; \frac{(q+k)(p+k)}{(q-k)(p-k)}\right) \Bigg]. \end{aligned} \tag{45}$$

The results in eqs (44) and (45) are in exact agreement with that of van Haeringen and van Wageningen [5] and of ref. [6]. However, van Haeringen and van Wageningen derived these results in the momentum space approach in contrast to r-space approach.

Now a slightly different method will be described for obtaining the expression for the T-matrix for Coulomb and Coulomb–Yamaguchi potentials by generating a relation between the corresponding off-shell physical solution and the T-matrix which does not involve the related potential explicitly.

The s-wave off-shell physical/outgoing wave solution for motion in a potential field $V(r)$ satisfies the differential equation

$$\left[\frac{{\rm d}^2}{{\rm d}r^2} + k^2 - V(r)\right] \psi^{(+)}(k,q,r) = (k^2 - q^2)\sin qr. \tag{46}$$

Equation (46) can be rewritten in the following form:

$$V(r)\psi^{(+)}(k,q,r) = \left[\frac{{\rm d}^2}{{\rm d}r^2} + k^2\right] \left\{\psi^{(+)}(k,q,r) - \sin qr\right\}. \tag{47}$$

To remove the potential function from the expression for off-shell T-matrix, eq. (47) is inserted in the s-wave version of eq. (43) (without subscripts) to have

$$\begin{aligned} T(p,q,k^2) &= \frac{2(k^2-p^2)}{\pi pq} \\ &\times \left[\int_0^\infty {\rm d}r\, \sin(pr)\psi^{(+)}(k,q,r) - \int_0^\infty {\rm d}r\, \sin(pr)\sin(qr)\right]. \end{aligned} \tag{48}$$

In deriving (48) transposed operator relation [24] $\langle\varphi|\hat{o}|\psi\rangle = \langle\psi|\tilde{o}|\varphi\rangle$, where the operator $\hat{o}$ is Hermitian, i.e. $\tilde{o} = \hat{o}$ ($\tilde{o}$ is the transposed conjugate of operator $\hat{o}$), is used judiciously. The second integral in the above equation extends no contribution to T-matrix. The above relation between T-matrix and $\psi^{(+)}(k,q,r)$ is equally applicable for local (Coulomb) and non-local (Coulomb–Yamaguchi) potentials. Therefore, in view of eqs (37), (38) and (48) $T_{\rm CY}(p,q,k^2)$ is obtained as

$$\begin{aligned} T_{\rm CY}(p,q,k^2) &= T_{\rm C}(p,q,k^2) \\ &\quad + \frac{(k^2-p^2)}{\pi ipq} K_{\rm CY}(\beta,q,k^2)[L_5(p,\beta,k^2) - L_6(p,\beta,k^2)] \end{aligned} \tag{49}$$

with

$$\begin{aligned} T_{\rm C}(p,q,k^2) &= \frac{(k^2-p^2)}{\pi ipq}[L_1(p,q,k^2) \\ &\quad -L_2(p,q,k^2) - L_3(p,q,k^2) + L_4(p,q,k^2)] \end{aligned} \tag{50}$$

and

$$\begin{aligned} K_{\rm CY}(\beta,q,k^2) &= \lambda \frac{(\beta+ik)(\beta+iq)}{2D^{(+)}(k)(1+i\eta)(\beta_2+k^2)(\beta^2+q^2)} \\ &\quad \times \left[(k-q)\,{}_2F_1\left(1,i\eta;2+i\eta;\frac{(q-k)(\beta+ik)}{(q+k)(\beta-ik)}\right)\right. \\ &\quad \left. -(k+q){}_2F_1\left(1,i\eta;2+i\eta;\frac{(q+k)(\beta+ik)}{(q-k)(\beta-ik)}\right)\right]. \end{aligned} \tag{51}$$

The quantities $L_1(p,q,k^2)$, $L_2(p,q,k^2)$, $L_3(p,q,k^2)$, $L_4(p,q,k^2)$, $L_5(p,\beta,k^2)$ and $L_6(p,\beta,k^2)$ in (49) and (50) are defined as

$$L_1(p,q,k^2) = \int_0^\infty {\rm d}r\, {\rm e}^{ipr}\psi_{\rm C}^{(+)}(k,q,r), \tag{52}$$

$$\begin{aligned} L_2(p,q,k^2) &= [L_1(p,q,k^2)]_{q\to -q}, \\ L_3(p,q,k^2) &= [L_1(p,q,k^2)]_{p\to -p}, \\ L_4(p,q,k^2) &= [L_1(p,q,k^2)]_{q\to -q,p\to -p}, \end{aligned} \tag{53}$$

and

$$L_5(p,\beta,k^2) = \int_0^\infty {\rm d}r\, {\rm e}^{ipr}U(\beta,k,r), \tag{54}$$

$$L_6(p,\beta,k^2) = [L_5(p,\beta,k^2)]_{p\to -p}. \tag{55}$$

To evaluate the integrals in eqs (52)–(55), to remove the infinite sums involved and express them in the maximal reduced form eqs (33), (34) and the following standard integrals, relations and integral representation [13,14,25–27]

$$\int_0^\infty \mathrm{d}z\, z^\nu \mathrm{e}^{-bz} \theta_\sigma(a, c; pz) = \frac{\Gamma(\nu + \sigma + 1) p^\sigma}{\sigma(\sigma + c - 1) b^{\nu+\sigma+1}}$$
$$\times {}_3F_2 \left(1, \sigma + a, \nu + \sigma + 1; \sigma + 1 \sigma + c; \frac{p}{b}\right),$$
$$\times \mathrm{Re}\, \sigma > 0, \quad \mathrm{Re}\, (\sigma + c) > 1, \quad \mathrm{Re} \nu > -1, \quad \mathrm{Re}\, b > \mathrm{Re}\, p, \tag{56}$$

$$F(a, b; c; z) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} F(a, b; a + b - c + 1; 1 - z)$$
$$+ (1 - z)^{c-a-b} \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)}$$
$$\times F(c - a, c - b; c - a - b + 1; 1 - z) \tag{57}$$

$$c\, [{}_2F_1(a, b; c; z) - {}_2F_1(a + 1, b; c; z)] + bz\, {}_2F_1(a + 1, b + 1; c + 1; z) = 0, \tag{58}$$

$${}_2F_1(a, b; c; z) = (1 - z)^{-a}\, {}_2F_1 \left(a, c - b; c; \frac{z}{z - 1}\right), \tag{59}$$

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 \mathrm{d}t\, t^{b-1} (1 - t)^{c-b-1} (1 - tz)^{-a} \tag{60}$$

and

$${}_2F_1(1, i\eta; 2 + i\eta; z) = (1 + i\eta) \left\{ \frac{1}{z} + \left(1 - \frac{1}{z}\right) {}_2F_1(1, i\eta; 1 + i\eta; z) \right\} \tag{61}$$

have been used to get

$$L_1(p, q, k^2) = -\frac{(q + k)(p + k)}{2(p + q)(p^2 - k^2)} + \frac{k}{(p^2 - k^2)}$$
$$\times {}_2F_1 \left(1, i\eta; 1 + i\eta; \frac{(q - k)(p - k)}{(q + k)(p + k)}\right) \tag{62}$$

and

$$L_5(p, \beta, k^2) = \frac{1}{(p^2 - k^2)} \left\{ \frac{i(p + k)}{(\beta + ik)(\beta - ip)} - \frac{2ik}{(\beta^2 + k^2)} \right.$$
$$\left. \times {}_2F_1 \left(1, i\eta; 1 + i\eta; \frac{(\beta + ik)(p - k)}{(\beta - ik)(p + k)}\right) \right\}. \tag{63}$$

Substitution of the expressions for $L_1(p, q, k^2)$ and $L_5(p, \beta, k^2)$ in conjunction with eqs (52)–(55) in eqs (49) and (50) reproduces the desired result for off-shell T-matrix already reported in eqs (44) and (45). The merit of the present approach in calculating T-matrices is that it is much more simpler and straightforward than the earlier approach [5,6] as the Coulomb interaction does not involve in eq. (48) explicitly.

5. Results, discussion and conclusion

The results for both $f_{\rm Y}(k,q)$ and $f_{\rm CY}(k,q)$ for $q < k$ and $q > k$ have been computed and plotted in figures 1 and 2 as a function of off-shell momentum q for laboratory energies 10 and 30 MeV respectively for (p–p) system in the $^1S_0$ channel. Here the numbers refer to a repulsive Coulomb potential with $(2k\eta)^{-1} = 28.8$ fm and attractive short-range potential with $\beta = 1.1$ fm^{-1} and $\lambda = -2.405$ fm^{-3}. The values of $f_{\rm Y}(k,q)$ and $f_{\rm CY}(k,q)$ provide a basis for investigating the role of long-range Coulomb interaction in (p–p) scattering off the energy shell. Real parts of both $f_{\rm Y}(k,q)$ and $f_{\rm CY}(k,q)$ are positive while imaginary parts of them are negative over the entire range of q. The quantity $f_{\rm Y}(k,q)$ is a continuous function of q. Real part of $f_{\rm Y}(k,q)$ increases whereas ${\rm Im}f_{\rm Y}(k,q)$ decreases smoothly with q for $E_{\rm Lab} = 10$ and 30 MeV. But $f_{\rm CY}(k,q)$ exhibits a discontinuity at the on-shell point $q = k$. However, the singular behaviour cannot be clearly seen in the scale of the figures. For $q < k$, ${\rm Re}f_{\rm CY}(k,q)$ increases with q and reaches discontinuity at $q = k$. Beyond on-shell point ($q > k$) ${\rm Re}f_{\rm CY}(k,q)$ reaches its minimum at $q = 0.37$ (10 MeV) and 0.62 (30 MeV) and then increases smoothly with q. In contrast, for $q < k$, ${\rm Im}f_{\rm CY}(k,q)$ initially decreases with q, reaches its minimum values at $q = 0.33$ (10 MeV) and 0.57 (30 MeV) then increases and reaches its discontinuity at $q = k$. For $q > k$, ${\rm Im}f_{\rm CY}(k,q)$ decreases with q. The values of $f_{\rm Y}(k,q)$ and $f_{\rm CY}(k,q)$ differ much for lower values of energy and momentum (q), the difference becomes insignificant for large energy and momentum values as the Coulomb distortion is predominant [28] at low and intermediate range of these parameters. The results for $f_{\rm CY}(k,q)$ exhibit characteristic discontinuity at the energy-shell arising from the fact that the Coulomb potential distorts not only the scattered wave but also the incident plane wave [29]. Sharma and Jain [30] and Kok *et al* [31] confirmed that off-shell effects are sizeable for $(\alpha, 2\alpha)$ reaction. The class of reactions like $(p,2p)$ and (p,p) bremsstrahlung are believed to probe the off-shell two-nucleon

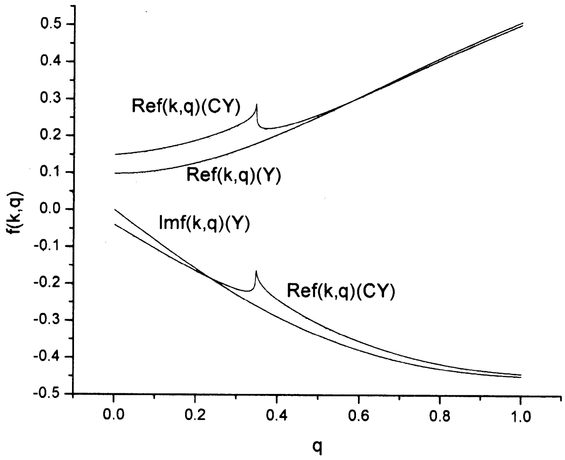

Figure 1. Off-shell Jost functions for Yamaguchi and Coulomb–Yamaguchi potentials as a function of q for laboratory energy 10 MeV.

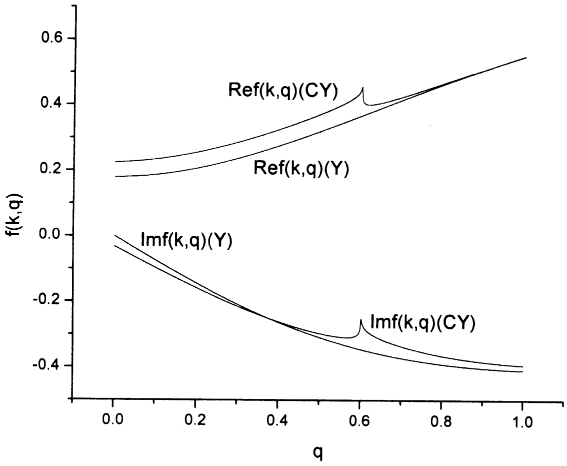

Figure 2. Off-shell Jost functions for Yamaguchi and Coulomb–Yamaguchi potentials as a function of q for laboratory energy 30 MeV.

force directly. The results in figures 1 and 2 are in order for $(p, 2p)$ reaction in which a single proton is knocked out of the nucleus and the momentum transfer distribution is measured [32].

By exploiting the relation that exists between off-shell physical solution and off-shell Jost solutions and functions, one will be in a position to write an expression for $\psi^{(+)}(k, q, r)$ and thereby off-shell T-matrix for Coulomb and Coulomb-like potentials. This alternative approach to the problem represents a straightforward method to deal with off-shell scattering on the same class of potentials using the full Green's function. The exact analytical expression for off-shell Jost solution for scattering by Coulomb–Yamaguchi potential is believed to be useful for the description of the charged particle scattering/reaction processes. The present approach can easily be extended to deal with potentials of higher rank and restriction to symmetric form factors is not compelling.

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