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Effects of three-body atomic interaction and optical lattice on solitons in quasi-one-dimensional Bose–Einstein condensate

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Abstract. We make use of a coordinate-free approach to implement Vakhitov–Kolokolov criterion for stability analysis in order to study the effects of three-body atomic recombination and lattice potential on the matter–wave bright solitons formed in Bose–Einstein condensates. We analytically demonstrate that (i) the critical number of atoms in a stable BEC soliton is just half the number of atoms in a marginally stable Townes-like soliton and (ii) an additive optical lattice potential further reduces this number by a factor of $\sqrt{1-bg_3}$ with g_3 the coupling constant of the lattice potential and $b = 0.7301$.

Keywords. Bose–Einstein condensates; solitons; Vakhitov–Kolokolov criterion; optical lattice; critical number of atoms.

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1. Introduction

Solitons of the quintic nonlinear Schrödinger (NLS) equation in one dimension (1D) are unstable and denote localized excitations similar to Townes solitons of the cubic NLS equation in two spatial dimensions (2D) [1]. As opposed to this, the NLS equation with purely cubic nonlinearity has a stable solution in the 1D case. The NLS equation containing both cubic and quintic terms has many applicative relevance. For example, in the case of Bose–Einstein condensation it models bright solitons in the condensate with two- and three-body interactions. Further, Bose–Einstein condensates (BECs) in the optical lattice allow interesting localized phenomena [2]. One finds localized states with energies lying in the gaps of the band structure that arise in the linear periodic problem. A BEC initially confined in a highly elongated harmonic trap if allowed for free propagation through an optical lattice along the x direction will be governed by the quasi-one-dimensional evolution equation [3]

$$i\psi_t + \psi_{xx} + g_1|\psi|^4\psi + g_2|\psi|^2\psi + g_3\cos 2x\psi - \psi = 0. \quad (1)$$

Equation (1) is often called the Gross–Pitaevskii equation. Here $\psi = \psi(x,t)$ is the wave function of the condensate. It is also called the order parameter. The last term in (1) gives the signature of the initial trap in the propagating BEC. Clearly, the term before the last stands for the optical lattice potential with a coupling constant g_3. The cubic (4th) and quintic (3rd) terms have their dynamical origin in the two- and three-body atomic interactions. The coupling constants g_2 and g_1 are parametrized by the s-wave scattering length for atom–atom collisions in the BEC.

The coupling constant $g_1 \ll g_2$. Thus the three-body interaction cannot destabilize the BEC soliton. Despite that, it may be quite interesting to examine if the perturbative effect of this interaction could be judiciously exploited to derive some new physical information for the Bose-condensed atoms. Moreover, it is always an interesting curiosity to investigate the influence of lattice potentials on stable BEC solitons. To achieve this goal we shall make use of the Vakhitov–Kolokolov criterion (VKC) [4] for stability analysis. Traditionally, the application of the VKC requires an explicit form of the stationary solution of the associated evolution equation in terms of the space variables. But such solution may not be always available. Keeping this in view we introduce in §2 a coordinate-free approach to implement VKC for stability analysis [5]. For the quintic NLS equation the stationary solution is available in closed form. We first use this solution in VKC to demonstrate the instability of the Townes-like soliton and then derive the same result by using our coordinate-free approach. In §3 we separately deal with the cubic–quintic nonlinear Schrödinger (CQNLS) equation and the full equation in (1) representing BEC soliton loaded in a lattice. The treatment of the CQNLS equation is quite straightforward and provides us with a basis to express the critical number of atoms ($N_{\rm c}$) in the soliton of the CQNLS equation with the mass of the marginally stable Townes-like soliton. The periodic potential in the full equation breaks the translational invariance of the problem. This introduces new mathematical complications even in our coordinate-free approach. To derive an analytical expression for the first integral of (1) we need to introduce an ansatz for the effect of the optical lattice on the BEC soliton.

2. Coordinate-free approach for VKC

The solitary wave solutions of the quintic NLS equation given by

$$i\psi_t + \psi_{xx} + g_1|\psi|^4\psi = 0 \tag{2}$$

for $g_1 > 0$ have stationary solutions of the form [6]

$$\psi(x,t) = \phi(x,\Lambda)\mathrm{e}^{i\Lambda t}. \tag{3}$$

Using (3) in (2) we write a nonlinear eigenvalue equation

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} + g_1\phi^5 - \Lambda\phi = 0. \tag{4}$$

Understandably, Λ represents the frequency of the phase or the so-called chemical potential. The solution

$$\phi(x,\Lambda) = \left(\frac{3\Lambda}{g_1}\right)^{1/4} \mathrm{sech}^{1/2}\left(2\sqrt{\Lambda}x\right) \tag{5}$$

of (4) gives the mass

$$N(\Lambda) \text{ or } N_{\mathrm{c}}^{Q} = \int_{-\infty}^{\infty} |\psi|^2 \mathrm{d}x = \int_{-\infty}^{\infty} \phi^2(x,\Lambda)\mathrm{d}x = \sqrt{\frac{3}{g_1}}\frac{\pi}{2}. \tag{6}$$

The Vakhitov–Kolokolov criterion tells us that the soliton solutions $\phi(x,\Lambda)$ for different Λ values are stable and unstable if $\frac{\mathrm{d}N}{\mathrm{d}\Lambda} > 0$ and $\frac{\mathrm{d}N}{\mathrm{d}\Lambda} < 0$ respectively. Since $N(\Lambda)$ in (6) does not depend on Λ, application of VKC implies marginal stability of these solitary solutions. Thus, if a solution is perturbed such that $N > N_{\mathrm{c}}^{Q}(=\sqrt{\frac{3}{g_1}}\frac{\pi}{2})$, a singularity appears in the intensity profile within a finite value of t and the solution collapses. On the other hand, a perturbed stationary solution with $N < N_{\mathrm{c}}$ cannot remain localized. The delocalization leads to complete dispersion of the solitary wave.

From the above it is clear that application of the VKC for stability analysis requires the explicit form of the stationary solution in terms of the x coordinate. But similar forms are not easily available for (1). Thus, it is quite urgent to have a theoretical framework in which use of the VKC will not call for a specific functional form of $\phi(x,\Lambda)$. In the following we describe a coordinate-free approach for stability analysis on the basis of VKC.

Equation (4) can be integrated to get

$$E = \frac{1}{2}\phi_x^2 + \frac{1}{6}g_1\phi^6 - \frac{1}{2}\Lambda\phi^2. \tag{7}$$

The first integral E of (4) is the Hamiltonian or energy density of the quintic NLS equation. By imposing the boundary condition $\phi \to 0$ for $x \to \infty$ one finds $E = 0$. Thus

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = \phi\left(\Lambda - \frac{g_1}{3}\phi^4\right)^{1/2}. \tag{8}$$

Substituting the value of $\mathrm{d}x$ from (8) in (6) we have

$$N(\Lambda) = \lim_{\phi\to 0}\left[\int \frac{\phi\,\mathrm{d}\phi}{\left(\Lambda - \frac{g_1}{3}\phi^4\right)^{1/2}}\right]. \tag{9}$$

The integral in (9) is elementary and the limit can easily be evaluated to verify that the result obtained from (9) is in exact agreement with that in (6). Formula (9) which is free from the coordinate x, forms the basis of our subsequent analysis.

3. Cubic–quintic BEC solitons in the lattice potential

First consider the BEC equation (1) with cubic and quintic terms only. In this case we shall have

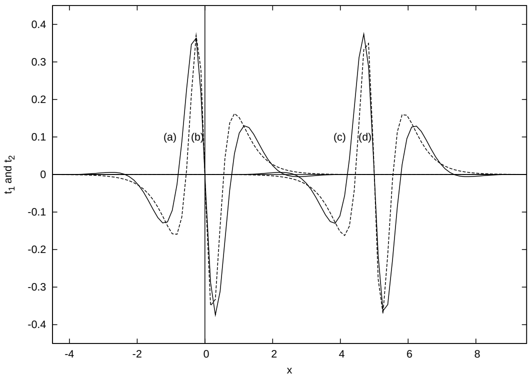

Figure 1. t_1 and t_2 as a function x.

$$E = \frac{1}{2}\phi_x^2 + \frac{g_1}{6}\phi^6 + \frac{g_2}{4}\phi^4 - \frac{1}{2}(1+\Lambda)\phi^2. \tag{10}$$

For the value of energy density in (10), the mass $N(\Lambda)$ is obtained as

$$N(\Lambda) = \frac{\sqrt{3}}{2\sqrt{g_1}}\left[\cos^{-1}\left(\frac{\frac{3}{4}\frac{g_2}{g_1}}{\sqrt{\frac{3}{g_1}\left(1+\Lambda+\frac{3}{16}\frac{g_2^2}{g_1}\right)}}\right) - \frac{\pi}{2}\right]. \tag{11}$$

From (11)

$$\frac{\mathrm{d}N}{\mathrm{d}\Lambda} = \frac{3}{16}\frac{g_2}{g_1}\frac{1}{\sqrt{1+\Lambda}}\frac{1}{\left(1+\Lambda+\frac{3}{16}\frac{g_2^2}{g_1}\right)}. \tag{12}$$

The derivative $\mathrm{d}N/\mathrm{d}\Lambda$ is positive for any value of $\Lambda > -1$. Obviously, $\Lambda > -1$ represents the stability condition of a BEC soliton with interacting atoms. The condition $\Lambda > -1 - \frac{3}{16}\frac{g_2^2}{g_1}$ is not valid because this will make the radical in (12) imaginary. Equation (11) can be used to obtain the critical number of Bose-condensed atoms in terms of the coupling constant for three-body interactions and we have

$$N_{\mathrm{c}}^{\mathrm{CQ}} = \sqrt{\frac{3}{g_1}}\frac{\pi}{4}. \tag{13}$$

Comparison of (6) and (13) shows that the critical number of atoms in a stable soliton of the CQNLS equation is equal to half the number of particles in a marginally stable Townes-like soliton. It is important to note that we could arrive at this conclusion only by applying VKC on the cubic–quintic quasi-one-dimensional Gross–Pitaevskii equation.

The optical lattice potential in (1) provides an awkward analytical constraint for application of the above procedure to the full BEC equation. For example, the energy density becomes an explicit function of x such that construction of an analytic expression for $N(\Lambda)$ similar to that in (9) or (11) is no longer possible. This difficulty cannot be circumvented even by writing an action functional for (1) and then constructing a Hamiltonian density by the use of Legendré map. To implement our coordinate-free approach for the stability analysis of (1) we, therefore, take recourse to the use of the following method:

Combining (1) and (3) and multiplying the resultant equation by ϕ_x we write

$$\phi_x \phi_{xx} + g_1 \phi^5 \phi_x + g_2 \phi^3 \phi_x + g_3 \phi \phi_x \cos(2x) - (1+\Lambda)\phi\phi_x = 0. \tag{14}$$

We venture to suggest that the term $\phi\phi_x \cos(2x) = t_1$(say) can be approximated by $-\left(a\phi^3\phi_x + b\phi_x\phi_{2x}\right) = t_2$(say). In figure 1 we display t_1 and t_2 as a function of x for ϕ given in (5) with $\Lambda = 1$. The solid curve (a) gives the variation of t_1 with respect to x and dashed curve (b) denotes similar variation for t_2. The observed fitting is done for $a = 0.0913$ and $b = 0.7301$. One may be interested to see if $t_1 = t_2$ remains translationally invariant. To that end we have shifted the x coordinate by an amount of 5 units and replotted the curves t_1 and t_2. The curves corresponding to a and b are now denoted by c and d. Comparison between the two sets of curves clearly exhibits the translational invariance of $t_1 = t_2$. However, we note that the agreement between a and b or c and d is qualitative. A better fit could perhaps be obtained by replacing ϕ^3 in t_2 by ϕ^n and then carrying out the fitting procedure by varying a, b and n simultaneously. But we shall work with t_2 that reproduces the plot in figure 1. For our choice of t_2 we obtain

$$E = \frac{1}{2}\left(1 - bg_3\right)\phi_x^2 + \frac{g_1}{6}\phi^6 + \frac{1}{4}\left(g_2 - ag_3\right)\phi^4 - \frac{1}{2}\left(1+\Lambda\right)\phi^2 \tag{15}$$

and

$$N\left(\Lambda\right) = \frac{1}{2}\sqrt{\frac{3}{g_1}\left(1 - bg_3\right)}\left[\cos^{-1}\left(\frac{k}{\sqrt{\frac{3}{g_1}(1+\Lambda) + k^2}}\right) - \frac{\pi}{2}\right], \tag{16}$$

where $k = \frac{3}{4g_1}\left(g_2 - ag_3\right)$. From (15) we get the conditions

$$g_3 \gg g_2 \tag{17a}$$

and

$$\Lambda > -1 \tag{17b}$$

for the stability of the BEC soliton. In writing (17a) we have used the values of a and b. While the inequality in (17b) is the same as that found for the CQNLS equation, the relation (17a) sets a criterion for choosing the coupling constant for the lattice potential to produce a stable BEC. For (1) we obtain the critical mass as

$$N_{\mathrm{c}}^{\mathrm{CQOL}} = \frac{\pi}{4}\sqrt{\frac{3}{g_1}}\sqrt{1 - bg_3}. \tag{18}$$

The subscript CQOL on $N_{\rm c}$ has been used to indicate that (1) involves all interactions (cubic, quintic and optical lattice). From (13) and (18) we find that $N_{\rm c}^{\rm CQOL} < N_{\rm c}^{\rm CQ}$. A plausible explanation for why a lattice leads to reduced value of $N_{\rm c}$ for which stability could be maintained is as follows.

When a BEC is confined jointly in an axial and optical traps, the latter introduces a new length scale to the system, namely, the lattice spacing which is much smaller than the BEC itself. Thus we have a periodic potential sitting on the harmonic trap [7] to provide a large number of new local trapping centres. The BEC will be squeezed towards the centre of each of these confining traps with a region of low density between the traps. As a result of this squeezing, the BEC collapses and leads to a value of $N_{\rm c}$ smaller than that in the absence of the optical lattice. If the centre of the quasi-one-dimensional magnetic trap, instead of keeping stationary with respect to the lattice, is suddenly shifted along the longitudinal direction, the stability is further affected by the onset of dissipative processes up to the full removal of the superfluid component [8]. Very recently, in an interesting work, Vyas *et al* [9] derived an analytical expression for the order parameter of the superfluid phase for a BEC loaded in an optical lattice when both two- and three-body interactions are operative. It appears that the mathematical framework for such studies was given by Bronski *et al* [10]. However, we have found that the wave functions of ref. [9] can be judiciously used to verify our claim regarding the atom density of a BEC in an optical lattice.

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