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Locally-rotationally-symmetric Bianchi type-V cosmology in general relativity

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Abstract. A spatially homogeneous locally-rotationally-symmetric (LRS) Bianchi type-V cosmological model is considered with a perfect fluid in general relativity. We present two types of cosmologies (power-law and exponential forms) by using a law of variation for the mean Hubble parameter that yields a constant value for the deceleration parameter. We discuss the physical properties of the non-flat and flat models in each cosmology. Exact solutions that correspond to singular and non-singular models are presented. In a generic situation, models can be interpolated between different phases of the Universe. We find that a constant value for the deceleration parameter is reasonable for a description of different phases of the Universe. We arrive at the conclusion that the Universe decelerates when the value of the deceleration parameter is positive whereas it accelerates when the value is negative. The dynamical behaviours of the solutions and kinematical parameters like expansion, shear and the anisotropy parameter are discussed in detail in each section. Exact expressions for look-back time, luminosity distance and event horizon vs. redshift are derived and their significances are discussed in some detail. It has been observed that the solutions are compatible with the results of recent observations.

Keywords. Bianchi type models; Hubble parameter; deceleration parameter; inflationary model.

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1. Introduction

The cosmological problem within the framework of general relativity consists of finding a model of the physical Universe which correctly predicts the result of astronomical observations and which is determined by those physical laws that describe the behaviour of matter on scales up to those of clusters of galaxies. The simplest models of the expanding Universe are those which are spatially homogeneous and isotropic at each instance of time.

The Bianchi cosmologies which are spatially homogeneous and anisotropic play an important role in theoretical cosmology and have been much studied since the 1960s. For simplification and description of the large scale behaviour of the actual

Universe, LRS Bianchi models have great importance. Lidsey [1] showed that these models are equivalent to Friedmann–Robertson–Walker (FRW) Universes.

The study of Bianchi type-V cosmological models create more interest as these models contain isotropic special cases and permit arbitrarily small anisotropy levels at some instant of cosmic time. These properties make them suitable as models of our Universe. Also the Bianchi type-V models are more complicated than the simplest Bianchi type models. Space-time models of Bianchi type-I and V Universes are generalizations of FRW models and it will be interesting to construct cosmological models of types which are of class one. Roy and Prasad [2] investigated Bianchi type-V Universes, which are LRS and are of embedding class one filled with perfect fluid with heat conduction and radiation. Bianchi type-V cosmological models have been studied by Farnsworth [3], Maartenes and Nel [4], Wainwraight *et al* [5], Collins [6], Coley and Dunn [7], Coley and Hoogen [8], Meena and Bali [9] and Pradhan and Rai [10], amongst others.

In recent years, the solutions of Einstein's field equations (EFEs) for homogeneous and anisotropic Bianchi type models have been studied by several authors, e.g., Hajj-Boutros [11,12], ShriRam [13,14], Mazumder [15], Camci *et al* [16] and Pradhan and Kumar [17] using different generating techniques. Solutions of the field equations may also be generated by applying a law of variation for the Hubble parameter, which was initially proposed by Berman [18] for FRW models. The law yields a constant value of the deceleration parameter. The theory of the constant deceleration parameter has been further developed by Berman and Gomide [19]. It should be remarked that the formula is independent of the particular gravitational theory being considered. It is a property valid for FRW metric, and it is approximately valid also for slowly time varying deceleration parameter.

In literature, cosmological models with a constant deceleration parameter have been studied by Johri and Desikan [20], Singh and Desikan [21], Maharaj and Naidoo [22], Pradhan *et al* [23], Pradhan and Vishwakarma [24,25], Rahaman *et al* [26], Reddy *et al* [27] and others in different theories of FRW and Bianchi type-I models. Recently, in a series of work, Singh and Kumar [28–30] and Kumar and Singh [31] extended Berman's work for the anisotropic Bianchi type-I and II space-time models by formulating a law of variation for the mean Hubble parameter and found the solutions to EFEs in the simplest way.

In this paper we extend the work to a spatially homogeneous LRS Bianchi type-V model with perfect fluid as a source. In §2 we outline LRS Bianchi type-V model and their field equations for perfect fluid, and a law for variation for the mean Hubble parameter that yields a constant value of the deceleration parameter. Solutions of the field equations are presented for two cosmologies using two forms of the average scale factor in §3. We discuss the physical and geometrical properties of non-flat and flat models in each cosmology. Exact solutions that correspond to singular and non-singular models are found. The behaviour of observationally important parameters like expansion scalar, mean anisotropy parameter and shear scalar are discussed in some detail in each section. Exact expressions for look-back time, luminosity distance and event horizon vs. redshift are derived and their significances are discussed in detail in §4. It has been observed that the solutions are compatible with the results of recent observations. In §5 the phases of the Universe are discussed. Section 6 contains the concluding remarks.

2. Model and field equations

We consider a locally-rotationally-symmetric (LRS) Bianchi type-V space-time with metric [32]

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + A^2(t)\mathrm{d}x^2 + \mathrm{e}^{2x}B^2(t)(\mathrm{d}y^2 + \mathrm{d}z^2), \tag{1}$$

where $A(t)$ and $B(t)$ are the cosmic scale functions. The Einstein's field equations are given by

$$R_{ij} - \frac{1}{2}g_{ij}R = 8\pi G T_{ij}, \tag{2}$$

where the energy–momentum tensor T_{ij} is

$$T_{ij} = (\rho + p)\, u_i u_j + p g_{ij}. \tag{3}$$

Here, ρ is the matter density and p is the thermodynamics pressure. Taking into account the conservation principle, i.e. $\mathrm{div}(T_i^j) = 0$, and co-moving system of coordinates ($u_i = \delta_0^i$), the resulting field equations for the metric (1) are as follows:

$$2\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} - \frac{1}{A^2} = -8\pi G p, \tag{4}$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}}{A}\frac{\dot{B}}{B} - \frac{1}{A^2} = -8\pi G p, \tag{5}$$

$$2\frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{B}^2}{B^2} - \frac{3}{A^2} = 8\pi G \rho, \tag{6}$$

$$\frac{\dot{A}}{A} - \frac{\dot{B}}{B} = 0, \tag{7}$$

$$\dot{\rho} + (\rho + p)\left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right) = 0, \tag{8}$$

where an overdot denotes ordinary derivative with respect to the cosmic time t.

Now, we define the following physical and geometrical parameters to be used in formulating the law and further in solving the field eqs (4)–(7).

The average scale factor a for the LRS Bianchi type-V model is defined as

$$a = \left(AB^2\right)^{1/3}. \tag{9}$$

A volume scale factor is given by

$$V = a^3 = AB^2. \tag{10}$$

The generalized mean Hubble parameter H is given by

$$H = \frac{1}{3}\left(H_1 + H_2 + H_3\right), \tag{11}$$

where $H_1 = \dot{A}/A$, $H_2 = H_3 = \dot{B}/B$ are the directional Hubble parameters in the directions of x-, y- and z-axes, respectively. The physical quantities of observational interest in cosmology are the expansion scalar θ, the average anisotropy parameter Ap and the shear scalar σ^2. These are defined as

$$\theta = u^i_{;i} = \left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right), \tag{12}$$

$$Ap = \frac{1}{3}\sum_{i=1}^{3}\left(\frac{\Delta H_i}{H}\right)^2, \tag{13}$$

where $\Delta H_i = H_i - H$ $(i = 1, 2, 3)$.

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} = \frac{1}{3}\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right)^2. \tag{14}$$

From eqs (9)–(11), we obtain an important relation

$$H = \frac{\dot{a}}{a} = \frac{1}{3}\left(\frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}\right). \tag{15}$$

For any physically relevant model, the Hubble parameter H and deceleration parameter q are the most important observational quantities in cosmology. The first quantity sets the present time scale of the expansion while the second one is telling us that the present stage is speeding up instead of slowing down as expected before the supernovae type Ia observations. The values of the deceleration parameter separate decelerating ($q > 0$) from accelerating ($q < 0$) periods in the evolution of the Universe. Determination of the deceleration parameter from the count magnitude relation for galaxies is a difficult task due to evolutionary effects. The present value q_0 of the deceleration parameter obtained from observations [33] are $-1.27 \leq q_0 \leq 2$. Studies of galaxy counts from redshift surveys provide a value of $q_0 = 0.1$, with an upper limit of $q_0 < 0.75$ [33]. Recent observations by Perlmutter *et al* [34,35] and Riess *et al* [36] show that the deceleration parameter of the Universe is in the range $-1 \leq q \leq 0$, and the present day Universe is undergoing accelerated expansion. It may be noted that though the current observations of SNe Ia and the CMBR favour accelerating models ($q < 0$), they do not altogether rule out the existence of the decelerating phase in the early history of our Universe which are also consistent with these observations [37].

Now, the law to be examined in this paper for LRS Bianchi type-V space-time model is

$$H = Da^{-n} = D\left(AB^2\right)^{-n/3}, \tag{16}$$

where $D(>0)$ and $n(\geq 0)$ are constants. The above law is valid since the line element (1) is completely characterized by Hubble parameter H.

The deceleration parameter (q) is defined as

$$q = -\frac{\ddot{a}a}{\dot{a}^2}. \tag{17}$$

From (15) and (16), we get

$$\dot{a} = Da^{-n+1}, \tag{18}$$

$$\ddot{a} = -D^2\,(n-1)\,a^{-2n+1}. \tag{19}$$

Using (18) and (19), eq. (17) gives

$$q = n - 1. \tag{20}$$

It is noted that q is constant. The theory of constant deceleration parameter has been developed by Berman [18], and Berman and Gomide [19], who found that the age of Universe, t, Hubble parameter H, and constant n are related by $H = (nt)^{-1}$. Later on, many authors (see, Singh and Kumar [28] and references therein) have studied flat FRW and Bianchi type models in different physical context. The sign of q indicates whether the model inflates or not. A positive sign for q, i.e., $n > 1$ corresponds to the 'standard' decelerating model whereas the negative sign $-1 \leq q < 0$ for $0 \leq n < 1$ indicates inflation.

From (18), we obtain the law for average scale factor a as

$$a = (nDt)^{1/n} \tag{21}$$

for $n \neq 0$ and

$$a = c\,\exp(Dt) \tag{22}$$

for $n = 0$, where c is a constant of integration. Here, in eq. (21) we have assumed that for $t = 0$ the value $a = 0$ so that the constant of integration turns out to be zero. Thus we have derived two types of models depending upon whether $n \neq 0$ or $n = 0$. It is however possible to have $D = 0$ in eq. (16) for which we would have a static Universe. But $D > 0$ is consistent with observation for which the Universe must be expanding. Hence we disregard a static Universe. The present day Universe has been thought of as Einstein–de Sitter, with a constant deceleration parameter $q = 1/2$. Since the recent observations of Supernovae data [34–36] confirm that the Universe is accelerating, we may ask whether the value of the deceleration parameter could be different from the de Sitter Universe, say as defined in eq. (20). In the case of an accelerating Universe, the second case for $n = 0$ becomes very relevant. Equation (21) implies that the condition for the expansion of the Universe is $n = q + 1 > 0$.

The age of Universe, in the first case, is

$$t_0 = \frac{1}{n}H_0^{-1} = \frac{1}{1+q}H_0^{-1}. \tag{23}$$

For $n = 0$

$$t_0 = \ln \left(\frac{a_0}{c} \right)^{3/D}, \tag{24}$$

where subscript 0 denotes the present phase. A numerical calculation can be made to estimate the present age of the Universe, the value of deceleration parameter compatible with the Supernovae observations. It should be remarked that eqs (21) and (22) are independent of the particular gravitational theory being considered. It is approximately valid also for slowly time-varying deceleration parameter. If $n > 0$, we expect that

$$\lim_{t\to\infty} a = \infty, \quad \lim_{a\to\infty} p = 0, \quad \lim_{a\to\infty} \rho = 0. \tag{25}$$

It deserves to mention here that eq. (16) refers to Bianchi type-V space-time in any physical theory. Our intention is to solve the Einstein's field equations for the Bianchi type-V model with a perfect fluid as the energy–momentum tensor using the above law of variation for the mean Hubble parameter (16) that yields a constant value of the deceleration parameter.

3. Solution of field equations

From eq. (7) it follows that

$$A = kB, \tag{26}$$

where k is a constant of integration. In this scenario the shear scalar and anisotropy parameter are zero, and the field equations (4)–(6) reduce to only two independent equations

$$\frac{2\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} - \frac{1}{k^2 B^2} = -8\pi G p, \tag{27}$$

$$\frac{3\dot{B}^2}{B^2} - \frac{3}{k^2 B^2} = 8\pi G \rho. \tag{28}$$

The conservation equation (8) reduces to

$$\dot{\rho} + 3\,(\rho + p)\,\frac{\dot{B}}{B} = 0. \tag{29}$$

Equations (27) and (28) can be rewritten as

$$3\dot{B}^2 = 8\pi G \rho B^2 + 3/k^2 \tag{30}$$

$$6\ddot{B} = -8\pi G\,(\rho + 3p) B. \tag{31}$$

If we consider that the gravitational-mass density is $\rho + 3p$, then from eq. (31), we see that $\ddot{B} < 0$, or that, from (19), $n > 1$, i.e. $q > 0$.

Using (18) and (19), the field equations (27) and (28) reduce to

$$8\pi G\rho = 3D^2 a^{-2n} - \frac{3}{k^{4/3}} a^{-2} \tag{32}$$

$$8\pi G p = (2n-3)D^2 a^{-2n} + \frac{1}{k^{4/3}} a^{-2}. \tag{33}$$

For $p = 0$ (matter-dominated era), we have

$$a_0 = \left[D\sqrt{k^{4/3}\left(3-2n\right)} \right]^{1/(n-1)}. \tag{34}$$

For this case we must have $1 < n < 1.5$ and $k > 0$ for the expansion of the Universe.

Now, the field equations (30) and (31) are a system of two equations with three unknown variables, namely, B, ρ and p. For complete solutions of the system, one extra condition is needed. For this we use eqs (21) and (22), defined for the average scale factor to solve the field equations (30) and (31).

3.1 *When $n \neq 0$*

In this case, from (9), (21) and (26), we get the solution of scale factors as

$$B(t) = k^{-1/3}\left(nDt\right)^{1/n} \tag{35}$$

$$A(t) = k^{2/3}\left(nDt\right)^{1/n}. \tag{36}$$

Hence the geometry of the space-time (1) takes the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + (nDt)^{2/n}[k^{4/3}\mathrm{d}x^2 + \mathrm{e}^{2x}k^{-2/3}(\mathrm{d}y^2 + \mathrm{d}z^2)]. \tag{37}$$

Here, we consider the following two sub-cases.

3.1.1 *Non-flat model*

In this case the Ricci scalar is given by

$$R = 6[(2-n)(nt)^{-2} - k^{-4/3}(nDt)^{-2/n}]. \tag{38}$$

Using eq. (35) in eqs (30) and (31), the energy density and pressure are respectively given by

$$8\pi G\rho = 3\left(nt\right)^{-2} - 3k^{-4/3}\left(nDt\right)^{-2/n} \tag{39}$$

$$8\pi G p = (2n-3)(nt)^{-2} + k^{-4/3}\left(nDt\right)^{-2/n}. \tag{40}$$

The expansion scalar is given by

$$\theta = 3(nt)^{-1}. \tag{41}$$

The above set of solutions identically satisfy the conservation equation (29). It is well-accepted that, for the present Universe the scale factors can be represented by power-law of time. We have thus obtained the solutions of the model with a constant deceleration parameter. We find that a constant deceleration parameter is reasonable for the description of different phases of the Universe. In this scenario the average anisotropy parameter and shear scalar are zero, and the model involves at most energy density and pressure. A simple analysis in this model shows that $A(t)$ and $B(t)$ are positive and increasing function of cosmic time t, and the model is expanding (i.e. $\theta > 0$). The expansion parameters H_i in the directions of coordinate axes (x, y, z) are same, i.e. $H_1 = H_2 = H_3 = (nt)^{-1}$. The model represents non-shearing, non-rotating isotropic and conformally flat Universe. The model starts expanding with Big-Bang at $t = 0$, which is a point singularity of the model. The model exhibits the power-law expansion after Big-Bang. Both the scale factors tend to zero at the point of singularity but tend to infinity as $t \to \infty$. The Ricci scalar, energy density and pressure tend to infinity when $t \to 0$ and $R \to 0$, $\rho \to 0$, $p \to 0$ when $t \to \infty$. Thus, the model would give essentially empty Universe at large times. The rate of expansion in the model stops when $t \to \infty$. The dominant energy condition $\rho + p \geq 0$ requires $n \geq 1$ whereas the strong energy condition $\rho + 3p \geq 0$ given by Hawking and Ellis [38] requires that $(nDt)^{(1-n)/n} \geq k^{-2/3}/D\sqrt{n}$. The ratio $\lim_{t\to\infty} \rho/\theta^2 = \text{constant}$ for $0 < n < 1$, which shows that the Universe remains homogeneous with time. Since $\mathrm{d}\theta/\mathrm{d}t < 0$, we conclude that the model starts expanding from its singular state and the rate of expansion decreases to become zero as $t \to \infty$.

3.1.2 *Flat model*

In this case, Ricci scalar, $R = 0$ and hence we get

$$(2-n)(nt)^{-2} = k^{-4/3}(nDt)^{-2/n}. \tag{42}$$

The energy density and pressure are given by

$$8\pi G\rho = 3(n-1)(nt)^{-2} \tag{43}$$

$$8\pi Gp = (n-1)(nt)^{-2}. \tag{44}$$

We find that the model starts expanding with the Big-Bang at $t = 0$ as ρ and p both tend to infinity at this point. As $t \to \infty$, both ρ and p tend to zero. When density is large ($\rho \to \infty$), the model corresponds to a radiation era and when density is small ($\rho \to 0$) the model corresponds to vacuum phase. Therefore, in a generic situation, models can be interpolating between different phases of the Universe: from a radiation Universe with equation of state $p = \rho/3$ to vacuum $p = \rho = 0$. Also, for $n = 1$, we find the vacuum case $p = \rho = 0$ result. Thus, eqs (43) and (44) describe the mixture of radiation and vacuum phases. For the reality of $\rho \geq 0$ and $p \geq 0$, we must have $n \geq 1$. Elimination of cosmic time t gives an equation of state $p = \rho/3$, which is the case of radiation-dominated phase. The model represents an expanding, non-shearing and isotropic Universe.

3.2 *When $n = 0$*

In this case, the average scale factor a is given by eq. (22). From (22), (9) and (26), the scale factors can be written as

$$A(t) = k^{2/3} c \exp(Dt), \tag{45}$$

$$B(t) = k^{-1/3} c \exp(Dt). \tag{46}$$

The geometry of space-time (1) takes the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + c^{-2} \mathrm{e}^{2Dt} [k^{4/3} \mathrm{d}x^2 + \mathrm{e}^{2x} k^{-2/3} (\mathrm{d}y^2 + \mathrm{d}z^2)]. \tag{47}$$

Now, we consider the following two sub-cases:

3.2.1 *Non-flat model*

The Ricci scalar is given by

$$R = 6[2D^2 - k^{-4/3} c^{-2} \exp(-2Dt)]. \tag{48}$$

When $t = 0$, $R = 6(2D^2 - k^{-4/3} c^{-2})$ and when $t \to \infty$, $R = 12D^2$.

Using (45) and (46) in (30) and (31), the energy density and pressure respectively are given by

$$8\pi G\rho = 3D^2 - 3k^{-4/3} c^{-2} \exp(-2Dt) \tag{49}$$

$$8\pi G p = -3D^2 + k^{-4/3} c^{-2} \exp(-2Dt). \tag{50}$$

The above set of solutions identically satisfy the conservation equation (29). The expansion scalar is given by $\theta = 3D$, whereas all the directional Hubble's parameters are constant, i.e. $H_1 = H_2 = H_3 = D$ throughout the evolution. The average anisotropy and shear scalar are zero. The ratio $\lim_{t\to\infty} \rho/\theta^2$ tends to be constant, which shows that the Universe remains homogeneous with cosmic time. The scale factors, energy density and pressure are constant at $t = 0$ and the rate of expansion is also constant throughout the evolution. The model starts with a constant volume and expands exponentially with time. As time passes the volume and scale factors expand exponentially and Universe becomes infinitely large as $t \to \infty$. The pressure and energy density describe an equation of state $p = -\rho$ at later stage of evolution. The negative pressure predicts the accelerating expansion of Universe for which deceleration parameter is negative. In general relativity, the vacuum energy density can be included in the cosmological constant term.

3.2.2 *Flat model*

For flat model $R = 0$ and hence in this case, we have

$$2D^2 = k^{-4/3} c^{-2} \exp(-2Dt). \tag{51}$$

The energy density and pressure are given by

$$8\pi G\rho = -3D^2 \tag{52}$$

$$8\pi Gp = -D^2. \tag{53}$$

For flat model we find solutions of (30) and (31). Unfortunately, from (52) it can be seen that this solution is unphysical since it leads to a negative energy density. But we find that pressure and density are constant and describes a relation with an equation of state $p = \rho/3$, which shows that the Universe is dominated by radiation. The solutions identically satisfy the conservation equation.

4. Kinematics tests

The formulas for $a(t)$ derived in (21) and (22) may be used to extend the kinematics tests for any arbitrary large redshifts. We now study the consistency of our models for both the cases with the observational parameters through kinematics tests.

4.1 *When* $n \neq 0$

4.1.1 *Look-back time – Redshift*

The look-back time, $\Delta t = t_0 - t(z)$, is the difference between the age of the Universe at present time ($z = 0$) and the age of the Universe when a particular light ray at redshift z was emitted. For a given redshift z, the expansion scale factor of the Universe $a(t_z)$ is related to a_0 by $1 + z = a_0/a$, where a_0 is the present scale factor. Therefore from (21), we get

$$1 + z = \left(\frac{t_0}{t}\right)^{1/n}, \quad n \neq 0. \tag{54}$$

The above equation gives

$$t = t_0(1 + z)^{-n}. \tag{55}$$

This equation can also be expressed as

$$H_0(t_0 - t) = \frac{1}{n}[1 - (1 + z)^{-n}], \tag{56}$$

where H_0 is the Hubble's constant at present in km s^{-1} Mpc^{-1} and its value is believed to be somewhere between 50 and 100 km s^{-1} Mpc^{-1}. However, the reciprocal of Hubble's constant is called the Hubble time $T_{\rm H}$: $T_{\rm H} = H_0^{-1}$, where $T_{\rm H}$ is expressed in s and H_0 in s^{-1}.

For small z, eq. (56) gives

$$H_0(t_0 - t) = \frac{1}{n}\left[nz - \frac{n(n-1)}{2}z^2 + \cdots\right]. \tag{57}$$

Using $q = (n-1)$, this transforms into

$$H_0(t_0 - t) = z - \frac{q}{2}z^2 + \cdots. \tag{58}$$

Taking limit $z \to \infty$ in (56), the present age of Universe (the extrapolated time back to the bang) is

$$t_0 = \frac{H_0^{-1}}{n} = \frac{{H_0}^{-1}}{1+q}, \tag{59}$$

which is the same as expected in (23). For $n = 3/2$, we get the well-known Einstein–de Sitter result

$$H_0(t_0 - t) = \frac{2}{3}[1 - (1+z)^{-3/2}], \tag{60}$$

which is used to describe look-back time in Einstein–de Sitter Universe. In the limit as $z \to \infty$, we obtain

$$t_0 = \frac{2}{3}H_0^{-1} = \frac{2}{3}T_{\mathrm{H}}. \tag{61}$$

4.1.2 *Luminosity distance – Redshift*

The luminosity distance of a light source is derived as the ratio of the detected energy flux L and the apparent luminosity l_*, i.e., $\mathrm{d}L^2 = L/4\pi l_*$. It takes the form

$$d_L = a_0 r_1(z)(1+z), \tag{62}$$

where $r_1(z)$ is the radial coordinate distance of the object at light emission and is given by

$$r_1(z) = \int_{t_1}^{t_0} \frac{\mathrm{d}t}{a} = \frac{{H_0}^{-1}{a_0}^{-1}}{(n-1)}[1 - (1+z)^{1-n}]. \tag{63}$$

Using eqs (63) into (62), we get

$$H_0 d_L = \frac{(1+z)}{n-1}[1 - (1+z)^{1-n}], \quad n \neq 1. \tag{64}$$

For small z, eq. (64) gives

$$H_0 d_L = z + \frac{(1-q)}{2}z^2 + \cdots. \tag{65}$$

For $q = 1$,

$$H_0 d_L = z \tag{66}$$

which shares linear relationship between luminosity distance and redshift, and for $q = 0$,

$$d_L = \frac{1}{H_0}\left(z + \frac{z^2}{2}\right). \tag{67}$$

4.2 *When $n = 0$*

4.2.1 *Look-back time – Redshift*

$$H_0(t_0 - t) = \log(1 + z). \tag{68}$$

For small z, we have

$$H_0(t_0 - t) = \left[z - \frac{z^2}{2} + \cdots\right]. \tag{69}$$

4.2.2 *Luminosity distance – Redshift*

$$d_L = \frac{1}{H_0}(z + z^2), \tag{70}$$

which shows that the luminosity distance increases faster with redshift z for $q = -1$.

4.2.3 *Event horizon*

The event horizon is given by

$$r_E = a(t_0)\int_{t_0}^{\infty} \frac{\mathrm{d}t}{a(t)} = \frac{1}{c_2 H}, \tag{71}$$

which shows that the event horizon exists in this model. This value of the limit gives the event horizon where no observer beyond a proper distance r_E at $t = t_0$ can communicate with another observer.

5. The phases of the Universe

Grøn [39] has remarked that it is the current belief that the Universe had the following early phases:

$$a_1 \propto t^{1/2}, \quad p = \rho/3 \tag{72}$$

$$a_2 \propto \exp(Ht), \quad p = -\rho \tag{73}$$

$$a_3 \propto t^{1/2}, \quad p = \rho/3. \tag{74}$$

On the other hand, according to Schwarzschild [40], the matter-dominated phase of the Universe is given by

$$a_4 \propto t^{2/3}, \quad p = 0. \tag{75}$$

The current accelerated phase of the Universe can be represented by

$$a_5 \propto t^{4/3}, \tag{76}$$

or, alternatively, by eq. (73).

We have already discussed in previous sections that the law of variation for the mean Hubble parameter (16) yields a constant value of the deceleration parameter $n = q + 1$ and generates two forms of cosmologies (21) and (22). We find out that the above five phases are particular cases of the constant deceleration parameter type. It is evident from eqs (21) and (22) that $n = 0$ stands for inflation, while $n = 2$, $n = 3/2$ and $3/4$ stand, respectively, for the first, third, fourth and the fifth phases. A relation between pressure and energy density for $n \neq 0$ can be written as

$$p = \left(\frac{2n-3}{3}\right)\rho, \tag{77}$$

which is the perfect gas law equation of state. Comparing eq. (77) with the equation of state $p = \omega\rho$, we find that $n = \frac{3}{2}(1+\omega)$ and consequently the deceleration parameter and the age of the Universe are given by

$$q = \frac{1+3\omega}{2}; \quad t_0 = \frac{2H_0^{-1}}{3(1+\omega)}. \tag{78}$$

The above equation yields for a radiation-dominated Universe ($\omega = 1/3, q = 1$) an age of $H_0t_0 = 1/2$ while for dust ($\omega = 0, q = 1/2$) we have $H_0t_0 = 2/3$, and, finally, for a flat Universe dominated by K-matter ($\omega = -1/3, q = 0$) one finds $H_0t_0 = 1$. All these results agree with the expressions (72)–(76).

On the other hand, for $n = 0$, we have

$$p = -\rho. \tag{79}$$

Thus, we observed that for $n \neq 0$ phases, we have a perfect gas equation of state, while for the $n = 0$ (inflation) case, we have negative pressure. The equation of state described by eq. (77) can be taken to get the solutions for different phases of Universe.

6. Conclusion

In this paper we have discussed a law of variation for the mean Hubble parameter in homogeneous LRS Bianchi type-V space-time model that yields a constant value of the deceleration parameter. We have obtained two types of exact non-singular and singular solutions of Einstein's field equations with the constant deceleration parameter. The law (16) gives the explicit solution of scale factors in a very simple manner, which are regarded as physically viable to describe the evolution of Universe. The law (16) generates two type of cosmologies for $n \neq 0$ and $n = 0$. We conclude that for the power-law solutions, as $t = 0$ the proper volume and scale factors vanish, the expansion scalar $\theta \to \infty$, and in consequence $\rho \to \infty$, $p \to \infty$, $R \to 0$. It is a point singularity. On the other hand, as $t \to \infty$, we have found that the scale factors become infinity, $\theta \to 0$, $\rho \to 0$, $p \to 0$ and proper volume tends to infinity. The models explode from singularity stage and approach an infinite expansion stage at $t \to \infty$. For the flat models, the Universe interpolates between the radiation phase to vacuum phase. The model is shear free, non-rotating and conformally flat of Petrov type 0 and of Segré-type $\{1,(111)\}$.

For exponential models as $t \to 0$, all the physical parameters such as the proper volume, scale factors, expansion scalar, energy density and pressure are constant. The rate of expansion is constant throughout the evolution. The average anisotropy parameter and shear scalar are zero. The model is non-singular at $t = 0$. On the other hand, as $t \to \infty$, we have $A(t) \to \infty$, $B(t) \to \infty$, $\theta =$ const., $\rho =$ const., $p =$ const. and $R =$ const. It is noted that the Universe describes the equation of state $p = -\rho$ for the non-flat model and $p = \rho/3$ for the flat models. In general relativity the vacuum energy density can be included in the cosmological constant term, which is supposed to be responsible for the cosmic expansion of the present day Universe. This class of solution is consistent with the recent observations of Supernovae Ia [34–36,41,42] that require the present Universe to be accelerating. A positive cosmological constant resists the attractive gravity of matter due to its negative pressure. For most Universes, the positive cosmological constant eventually dominates over the attraction of matter and drives the Universe to expand exponentially. Both the power-law and exponential solutions satisfy the conservation equation (29). Therefore, the law (16) provides an alternative and easy approach to get exact solutions in a very simple manner since nature strives for simplicity. Under the law (16) we have investigated the models, arriving at the conclusion that if $q > 0$, the model expands but always decelerates whereas $q < 0$ gives exponential expansion and later accelerates the Universe. It is noted that the law (16) refers to the anisotropic LRS Bianchi type-V model in any physical theory. We have observed that for the $n \neq 0$ phases, we have a perfect gas equation of state, while for the $n = 0$ (inflation) case, we have negative pressure.

We have also discussed the well-known astrophysical phenomena, namely the look-back time, luminosity distance and event horizon with redshift. It has been observed that such models are compatible with present observations. We have observed that luminosity distance increases linearly with redshift for $q = 1$ whereas it increases faster with redshift z for $q = 0$ and -1. The solutions obtained in the present paper could give an appropriate description of the evolution of Universe. More realistic models may be analysed using this technique, which may lead to interesting and different physical behaviour of the evolution of Universe. Further, we are studying whether a variation for the Hubble parameter could explain the continuous transition from a decelerated Universe to an accelerated one.

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