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Exact solutions to the generalized Lienard equation and its applications

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Abstract. Some new exact solutions of the generalized Lienard equation are obtained, and the solutions of the equation are applied to solve nonlinear wave equations with nonlinear terms of any order directly. The generalized one-dimensional Klein–Gordon equation, the generalized Ablowitz (A) equation and the generalized Gerdjikov–Ivanov (GI) equation are investigated and abundant new exact travelling wave solutions are obtained that include solitary wave solutions and triangular periodic wave solutions.

Keywords. The generalized Lienard equation; the generalized one-dimensional Klein–Gordon equation; the generalized Ablowitz equation; the generalized Gerdjikov–Ivanov equation.

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1. Introduction

Recently Kong [1] studied the following Lienard equation:

$$a''(\xi) + la(\xi) + ma^3(\xi) + na^5(\xi) = 0, \tag{1}$$

where l, m and n are constant coefficients. Kong has obtained a kind of solitary wave solution for Lienard equation by the method of undetermined coefficients. By means of this solution, he has obtained a kind of explicit exact solitary wave solution to the Rangwala–Rao (RR) equation, the Ablowitz (A) equation and the Gerdjikov–Ivanov (GI) equation. Zhang [2,3] has obtained two kinds of explicit exact solitary wave solutions for the Lienard equation, and by using the result, a kind of explicit exact solitary wave solutions to Chen–Lee–Lin equation, the nonlinear derivative Schrödinger equation and equations mentioned previously, two kinds of explicit exact solitary wave solutions for the Kundu equation, the nonlinear wave equation and the generalized Pochhammer–Chree (PC) equation were obtained. Feng [4] pointed out that the same solutions for eq. (1) in ref. [3] can be obtained by means of a direct method and studied the nonlinear Schrödinger equation and

PC equation by means of the solutions to eq. (1). Feng [5] used the phase-plane analysis method to obtain some explicit exact solitary wave solutions to eq. (1) and applied it to the nonlinear Schrödinger equation. Dey *et al* [6] have obtained various topological and periodic solitary wave solutions of eq. (1) by identifying the Lienard equation with the equation of motion of the nonlinear ϕ^6 field theory in (1+1) dimensions. Inspired by the success of the Lienard equation (1), the following generalized Lienard equations with nonlinear terms of any order was investigated.

$$a''(\xi) + la(\xi) + ma^{p+1}(\xi) + na^{2p+1}(\xi) = 0, \tag{2}$$

where l, m, n are constant coefficients and $p = 1, 2, 3, \ldots$. Lienard equation (1) corresponds to the $p = 2$ case of the generalized Lienard equation. Some exact solutions of the generalized Lienard equation (2) and their applications have been reported in refs [7,8]. Dey *et al* [6] have presented four exact solutions of eq. (2) by mapping them to the field equation of the ϕ^6 field theory. Zhang *et al* [9–11] have obtained two bell-profile solitary wave solutions and a pair of kink-profile solitary wave solutions of eq. (2) by the undetermined coefficients method, and further studied the compound KdV-type equation with nonlinear terms of any order, the generalized modified Boussinesq equation without dissipative term, the generalized one-dimensional Klein–Gordon equation, the generalized Zakharov equations, the generalized BBM equation and the generalized (2+1)-dimensional Klein–Gordon equation by using exact solutions of eq. (2). Feng [12] obtained more general solutions of eq. (2) than those described in refs [6–11] by a direct method, and applied those solutions to the generalized RR equation and the compound KdV-type equation with nonlinear terms of any order.

Evidence shows that the study of the Lienard equation is very useful, because finding solitary wave solutions of some nonlinear wave equations can be reduced to finding the solutions of the Lienard equations (1) and (2). Moreover, the generalized Lienard equation (2) is more important than Lienard equation (1) for Lienard equation (1) is a special case of eq. (2). It is very natural to ask whether the generalized Lienard equation (2) has some other new explicit exact solution except those shown in refs [6–12]. One purpose of this paper is to answer the problem.

The rest of this paper is organized as follows. In §2, some new exact solutions of the generalized Lienard equation (2) are obtained. In §3, by means of these solutions, the travelling wave solutions for the generalized one-dimensional Klein–Gordon equation, the generalized A-equations and the generalized GI-equations are presented. In §4, some conclusions are provided.

2. Exact solutions of the generalized Lienard equation

In the general case, it is commonly believed that it is very difficult to find exact solutions of the generalized Lienard equation by usual ways [13]. In this paper, by means of the undetermined coefficient method [9–11] and the direct method [12] mentioned previously, trial function method [14], the form of the solutions of the auxiliary equation in ref. [15], and our previous experience [16], we obtain abundant exact solutions for the generalized Lienard equation and verify these solutions by putting them back into eq. (2) with the aid of *Mathematica* (for instance solution

(10) of eq. (2) are given in Appendix). For brevity, we omit the procedure of solving eq. (2) and list exact solutions of eq. (2) below:

Case 1. When $l < 0$, $(D)^{1/p}$ makes sense for arbitrary negative number D (for instance p is an odd number, if $D > 0$, p can be arbitrary real number, where D is an expression), eq. (2) has the following solution:

$$a_1(\xi) = \left(\frac{-2lm\,\mathrm{sech}^2\left(\pm\frac{p\sqrt{-l}}{2}\xi\right)}{\frac{4m^2}{2+p} - \frac{nl(2+p)}{p+1}\left(1-\tanh\left(\pm\frac{p\sqrt{-l}}{2}\xi\right)\right)^2} \right)^{1/p}. \tag{3}$$

Case 2. When $l < 0$, $n < 0$ and $(D)^{1/p}$ makes sense for arbitrary negative number D, eq. (2) has the following solutions:

$$a_2(\xi) = \left(\frac{-l\,\mathrm{csch}^2\frac{p\sqrt{-l}}{2}\xi}{-\frac{2m}{p+2} + 2\sqrt{\frac{nl}{p+1}}\coth\frac{p\sqrt{-l}}{2}\xi} \right)^{1/p}, \tag{4}$$

$$a_3(\xi) = \left(\frac{-4l(\cosh p\sqrt{-l}\xi + \sinh p\sqrt{-l}\xi)}{\frac{4nl}{p+1} - (-\frac{2m}{p+2} + \cosh p\sqrt{-l}\xi + \sinh p\sqrt{-l}\xi)^2} \right)^{1/p}, \tag{5}$$

$$\begin{aligned} a_4(\xi) = \Bigg(& 8l^2\mathrm{sech}\,p\sqrt{-l}\xi \Bigg/ \\ & \left(\frac{4m^2}{(p+2)^2} - 4l\left(-l + \frac{n}{p+1}\right) - \frac{8ml}{p+2}\mathrm{sech}\,p\sqrt{-l}\xi \right. \\ & \left.\left. + \left(\frac{4m^2}{(p+2)^2} - 4l\left(l + \frac{n}{p+1}\right)\right)\tanh p\sqrt{-l}\xi\right)\right)^{1/p}, \end{aligned} \tag{6}$$

$$a_5(\xi) = \left(\frac{-l(-1 + (\tanh p\sqrt{-l}\xi \pm i\,\mathrm{sech}\,p\sqrt{-l}\xi)^2)}{-\frac{2m}{p+2} + 2\sqrt{\frac{nl}{p+1}}(\tanh p\sqrt{-l}\xi \pm i\,\mathrm{sech}\,p\sqrt{-l}\xi)} \right)^{1/p}, \tag{7}$$

$$a_6(\xi) = \left(\frac{-l\,\mathrm{csch}\frac{p\sqrt{-l}}{2}\xi}{-\frac{2m}{p+2}\sinh\frac{p\sqrt{-l}}{2}\xi + 2\sqrt{\frac{nl}{p+1}}\cosh\frac{p\sqrt{-l}}{2}\xi} \right)^{1/p}, \tag{8}$$

$$a_7(\xi) = \left(\frac{-l\,\mathrm{sech}\frac{p\sqrt{-l}}{2}\xi}{2\sqrt{\frac{nl}{p+1}}\sinh\frac{p\sqrt{-l}}{2}\xi + \frac{2m}{p+2}\cosh\frac{p\sqrt{-l}}{2}\xi} \right)^{1/p}. \tag{9}$$

Case 3. When $l < 0, \frac{m^2}{(p+2)^2} - \frac{nl}{p+1} > 0$ and $(D)^{1/p}$ makes sense for arbitrary negative number$D$, eq. (2) has the following solution:

$$a_8(\xi) = \left(\frac{(p+2)l}{-m \pm (p+2)\sqrt{\frac{m^2}{(p+2)^2} - \frac{nl}{p+1}} \cosh p\sqrt{-l}\xi} \right)^{1/p}. \tag{10}$$

Case 4. When $l < 0, \frac{m^2}{(p+2)^2} - \frac{nl}{p+1} < 0$ and $(D)^{1/p}$ makes sense for arbitrary negative number D, eq. (2) has the following solution:

$$a_9(\xi) = \left(\frac{l \operatorname{csch} p\sqrt{-l}\xi}{\pm\sqrt{\frac{nl}{p+1} - \frac{m^2}{(p+2)^2}} - \frac{m}{p+2} \operatorname{csch} p\sqrt{-l}\xi} \right)^{1/p}. \tag{11}$$

Case 5. When $l < 0, \frac{m^2}{(p+2)^2} - \frac{nl}{p+1} = 0$ and $(D)^{1/p}$ makes sense for arbitrary negative number D, eq. (2) has the following solutions:

$$a_{10}(\xi) = \left(-\frac{l(p+2)}{2m} \left(1 \pm \tanh \left(\frac{p\sqrt{-l}}{2} \xi \right) \right) \right)^{1/p}, \tag{12}$$

$$a_{11}(\xi) = \left(-\frac{l(p+2)}{2m} \left(1 \pm \coth \left(\frac{p\sqrt{-l}}{2} \xi \right) \right) \right)^{1/p}. \tag{13}$$

Case 6. When $l > 0$, $n < 0$ and $(D)^{1/p}$ makes sense for arbitrary negative number D, eq. (2) has the following solutions:

$$a_{12}(\xi) = \left(\frac{(p+2)l}{-m \pm (p+2)\sqrt{\frac{m^2}{(p+2)^2} - \frac{nl}{p+1}} \sin p\sqrt{l}\xi} \right)^{1/p}, \tag{14}$$

$$a_{13}(\xi) = \left(\frac{(p+2)l}{-m \pm (p+2)\sqrt{\frac{m^2}{(p+2)^2} - \frac{nl}{p+1}} \cos p\sqrt{l}\xi} \right)^{1/p}, \tag{15}$$

$$a_{14}(\xi) = \left(\frac{-l \sec^2 \frac{p\sqrt{l}}{2}\xi}{\frac{2m}{p+2} + 2\sqrt{-\frac{nl}{p+1}} \tan \frac{p\sqrt{l}}{2}\xi} \right)^{1/p}, \tag{16}$$

$$a_{15}(\xi) = \left(\frac{-l \csc^2 \frac{p\sqrt{l}}{2}\xi}{\frac{2m}{p+2} + 2\sqrt{-\frac{nl}{p+1}} \cot \frac{p\sqrt{l}}{2}\xi} \right)^{1/p}, \tag{17}$$

$$a_{16}(\xi) = \left(\frac{l(1 + (\tan p\sqrt{l}\xi \pm \sec p\sqrt{l}\xi)^2)}{-\frac{2m}{p+2} - 2\sqrt{-\frac{nl}{p+1}}(\tan p\sqrt{l}\xi \pm \sec p\sqrt{l}\xi)} \right)^{1/p}, \tag{18}$$

$$a_{17}(\xi) = \left(\frac{l \csc \frac{p\sqrt{l}}{2}\xi}{-\frac{2m}{p+2} \sin \frac{p\sqrt{l}}{2}\xi + 2\sqrt{-\frac{nl}{p+1}} \cos \frac{p\sqrt{l}}{2}\xi} \right)^{1/p}, \tag{19}$$

$$a_{18}(\xi) = \left(\frac{-l \sec \frac{p\sqrt{l}}{2}\xi}{2\sqrt{-\frac{nl}{p+1}} \sin \frac{p\sqrt{l}}{2}\xi + \frac{2m}{p+2} \cos \frac{p\sqrt{l}}{2}\xi} \right)^{1/p}. \tag{20}$$

Case 7. When $m = 0$ and $(D)^{1/p}$ makes sense for arbitrary negative number D, eq. (2) has the following solutions:

$$a_{19}(\xi) = \left(\pm\sqrt{\frac{l(p+1)}{n}} \operatorname{csch} (p\sqrt{-l}\xi) \right)^{1/p}, \quad (l < 0, n < 0) \tag{21}$$

$$a_{20}(\xi) = \left(\pm\sqrt{-\frac{l(p+1)}{n}} \operatorname{sech} (p\sqrt{-l}\xi) \right)^{1/p}, \quad (l < 0, n > 0) \tag{22}$$

$$a_{21}(\xi) = \left(\pm\sqrt{-\frac{l(p+1)}{n}} \csc (p\sqrt{l}\xi) \right)^{1/p}, \quad (l > 0, n < 0). \tag{23}$$

Case 8. When $n = 0$ and $(D)^{1/p}$ makes sense for arbitrary negative number D, eq. (2) has the following solutions:

$$a_{22}(\xi) = \left(-\frac{l(p+2)}{2m} \operatorname{sech}^2 \left(\frac{p\sqrt{-l}}{2}\xi \right) \right)^{1/p}, \quad (l < 0) \tag{24}$$

$$a_{23}(\xi) = \left(\frac{l(p+2)}{2m} \operatorname{csch}^2 \left(\frac{p\sqrt{-l}}{2}\xi \right) \right)^{1/p}, \quad (l < 0) \tag{25}$$

$$a_{24}(\xi) = \left(-\frac{l(p+2)}{2m} \sec^2 \left(\frac{p\sqrt{l}}{2}\xi \right) \right)^{1/p}, \quad (l > 0). \tag{26}$$

Remark. The solutions (3)–(13), (21), (22), (24) and (25) are solitary wave solutions that include bell-profile and kink-profile solitary wave solutions, and the solutions (14)–(20), (23) and (26) are triangular periodic wave solutions. The solutions (10), (12) and (14) can be found in refs [6–12] and the others are new, which cannot be found in literature to our knowledge.

3. The travelling wave solutions for the Klein–Gordon equation, the A-equation and the GI-equation

In this section, we will apply the exact solutions to eq. (2) obtained in the second section to find the exact solutions for three nonlinear wave equations with nonlinear terms of any order.

3.1 *The generalized one-dimensional Klein–Gordon equation*

Consider the generalized one-dimensional Klein–Gordon equation

$$u_{tt} - ku_{xx} + b_1 u + b_2 u^{p+1} + b_3 u^{2p+1} = 0, \quad p = 1, 2, 3, \ldots. \tag{27}$$

We perform the following travelling wave transformation:

$$u(x,t) = u(\xi), \quad \xi = x - vt, \tag{28}$$

where v is a constant. Substituting eq. (28) into eq. (27) yields a nonlinear ordinary differential equation

$$u''(\xi) + \frac{b_1}{v^2-k}u(\xi) + \frac{b_2}{v^2-k}u^{p+1}(\xi) + \frac{b_3}{v^2-k}u^{2p+1}(\xi) = 0. \tag{29}$$

Comparing eq. (29) with the generalized Lienard equation (2) and take

$$l = \frac{b_1}{v^2-k}, \quad m = \frac{b_2}{v^2-k}, \quad n = \frac{b_3}{v^2-k}, \tag{30}$$

where $v^2 \neq k$. Substituting this expression (30) into expressions (3)–(26) yields the following solitary wave solutions and triangular periodic wave solutions for eq. (27). For brevity in the following expressions, $\xi = x - vt$ and $(D)^{1/p}$ makes sense for arbitrary negative number D.

$$u_1(x,t) = \left(\frac{-2b_1 b_2 \operatorname{sech}^2\left(\pm\frac{p\sqrt{-\frac{b_1}{v^2-k}}}{2}\xi\right)}{\frac{4b_2^2}{2+p} - \frac{b_1 b_3(2+p)}{p+1}\left(1-\tanh\left(\pm\frac{p\sqrt{-\frac{b_1}{v^2-k}}}{2}\xi\right)\right)^2} \right)^{1/p}, \tag{31}$$

where $\frac{b_1}{v^2-k} < 0$;

$$u_2(x,t) = \left(\frac{-b_1 \operatorname{csch}^2 \frac{1}{2}p\sqrt{-\frac{b_1}{v^2-k}}\xi}{-\frac{2b_2}{p+2} + 2(v^2-k)\sqrt{\frac{b_1 b_3}{(v^2-k)^2(p+1)}}\coth\frac{1}{2}p\sqrt{-\frac{b_1}{v^2-k}}\xi} \right)^{1/p}, \tag{32}$$

where $\frac{b_1}{v^2-k}<0,\ \frac{b_3}{v^2-k}<0$;

$$u_3(x,t)=\left(-4b_1\left(\cosh p\sqrt{-\frac{b_1}{v^2-k}}\xi+\sinh p\sqrt{-\frac{b_1}{v^2-k}}\xi\right)\Bigg/\left(\frac{4b_1b_3}{(v^2-k)(p+1)}-(v^2-k)\left(-\frac{2b_2}{(v^2-k)(p+2)}+\cosh p\sqrt{-\frac{b_1}{v^2-k}}\xi\ +\sinh p\sqrt{-\frac{b_1}{v^2-k}}\xi\right)^2\right)\right)^{1/p},\qquad(33)$$

where $\frac{b_1}{v^2-k}<0,\ \frac{b_3}{v^2-k}<0$;

$$u_4(x,t)=\left(8b_1^2\,\mathrm{sech}\,p\sqrt{\frac{-b_1}{v^2-k}}\xi\Bigg/\left(\frac{4b_2^2}{(p+2)^2}-4b_1\left(-b_1+\frac{b_3}{p+1}\right)-\frac{8b_1b_2}{p+2}\,\mathrm{sech}\,p\sqrt{\frac{-b_1}{v^2-k}}\xi+\left(\frac{4b_2^2}{(p+2)^2}-4b_1\left(b_1+\frac{b_3}{p+1}\right)\right)\tanh p\sqrt{\frac{-b_1}{v^2-k}}\xi\right)\right)^{1/p},\qquad(34)$$

where $\frac{b_1}{v^2-k}<0,\ \frac{b_3}{v^2-k}<0$;

$$u_5(x,t)=\left(-b_1\left(-1+\left(\tanh p\sqrt{\frac{-b_1}{v^2-k}}\xi\pm i\,\mathrm{sech}\,p\sqrt{\frac{-b_1}{v^2-k}}\xi\right)^2\right)\Bigg/\left(\frac{-2b_2}{p+2}+2(v^2-k)\sqrt{\frac{b_1b_3}{(v^2-k)^2(p+1)}}\times\left(\tanh p\sqrt{\frac{-b_1}{v^2-k}}\xi\pm i\mathrm{sech}\,p\sqrt{\frac{-b_1}{v^2-k}}\xi\right)\right)\right)^{1/p},\qquad(35)$$

where $\frac{b_1}{v^2-k}<0,\ \frac{b_3}{v^2-k}<0$;

$$u_6(x,t)=\left(-b_1\,\mathrm{csch}\,\frac{p}{2}\sqrt{\frac{-b_1}{v^2-k}}\xi\Bigg/\left(-\frac{2b_2}{p+2}\sinh\frac{p}{2}\sqrt{\frac{-b_1}{v^2-k}}\xi+2(v^2-k)\times\sqrt{\frac{b_1b_3}{(v^2-k)^2(p+1)}}\cosh\frac{p}{2}\sqrt{\frac{-b_1}{v^2-k}}\xi\right)\right)^{1/p},\qquad(36)$$

where $\frac{b_1}{v^2-k} < 0$, $\frac{b_3}{v^2-k} < 0$;

$$u_7(x,t) = \left(-b_1 \operatorname{sech} \frac{1}{2} p \sqrt{\frac{-b_1}{v^2-k}}\xi \Bigg/ \left(2(v^2-k)\sqrt{\frac{b_1 b_3}{(v^2-k)^2(p+1)}} \sinh \frac{1}{2} p \times \sqrt{\frac{-b_1}{v^2-k}}\xi + \frac{2b_2}{p+2} \cosh \frac{1}{2} p \sqrt{\frac{-b_1}{v^2-k}}\xi \right)\right)^{1/p}, \tag{37}$$

where $\frac{b_1}{v^2-k} < 0$, $\frac{b_3}{v^2-k} < 0$;

$$u_8(x,t) = \left(\frac{b_1}{-\frac{b_2}{p+2} \pm (v^2-k)\sqrt{\frac{b_2^2(p+1)-b_3 b_1(p+2)^2}{(p+2)^2(v^2-k)^2(p+1)}} \cosh p \sqrt{-\frac{b_1}{v^2-k}}\xi}\right)^{1/p}, \tag{38}$$

where $\frac{b_1}{v^2-k} < 0$, $b_2^2(p+1) - b_1 b_3 (p+2)^2 > 0$;

$$u_9(x,t) = \left(\frac{b_1 \operatorname{csch} p \sqrt{\frac{-b_1}{v^2-k}}\xi}{\pm(v^2-k)\sqrt{\frac{b_3 b_1 (p+2)^2 - b_2^2(p+1)}{(p+2)^2(v^2-k)^2(p+1)}} - \frac{b_2}{(p+2)} \operatorname{csch} p \sqrt{\frac{-b_1}{v^2-k}}\xi}\right)^{1/p}, \tag{39}$$

where $\frac{b_1}{v^2-k} < 0$, $b_2^2(p+1) - b_1 b_3 (p+2)^2 < 0$;

$$u_{10}(x,t) = \left(-\frac{b_1(p+2)}{2b_2}\left(1 \pm \tanh \frac{1}{2} p \sqrt{\frac{-b_1}{v^2-k}}\xi\right)\right)^{1/p}, \tag{40}$$

where $\frac{b_1}{v^2-k} < 0$, $b_2^2(p+1) - b_1 b_3 (p+2)^2 = 0$;

$$u_{11}(x,t) = \left(-\frac{b_1(p+2)}{2b_2}\left(1 \pm \coth \frac{1}{2} p \sqrt{\frac{-b_1}{v^2-k}}\xi\right)\right)^{1/p}, \tag{41}$$

where $\frac{b_1}{v^2-k} < 0$, $b_2^2(p+1) - b_1 b_3 (p+2)^2 = 0$;

$$u_{12}(x,t) = \left(\frac{b_1}{\frac{-b_2}{p+2} \pm (v^2-k)\sqrt{\frac{b_2^2(p+1)-b_1 b_3(p+2)^2}{(p+1)(p+2)^2(v^2-k)^2}} \sin p \sqrt{\frac{b_1}{v^2-k}}\xi}\right)^{1/p}, \tag{42}$$

where $\frac{b_1}{v^2-k} > 0,\ \frac{b_3}{v^2-k} < 0$;

$$u_{13}(x,t) = \left(\frac{b_1}{\frac{-b_2}{p+2} \pm (v^2-k)\sqrt{\frac{b_2^2(p+1)-b_1b_3(p+2)^2}{(p+1)(p+2)^2(v^2-k)^2}}\cos p\sqrt{\frac{b_1}{v^2-k}}\xi}\right)^{1/p}, \tag{43}$$

where $\frac{b_1}{v^2-k} > 0,\ \frac{b_3}{v^2-k} < 0$;

$$u_{14}(x,t) = \left(\frac{-b_1\sec^2\frac{1}{2}p\sqrt{\frac{b_1}{v^2-k}}\xi}{\frac{2b_2}{p+2} + 2(v^2-k)\sqrt{\frac{-b_1b_3}{(v^2-k)^2(p+1)}}\tan\frac{1}{2}p\sqrt{\frac{b_1}{v^2-k}}\xi}\right)^{1/p}, \tag{44}$$

where $\frac{b_1}{v^2-k} > 0,\ \frac{b_3}{v^2-k} < 0$;

$$u_{15}(x,t) = \left(\frac{-b_1\csc^2\frac{1}{2}p\sqrt{\frac{b_1}{v^2-k}}\xi}{\frac{2b_2}{p+2} + 2(v^2-k)\sqrt{\frac{-b_1b_3}{(v^2-k)^2(p+1)}}\cot\frac{1}{2}p\sqrt{\frac{b_1}{v^2-k}}\xi}\right)^{1/p}, \tag{45}$$

where $\frac{b_1}{v^2-k} > 0,\ \frac{b_3}{v^2-k} < 0$;

$$\begin{aligned} u_{16}(x,t) = {} & \left(b_1\left(1+\left(\tan p\sqrt{\frac{b_1}{v^2-k}}\xi \pm \sec p\sqrt{\frac{b_1}{v^2-k}}\xi\right)^2\right)\Bigg/\right. \\ & \left(\frac{-2b_2}{p+2} - 2(v^2-k)\sqrt{\frac{-b_1b_3}{(v^2-k)^2(p+1)}}\right. \\ & \left.\left.\times\left(\tan p\sqrt{\frac{b_1}{v^2-k}}\xi \pm \sec p\sqrt{\frac{b_1}{v^2-k}}\xi\right)\right)\right)^{1/p}, \end{aligned} \tag{46}$$

where $\frac{b_1}{v^2-k} > 0,\ \frac{b_3}{v^2-k} < 0$;

$$\begin{aligned} u_{17}(x,t) = {} & \left(b_1\csc\frac{1}{2}p\sqrt{\frac{b_1}{v^2-k}}\xi\Bigg/\right. \\ & \left(-\frac{2b_2}{p+2}\sin\frac{1}{2}p\sqrt{\frac{b_1}{v^2-k}}\xi + 2(v^2-k)\right. \\ & \left.\left.\times\sqrt{\frac{-b_1b_3}{(v^2-k)^2(p+1)}}\cos\frac{1}{2}p\sqrt{\frac{b_1}{v^2-k}}\xi\right)\right)^{1/p}, \end{aligned} \tag{47}$$

where $\frac{b_1}{v^2-k} > 0,\ \frac{b_3}{v^2-k} < 0$;

$$u_{18}(x,t)=\left(-b_1\sec\frac{1}{2}p\sqrt{\frac{b_1}{v^2-k}}\xi\Bigg/\left(2(v^2-k)\sqrt{\frac{-b_1b_3}{(v^2-k)^2(p+1)}}\sin\frac{1}{2}p\sqrt{\frac{b_1}{v^2-k}}\xi+\frac{2b_2}{p+2}\cos\frac{1}{2}p\sqrt{\frac{b_1}{v^2-k}}\xi\right)\right)^{1/p}, \tag{48}$$

where $\frac{b_1}{v^2-k}>0$, $\frac{b_3}{v^2-k}<0$;

$$u_{19}(x,t)=\left(\pm\sqrt{\frac{b_1(p+1)}{b_3}}\operatorname{csch}p\sqrt{-\frac{b_1}{v^2-k}}\xi\right)^{1/p}, \tag{49}$$

where $b_2=0$, $\frac{b_1}{v^2-k}<0$, $\frac{b_3}{v^2-k}<0$;

$$u_{20}(x,t)=\left(\pm\sqrt{-\frac{b_1(p+1)}{b_3}}\operatorname{sech}\left(p\sqrt{-\frac{b_1}{v^2-k}}\xi\right)\right)^{1/p}, \tag{50}$$

where $b_2=0$, $\frac{b_1}{v^2-k}<0$, $\frac{b_3}{v^2-k}>0$;

$$u_{21}(x,t)=\left(\pm\sqrt{-\frac{b_1(p+1)}{b_3}}\csc\left(p\sqrt{\frac{b_1}{v^2-k}}\xi\right)\right)^{1/p}, \tag{51}$$

where $b_2=0$, $\frac{b_1}{v^2-k}>0$, $\frac{b_3}{v^2-k}<0$;

$$u_{22}(x,t)=\left(-\frac{b_1(p+2)}{2b_2}\operatorname{sech}^2\frac{1}{2}p\sqrt{-\frac{b_1}{v^2-k}}\xi\right)^{1/p}, \tag{52}$$

where $b_3=0$, $\frac{b_1}{v^2-k}<0$;

$$u_{23}(x,t)=\left(\frac{b_1(p+2)}{2b_2}\operatorname{csch}^2\frac{1}{2}p\sqrt{-\frac{b_1}{v^2-k}}\xi\right)^{1/p}, \tag{53}$$

where $b_3=0$, $\frac{b_1}{v^2-k}<0$;

$$u_{24}(x,t)=\left(-\frac{b_1(p+2)}{2b_2}\sec^2\frac{1}{2}p\sqrt{\frac{b_1}{v^2-k}}\xi\right)^{1/p}, \tag{54}$$

where $b_3=0$, $\frac{b_1}{v^2-k}>0$.

Remark. The solutions (31)–(41), (49), (50), (52) and (53) are solitary wave solutions that include bell-profile and kink-profile solitary wave solutions, and (42)–(48), (51) and (54) are triangular periodic wave solutions. The solutions (38) and (40) can be found in ref. [10], and the others have not been given in literature to our knowledge.

3.2 *The generalized A-equation*

Consider the following generalized A-equation with nonlinear terms of any order

$$iu_t = u_{xx} - 4i|u|^{p-2}u^2\bar{u}_x + 8|u|^{2p}u, \quad p = 1, 2, 3, \ldots. \tag{55}$$

Suppose that eq. (55) has solutions of the form

$$u(x,t) = \mathrm{e}^{i(\psi(\xi)-\omega t)}a(\xi), \quad \xi = x - vt, \tag{56}$$

where ω, v are constants. Substituting the expression (56) into eq. (55) and letting the real part and imaginary part to be zero yields

$$a(\xi)(\psi'(\xi)v + \psi'^2(\xi) + \omega) - a''(\xi) + 4a^{p+1}(\xi)\psi'(\xi) - 8a^{2p+1}(\xi) = 0, \tag{57}$$

$$-2a'(\xi)\psi'(\xi) - va'(\xi) - a(\xi)\psi''(\xi) + 4a^p(\xi)a'(\xi) = 0. \tag{58}$$

Let

$$\psi'(\xi) = A + Ba^p(\xi). \tag{59}$$

In order to make the left-hand side of eq. (58) equal to zero identically, we set

$$A = -\frac{v}{2}, \quad B = \frac{4}{p+2}. \tag{60}$$

Utilizing the expressions (59) and (60), eq. (57) becomes

$$a''(\xi) - \left(\omega - \frac{v^2}{4}\right)a(\xi) + 2va^{p+1}(\xi) + \frac{8(p^2+2p-2)}{(p+2)^2}a^{2p+1}(\xi) = 0. \tag{61}$$

Comparing eq. (61) with the generalized Lienard equation (2), take

$$l = -\left(\omega - \frac{v^2}{4}\right), \quad m = 2v, \quad n = \frac{8(p^2+2p-2)}{(p+2)^2}. \tag{62}$$

By means of the expressions (3)–(26) and (62), the following solitary wave solutions for eq. (61) are obtained.

$$a_1(\xi) = \left(\frac{v\left(\omega - \frac{v^2}{4}\right)\mathrm{sech}^2\left(\pm\frac{p\sqrt{\omega - \frac{v^2}{4}}}{2}\xi\right)}{\frac{4v^2}{2+p} + \frac{2(p^2+2p-2)(\omega - \frac{v^2}{4})}{(p+1)(p+2)}\left(1 - \tanh\left(\pm\frac{p\sqrt{\omega - \frac{v^2}{4}}}{2}\xi\right)\right)^2}\right)^{1/p}, \tag{63}$$

where $\xi = x - vt$, $\omega - v^2/4 > 0$ and $(D)^{1/p}$ makes sense for arbitrary negative number D. Then eq. (55) has an exact solution in the form of expression (56), where $\psi(\xi)$ is expressed as expression (59), $a(\xi)$ is described as expression (63).

$$a_2(\xi) = \left(\frac{-\omega + \frac{v^2}{4}}{-\frac{2v}{p+2} \pm \sqrt{\frac{4v^2}{(p+2)^2} + \frac{8(p^2+2p-2)(\omega - \frac{v^2}{4})}{(p+2)^2(p+1)}} \cosh p\sqrt{\omega - \frac{v^2}{4}}\xi} \right)^{1/p}, \tag{64}$$

where $\xi = x - vt$, $\omega - v^2/4 > 0$ and $(D)^{1/p}$ makes sense for arbitrary negative number D. Then eq. (55) admits an exact solution in the form of expression (56), where $\psi(\xi)$ is expressed as expression (59), $a(\xi)$ is described as expression (64).

Evidently, by means of expressions (4)–(9) and (11)–(26), we cannot find solutions of eq. (55), because l, m and n in expression (62) do not satisfy their conditions. In addition, solution (64) can be obtained using the method mentioned in ref. [10,12], and the solution (63) are new, which is unavailable in literature.

3.3 *The generalized GI-equation*

Consider the generalized GI-equation with nonlinear terms of any order

$$iu_t + u_{xx} + 2i\delta|u|^{p-2}u^2\bar{u}_x + \beta|u|^p u + 2\delta^2|u|^{2p}u = 0, \quad p = 1, 2, 3, \ldots. \tag{65}$$

Assume that eq. (65) has solutions in the form

$$u(x,t) = \mathrm{e}^{i(\psi(\xi) - \omega t)}a(\xi), \quad \xi = x - vt, \tag{66}$$

where ω, v are constants. Substituting expression (66) into eq. (65) and letting the real part and imaginary part to be zero respectively, then yields

$$\begin{aligned} &a(\xi)(\psi'(\xi)v - \psi'^2(\xi) + \omega) + a''(\xi) \\ &\quad + (\beta + 2\delta\psi'(\xi))a^{p+1}(\xi) + 2\delta^2 a^{2p+1}(\xi) = 0, \end{aligned} \tag{67}$$

$$2a'(\xi)\psi'(\xi) - va'(\xi) + a(\xi)\psi''(\xi) + 2\delta a'(\xi)a^p(\xi) = 0. \tag{68}$$

Let

$$\psi'(\xi) = A + Ba^p(\xi). \tag{69}$$

In order to make the left-hand side of eq. (68) equal to zero identically, we set

$$A = \frac{v}{2}, \quad B = \frac{-2\delta}{p+2}. \tag{70}$$

Utilizing expressions (69) and (70), eq. (67) becomes

$$a''(\xi)+\left(\omega+\frac{v^2}{4}\right)a(\xi)+(\beta+\delta v)a^{p+1}(\xi)$$
$$+\frac{2(p^2+2p-2)}{(p+2)^2}\delta^2 a^{2p+1}(\xi)=0. \tag{71}$$

This is the form of the generalized Lienard equation (2) and take

$$l=\omega+\frac{v^2}{4},\quad m=\beta+\delta v,\quad n=\frac{2(p^2+2p-2)}{(p+2)^2}\delta^2. \tag{72}$$

By means of expressions (3)–(26) and (72), the following solitary wave solutions for eq. (71) are obtained.

$$a_1(\xi)=\left(\frac{-\left(\omega+\frac{v^2}{4}\right)(\beta+\delta v)\,\mathrm{sech}^2\left(\pm\frac{p\sqrt{-\omega-\frac{v^2}{4}}}{2}\xi\right)}{\frac{2(\beta+\delta v)^2}{2+p}-\frac{(p^2+2p-2)\delta^2(\omega+\frac{v^2}{4})}{(p+1)(p+2)}\left(1-\tanh\left(\pm\frac{p\sqrt{-\omega-\frac{v^2}{4}}}{2}\xi\right)\right)^2}\right)^{1/p}, \tag{73}$$

where $\xi=x-vt$, $\omega+v^2/4<0$ and $(D)^{1/p}$ makes sense for arbitrary negative number D. Then eq. (65) has an exact solution in the form of expression (66), where $\psi(\xi)$ is expressed as expression (69) and $a(\xi)$ is described as expression (73).

$$a_2(\xi)=\left[\omega+\frac{v^2}{4}\Bigg/\left(-\frac{\beta+\delta v}{p+2}\right.\right.$$
$$\left.\left.\pm\sqrt{\frac{(\beta+\delta v)^2}{(p+2)^2}-\frac{2(p^2+2p-2)\delta^2(\omega+\frac{v^2}{4})}{(p+1)(p+2)^2}}\cosh p\sqrt{-\omega-\frac{v^2}{4}}\xi\right)\right]^{1/p}, \tag{74}$$

where $\xi=x-vt$, $\omega+v^2/4<0$ and $(D)^{1/p}$ makes sense for arbitrary negative number D. Then eq. (65) admits an exact solution in the form of expression (66), where $\psi(\xi)$ is expressed as expression (69) and $a(\xi)$ is described as expression (74).

In the same way, by means of expressions (4)–(9) and (11)–(26), we cannot find solutions of eq. (65), because l,m and n in expression (72) do not satisfy their conditions. In addition, the solution (74) can be obtained by using the method mentioned in refs [10,12], and the solution (73) are new, which is unavailable in literature.

Finally, it should be pointed out that we have verified these solutions obtained in this paper by putting them back into the original equation with the aid of *Mathematica*.

4. Conclusions

In summary, in this paper, first we obtain some new exact solutions for the generalized Lienard equation, which are unavailable in literature to our knowledge.

Then by means of the solutions of the generalized Lienard equation, abundant exact travelling wave solutions for the generalized one-dimensional Klein–Gordon equation, the generalized A-equation and the generalized GI-equation are obtained that include solitary wave solutions and triangular periodic wave solutions.

It is obvious that by using proper transform, some of the other nonlinear evolution equations can lead to the Lienard equation (1) or the generalized Lienard equation (2), for example, except the equations mentioned in first section, there are 15 nonlinear equations listed in ref. [12]. In addition, letting $p = 2$ and selecting D makes $(D)^{1/2}$ meaningful, we can obtain exact solutions for the Lienard equation (1) from the expressions (3)–(26). Hence, we can apply the Lienard equation (1), (2) and their solutions to look for the explicit exact solutions of some nonlinear wave equations.

Appendix

We consider the bell profile solution for eq. (2). Performing the following transformation:

$$a(\xi) = \phi(\xi)^{1/p}, \quad p \neq 0 \tag{A1}$$

and then inserting eq. (A1) into eq. (2) implies that

$$\frac{1}{p}\phi(\xi)\phi''(\xi) - \frac{p-1}{p^2}[\phi'(\xi)]^2 + l\phi^2(\xi) + m\phi^3(\xi) + n\phi^4(\xi) = 0. \tag{A2}$$

We assume that solution of eq. (A2) has the following form:

$$\phi(\xi) = \frac{A\mathrm{e}^{\alpha\xi}}{(1+\mathrm{e}^{\alpha\xi})^2 + B\mathrm{e}^{\alpha\xi}} = \frac{A\,\mathrm{sech}^2\alpha\xi/2}{4 + B\,\mathrm{sech}^2\alpha\xi/2}, \tag{A3}$$

where A, B and α are constants to be determined later. Substituting (A3) into eq. (A2), we obtain

$$\begin{cases} lp^2 + \alpha = 0, \\ 2(2+m)lp + Amp - (2+B)\alpha^2 = 0, \\ (6+4B+B^2)lp^2 + A(2+B)mp^2 + A^2np - 2(1+2p)\alpha^2 = 0. \end{cases} \tag{A4}$$

Solving eq. (A4) leads to the following result:

$$\begin{cases} \alpha = p\sqrt{-l}, \quad l < 0, \quad p > 0, \\ A = \pm\frac{2l(p+2)\sqrt{1+p}}{\sqrt{m^2(1+p)-nl(p+2)^2}}, \\ B = 2\left(-1 \mp \frac{m\sqrt{1+p}}{\sqrt{m^2(1+p)-nl(p+2)^2}}\right), \end{cases} \tag{A5}$$

where $m^2(1+p) - nl(p+2)^2 > 0$. Inserting (A5) into (A3), we obtain

$$\phi(\xi) = \frac{\pm \frac{l(p+2)\sqrt{1+p}}{\sqrt{m^2(1+p)-nl(p+2)^2}} \operatorname{sech}^2 p\xi\sqrt{-l}/2}{2 + \left(-1 \mp \frac{m\sqrt{1+p}}{\sqrt{m^2(1+p)-nl(p+2)^2}}\right)\operatorname{sech}^2 p\xi\sqrt{-l}/2}$$
$$= \frac{(p+2)l}{-m \pm (p+2)\sqrt{\frac{m^2}{(p+2)^2} - \frac{nl}{p+1}}\cosh p\sqrt{-l}\xi}. \tag{A6}$$

From (A1), we can get the solution (10) of eq. (2).

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