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Pancharatnam geometric phase originating from successive partial projections

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Abstract. The spin of a polarized neutron beam subjected to a partial projection in another direction, traces a geodesic arc in the 2-sphere ray space. We delineate the geometric phase resulting from two successive partial projections on a general quantal state and derive the direction and strength of the third partial projection that would close the geodesic triangle. The constraint for the three successive partial projections to be identically equivalent to a net spin rotation regardless of the initial state, is derived.

Keywords. Geometric phase; Pancharatnam triangle phase; polarized neutrons; partial projection.

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1. Introduction

Pancharatnam [1,2] was the first to recognize that two successive phase-preserving projections on a quantal state yield a phase that depends solely on the geometry of the two geodesic arcs traversed in the ray space. This pure geometric phase equals minus half the solid angle enclosed by the triangle obtained by joining the final and initial rays with the shorter geodesic. Extending the result to the case of partial projections, Samuel and Sinha [3] showed that for a sequence of three partial projections to close, the axes of the three imperfect polarizers must lie on the same great circle on the Poincaré sphere and that the sequence then is equivalent to a rotation about the axis transverse to the great circle. Here we formulate a partial projection and enunciate the geometric phase arising from two successive partial projections. Next, we characterize the third projection that would close the geodesic triangle when the initial state is either a given state ψ_0, or its orthogonal state $\bar{\psi}_0$ or both. Finally, we derive analytic conditions satisfied by each partial projection in terms of the other two under which the sequence of three partial projections is universally equivalent to an $SU(2)$ rotation.

2. Partial projection

A normalized ray ψ_0 representing a quantal two-state corresponds to the 'spin' direction $\mathbf{s}_0 = \psi_0^\dagger \boldsymbol{\sigma} \psi_0 = \mathrm{Tr}\rho_0 \boldsymbol{\sigma}$. Here $\boldsymbol{\sigma}$ denotes the vector of Pauli spin operators,

$\rho_0 = \psi_0\psi_0^\dagger = (1 + \boldsymbol{\sigma}\cdot\mathbf{s}_0)/2$ is the pure state density operator and 1 represents the unity operator. An imperfect projection of this state onto a nonorthogonal ray ψ_1 with the spin $\mathbf{s}_1$, attenuates the component of ψ_0 orthogonal to ψ_1 only by a factor $\exp(-\alpha_1)$, say, without introducing any phase. This optically dichroic operation,

$$\begin{aligned}\rho_1 + \exp(-\alpha_1)(1-\rho_1) &= \exp(-\alpha_1/2)\cosh(\alpha_1/2)(1 + t_1\boldsymbol{\sigma}\cdot\mathbf{s}_1)\\ &= P(t_1, \mathbf{s}_1),\ \text{say},\end{aligned} \tag{1}$$

where $\rho_1 = \psi_1\psi_1^\dagger$ takes the state $\psi_0 = \cos(\theta_1/2)\psi_1 + \sin(\theta_1/2)\bar{\psi}_1$ along the shorter geodesic $0 \to 1$ on the two-sphere only part of the way to $\psi_p = \cos(\theta_1/2)\psi_1 + \exp(-\alpha_1)\sin(\theta_1/2)\bar{\psi}_1$ (figure 1b). Here θ_1 denotes the angle between $\mathbf{s}_1$ and $\mathbf{s}_0$ and $t_1 = \tanh(\alpha_1/2)$ represents the strength of the partial projection. The state ψ_p is attenuated by a factor $\{\cos^2(\theta_1/2) + \exp(-2\alpha_1)\sin^2(\theta_1/2)\}^{1/2}$ and has the spin $\mathbf{s}_p$ at an angle θ' from $\mathbf{s}_1$, given by $\tan(\theta'/2) = \exp(-\alpha_1)\tan(\theta_1/2)$. Pancharatnam connection [1,2] dictates that ψ_p is in phase ψ_0. The partial projection effects a spin rotation $\delta = \theta_1 - \theta'$ about $\mathbf{s}_0 \times \mathbf{s}_1$. Angles θ' and θ_1 are analogous to those of photon propagation, viz.

$$\cos\theta' = \frac{\cos\theta_1 + p_1}{1 + p_1\cos\theta_1} \quad\text{and}\quad \sin\theta' = \frac{\sin\theta_1\sqrt{1-p_1^2}}{1 + p_1\cos\theta_1}, \tag{2}$$

in two inertial frames with their relative velocity cp_1 [3]. Here $p_1 = \tanh\alpha_1 = 2t_1/(1 + t_1^2)$, denotes the polarization efficiency of the partial projection, i.e. an unpolarized beam subjected to this partial projection acquires a polarization p_1 along $\mathbf{s}_1$. Inverting the relation (2), we get

$$\cos\theta_1 = \frac{\cos\theta' - p_1}{1 - p_1\cos\theta'} \Rightarrow \cos(\pi - \theta_1) = \frac{\cos(\pi - \theta') + p_1}{1 + p_1\cos(\pi - \theta')}$$

and

$$\sin(\pi - \theta_1) = \frac{\sin(\pi - \theta')\sqrt{1-p_1^2}}{1 + p_1\cos(\pi - \theta')}. \tag{3}$$

Thus a partial projection $P(t_1, \mathbf{s}_1)$ on a state with the spin at an angle $\pi - \theta'$ with $\mathbf{s}_1$ brings it to an angle $\pi - \theta_1$ with $\mathbf{s}_1$. This is an equivalent of the inverse partial projection $P(t_1, -\mathbf{s}_1)$ along $\bar{\psi}_1$ returning the ray ψ_p to ψ_0, apart from a state attenuation. The spin rotation δ is given by

$$\tan\frac{\delta}{2} = \frac{\sin\theta_1}{t_1^{-1} + \cos\theta_1} = \frac{\sin(\pi - \theta')}{t_1^{-1} + \cos(\pi - \theta')} = \frac{\sin\theta'}{t_1^{-1} - \cos\theta'}. \tag{4}$$

Thus for initial spin angles θ_1 and the corresponding $\pi - \theta'$ (cf. eq. (2)), the same spin rotation δ occurs (figure 2). The two angles become equal at $\theta_{1m} = \pi - \cos^{-1}(t_1) = \pi - \theta'_m$ and the spin rotation becomes a maximum, viz. $\delta_m = 2\sin^{-1}(t_1)$. This maximum can be understood by expressing δ as

$$\sin\frac{\delta}{2} = \sin\frac{\theta_1 - \theta'}{2} = t_1\sin\frac{\theta_1 + \theta'}{2} \to \delta_m = 2\sin^{-1}(t_1). \tag{5}$$

For the initial or final spin angle of $\pi/2$, i.e. $\theta_1 = \pi/2$ ($\theta' = \cos^{-1}(p_1)$) or $\theta' = \pi/2(\theta_1 = \pi - \cos^{-1}(p_1))$, equivalent to mutually inverse operations (ignoring the state attenuations),

$$\delta_{\pi/2} = 2\tan^{-1}(t_1) = \sin^{-1}(p_1). \tag{6}$$

3. Two partial projections

A second partial projection $P(t_2, \mathbf{s}_2)$ will bring the state ψ_p part of the way along the shorter geodesic $p \to 2$ (figure 1b) to ψ_f given by $(1 + t_2\boldsymbol{\sigma}\cdot\mathbf{s}_2)(1 + t_1\boldsymbol{\sigma}\cdot\mathbf{s}_1)\psi_0$ up to a real multiplier. Here again, ψ_f is in phase with ψ_p but relative to ψ_0, has a pure geometric phase $-\Omega/2$ [1,2], since $\Phi = \arg \mathrm{Tr}\rho_0(1 + t_2\boldsymbol{\sigma}\cdot\mathbf{s}_2)(1 + t_1\boldsymbol{\sigma}\cdot\mathbf{s}_1)$, i.e.

$$\tan\Phi = -\frac{t_1 t_2[\mathbf{s}_0\mathbf{s}_1\mathbf{s}_2]}{1 + t_1 t_2\mathbf{s}_1\cdot\mathbf{s}_2 + t_1\mathbf{s}_0\cdot\mathbf{s}_1 + t_2\mathbf{s}_2\cdot\mathbf{s}_0} = -\tan\frac{\Omega}{2}, \tag{7}$$

Ω symbolizing the solid angle of the geodesic triangle $0 \to p \to f \to 0$ (figure 1b). For $t_1 = t_2 = 1$, Φ reduces to the well-known Pancharatnam triangle phase [1,2] for full projections (figure 1a). On reversing the order of two partial projections, the phase just changes sign as in the full projection case, but the ray traverses a different geodesic triangle $0 \to p' \to f'(\to 0)$, enclosing a solid angle $-\Omega$ (figure 1c).

Without any loss of generality, we may assume $\mathbf{s}_0 = \hat{\mathbf{z}}$, i.e. $\psi_0 = |z\rangle$, so that spins $\mathbf{s}_1$ and $\mathbf{s}_2$ have polar angular coordinates $(\theta_1, 0)$ and $(\theta_2,\ \phi_{12})$, respectively (figure 1a). Then polar coordinates $(\theta_\mathrm{f},\ \phi_\mathrm{f})$ of the final spin $\mathbf{s}_\mathrm{f}$ are given by

$$\begin{aligned}\tan\frac{\theta_\mathrm{f}}{2}\mathrm{e}^{i\phi_\mathrm{f}} &= \frac{\bar{\psi}_0^\dagger\psi_\mathrm{f}}{\psi_0^\dagger\psi_\mathrm{f}} \\ &= \frac{(t_1\mathbf{s}_1 + t_2\mathbf{s}_2 - it_1t_2\mathbf{s}_1\times\mathbf{s}_2)\cdot(\hat{\mathbf{x}} + i\hat{\mathbf{y}})}{(1 + t_1t_2\mathbf{s}_1\cdot\mathbf{s}_2 + t_1\mathbf{s}_0\cdot\mathbf{s}_1 + t_2\mathbf{s}_2\cdot\mathbf{s}_0 - it_1t_2[\mathbf{s}_0\mathbf{s}_1\mathbf{s}_2])},\end{aligned} \tag{8a}$$

so that

$$\phi_\mathrm{f} = \arg\{(t_1\mathbf{s}_1 + t_2\mathbf{s}_2 - it_1t_2\mathbf{s}_1\times\mathbf{s}_2)\cdot(\hat{\mathbf{x}} + i\hat{\mathbf{y}})\} + \Omega/2 \tag{8b}$$

and $\tan(\theta_\mathrm{f}/2)$ equals the modulus of the RHS of eq. (8a).

4. Pancharatnam triangle closure

We seek the third partial projection $P(t_3, \mathbf{s}_3)$ which will close the geodesic triangle by taking ψ_f back to ψ_0. The spin $\mathbf{s}_3$ must then lie beyond 0 on the geodesic $f \to 0$ of length π. Hence the azimuthal angle of $\mathbf{s}_3$, $\phi_{13} = \phi_\mathrm{f} + \pi$ and the polar angle θ_3 is constrained to lie between 0 and $\pi - \theta_\mathrm{f}$. The inverse projection $P(t_3, -\mathbf{s}_3) \propto (1 - t_3\boldsymbol{\sigma}\cdot\mathbf{s}_3)$ on ψ_0 must yield ψ_f up to an attenuation, i.e.

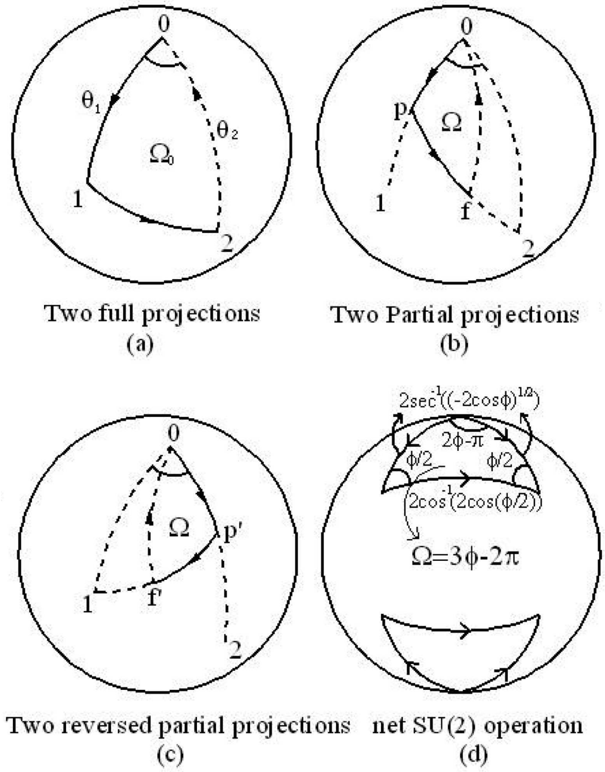

Figure 1. Pancharatnam triangles for two successive full (**a**), partial ((**b**) and (**c**)) projections and three identical ($t_1 = t_2 = t_3 = t = [-(1 + 2\cos\phi)]^{1/2}$) projections equivalent to an $SU(2)$ rotation (**d**).

$$\frac{t_3\mathbf{s}_3 \cdot (\hat{\mathbf{x}} + i\hat{\mathbf{y}})}{1 - t_3\mathbf{s}_3 \cdot \mathbf{s}_0} = -\tan\frac{\theta_\mathrm{f}}{2}\mathrm{e}^{i\phi_\mathrm{f}}, \quad \phi_{13} = \phi_\mathrm{f} + \pi$$

and

$$t_3 = \frac{\sin\frac{\theta_\mathrm{f}}{2}}{\sin(\theta_3 + \frac{\theta_\mathrm{f}}{2})}, \quad 0 < \theta_3 < \pi - \theta_\mathrm{f}. \tag{9}$$

At the centre $\theta_{3c} = (\pi - \theta_\mathrm{f})/2$ of the allowed domain $(0, \pi - \theta_\mathrm{f})$ of θ_3, the projection strength t_3 exhibits a minimum equal to $\sin(\theta_\mathrm{f}/2)$ and rises symmetrically on either side of $\theta_{3\mathrm{c}}$, approaching unity at both extremes of the θ_3 domain (figure 3).

The successive partial projections $P(t_3, -\mathbf{s}_3)$ followed by $P(t_2, -\mathbf{s}_2)$ take ψ_0 to the ray ψ_p (figure 1b) resulting in the triangle phase $\Omega/2$. Hence $\frac{\Omega}{2} = \arg \mathrm{Tr}\rho_0(1 - t_2\boldsymbol{\sigma} \cdot \mathbf{s}_2)(1 - t_3\boldsymbol{\sigma} \cdot \mathbf{s}_3)$, i.e.

$$\tan\frac{\Omega}{2} = \frac{t_2 t_3[\mathbf{s}_0\mathbf{s}_2\mathbf{s}_3]}{1 + t_2 t_3\mathbf{s}_2 \cdot \mathbf{s}_3 - t_3\mathbf{s}_0 \cdot \mathbf{s}_3 - t_2\mathbf{s}_2 \cdot \mathbf{s}_0}. \tag{10}$$

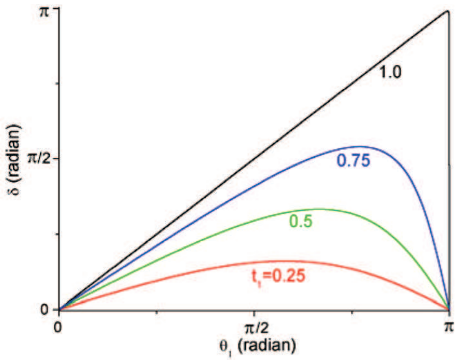

Figure 2. Spin deflection in a partial projection.

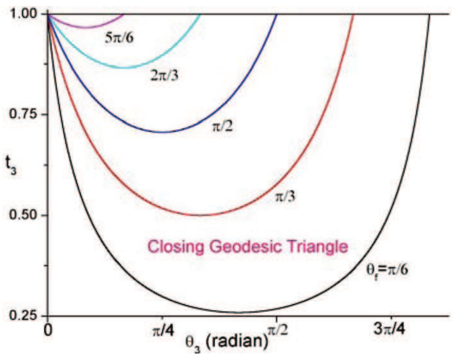

Figure 3. Variation of t_3 vs. θ_3 for several θ_{f}.

The triangle can now be closed by the partial projection $P(t_1, -\mathbf{s}_1)$ provided

$$\begin{aligned}
&\frac{t_1\mathbf{s}_1\cdot(\hat{\mathbf{x}}+i\hat{\mathbf{y}})}{1+t_1\mathbf{s}_1\cdot\mathbf{s}_0} = \tan\frac{\theta_{\mathrm{p}}}{2}\mathrm{e}^{i\phi_{\mathrm{p}}} \\
&= -\frac{(t_2\mathbf{s}_2+t_3\mathbf{s}_3-it_2t_3\mathbf{s}_2\times\mathbf{s}_3)\cdot(\hat{\mathbf{x}}+i\hat{\mathbf{y}})}{(1+t_2t_3\mathbf{s}_2\cdot\mathbf{s}_3-t_2\mathbf{s}_2\cdot\mathbf{s}_0-t_3\mathbf{s}_3\cdot\mathbf{s}_0+it_2t_3[\mathbf{s}_0\mathbf{s}_2\mathbf{s}_3])} \\
&\to t_1 = \frac{\sin\frac{\theta_{\mathrm{p}}}{2}}{\sin(\theta_1-\frac{\theta_{\mathrm{p}}}{2})},\quad \theta_{\mathrm{p}}<\theta_1<\pi,
\end{aligned} \tag{11}$$

yielding, for a given choice of t_3 and $\mathbf{s}_3$, a range $[\sin(\theta_{\mathrm{p}}/2), 1]$ of solutions t_1 as a function of θ_1 over the allowed domain on the extended geodesic $0 \to p$. We note that the state $P(t_1, \mathbf{s}_1)\psi_0$ bears a phase $-\Omega/2$ with respect to the state $P(t_3, -\mathbf{s}_3)\psi_0$, i.e.

$$\tan\frac{\Omega}{2} = \frac{t_3t_1[\mathbf{s}_0\mathbf{s}_3\mathbf{s}_1]}{1-t_3t_1\mathbf{s}_3\cdot\mathbf{s}_1+t_1\mathbf{s}_1\cdot\mathbf{s}_0-t_3\mathbf{s}_3\cdot\mathbf{s}_0}. \tag{12}$$

Combining eqs (7), (10) and (12), we arrive at the relation

$$\tan\frac{\Omega}{2} = \frac{\mathbf{s}_0\cdot\{t_1t_2\mathbf{s}_1\times\mathbf{s}_2+t_2t_3\mathbf{s}_2\times\mathbf{s}_3-t_3t_1\mathbf{s}_3\times\mathbf{s}_1\}}{1+t_1t_2\mathbf{s}_1\cdot\mathbf{s}_2+t_2t_3\mathbf{s}_2\cdot\mathbf{s}_3+t_3t_1\mathbf{s}_3\cdot\mathbf{s}_1}. \tag{13}$$

Equation (13) implies that the operations $P(t_1, \mathbf{s}_1)P(t_3, -\mathbf{s}_3)$ and $P(t_3, -\mathbf{s}_3)P(t_1, \mathbf{s}_1)$ on ψ_0 also yield the triangle phases $\Omega/2$ and $-\Omega/2$ respectively, albeit traversing geodesic triangles different from $0 \to p \to f$ or $0 \to f \to p$. $P(t_3, -\mathbf{s}_3)P(t_2, -\mathbf{s}_2)$ also traces another triangle yielding the phase $-\Omega/2$. Likewise, if with an initial state $\psi_0 = |-z\rangle$, we seek the closure of the geodesic triangle by a third partial projection $P(t_3, \mathbf{s}_3)$, we obtain

$$\begin{aligned}
&\frac{t_3\mathbf{s}_3\cdot(\hat{\mathbf{x}}-i\hat{\mathbf{y}})}{1+t_3\mathbf{s}_3\cdot\mathbf{s}_0} = \cot\frac{\theta_{\mathrm{f}}'}{2}\mathrm{e}^{-i\tilde{\phi}_{\mathrm{f}}} \\
&\quad = -\frac{(t_1\mathbf{s}_1+t_2\mathbf{s}_2-it_1t_2\mathbf{s}_1\times\mathbf{s}_2)\cdot(\hat{\mathbf{x}}-i\hat{\mathbf{y}})}{(1+t_1t_2\mathbf{s}_1\cdot\mathbf{s}_2-t_1\mathbf{s}_1\cdot\mathbf{s}_0-t_2\mathbf{s}_2\cdot\mathbf{s}_0-it_1t_2[\mathbf{s}_0\mathbf{s}_1\mathbf{s}_2])}.
\end{aligned} \tag{14}$$

For the third partial projection of strength t_3 and an azimuth angle ϕ_{13} to effect a triangle closure for both initial states $|z\rangle$ and $|-z\rangle$, we derive the constraint

$$t_1 \cos\theta_1 = -t_2 \cos\theta_2 = t_3 \cos\theta_3. \tag{15}$$

Solid angles enclosed by the geodesic triangles for initial states $|z\rangle$ and $|-z\rangle$ are then equal and opposite [4], viz.

$$\tan\frac{\Omega}{2} = -\tan\frac{\tilde{\Omega}}{2} = \frac{\sin\phi_{12}}{\frac{1-t_1^2\cos^2\theta_1}{t_1 t_2 \sin\theta_1 \sin\theta_2} + \cos\phi_{12}},$$
$$\phi_{13} = \pi + \frac{\Omega}{2} + \tan^{-1}\frac{\sin\phi_{12}}{\frac{t_1 \sin\theta_1}{t_2 \sin\theta_2} + \cos\phi_{12}} = \pi + \phi_{\mathrm{f}} = \pi + \tilde{\phi}_{\mathrm{f}}. \tag{16}$$

5. *SU*(2) operation

The three successive partial projections are identically equivalent to a net $SU(2)$ rotation about $\hat{\mathbf{z}}$ to within a real multiplier, $P(t_3, \mathbf{s}_3)P(t_2, \mathbf{s}_2)P(t_1, \mathbf{s}_1) \propto \mathrm{e}^{-i\sigma_z\Omega/2}$ regardless of ψ_0 only if

$$t_3(1 + t_1 t_2 \cos\phi_{12})\hat{\mathbf{s}}_3 + t_1(1 + t_2 t_3 \cos\phi_{23})\hat{\mathbf{s}}_1$$
$$+ t_2(1 - t_1 t_3 \cos\phi_{13})\hat{\mathbf{s}}_2 = 0$$

and

$$\theta_1 = \theta_2 = \theta_3 = \pi/2, \tag{17}$$

i.e. $\mathbf{s}_1$, $\mathbf{s}_2$ and $\mathbf{s}_3$ *must lie in same plane normal to* $\mathbf{s}_0 = \hat{\mathbf{z}}$, a result derived earlier [3] for the optical analogue of a Thomas rotation. For the third partial projection, we then have

$$t_3 \mathrm{e}^{i\phi_{13}} = -\frac{t_1 + t_2 \mathrm{e}^{i\phi_{12}}}{1 + t_1 t_2 \mathrm{e}^{-i\phi_{12}}}, \tag{18a}$$

$$t_3 = \frac{\sqrt{t_1^2 + t_2^2 + 2t_1 t_2 \cos\phi_{12}}}{\sqrt{1 + t_1^2 t_2^2 + 2t_1 t_2 \cos\phi_{12}}}, \quad \phi_{13} = \pi + \frac{\Omega}{2} + \tan^{-1}\frac{\sin\phi_{12}}{\frac{t_1}{t_2} + \cos\phi_{12}} \tag{18b}$$

and

$$\tan\frac{\Omega}{2} = \frac{\sin\phi_{12}}{(t_1 t_2)^{-1} + \cos\phi_{12}}. \tag{18c}$$

Likewise, projections 1 and 2 can be characterized respectively in terms of the other two as

$$t_1 \mathrm{e}^{-i\phi_{13}} = -\frac{t_2 \mathbf{e}^{-i\phi_{23}} + t_3}{1 + t_2 t_3 \mathrm{e}^{i\phi_{23}}}, \tag{19a}$$

$$t_1 = \frac{\sqrt{t_2^2 + t_3^2 + 2t_2 t_3 \cos\phi_{23}}}{\sqrt{1 + t_2^2 t_3^2 + 2t_2 t_3 \cos\phi_{23}}},$$
$$\phi_{13} = \pi + \tan^{-1}\left(\frac{\sin\phi_{23}}{t_2/t_3 + \cos\phi_{23}}\right) + \frac{\Omega}{2}, \tag{19b}$$

$$\tan\frac{\Omega}{2} = \frac{\sin\phi_{23}}{(t_2 t_3)^{-1} + \cos\phi_{23}} \tag{19c}$$

$$t_2 \mathrm{e}^{-i\phi_{12}} = -\frac{t_1 + t_3 \mathrm{e}^{i(\Omega - \phi_{13})}}{1 + t_1 t_3 \mathrm{e}^{i(\Omega - \phi_{13})}} = -\frac{\sinh\alpha_1 + \sinh\alpha_3 \mathrm{e}^{-i\phi_{13}}}{\cosh\alpha_1 + \cosh\alpha_3} \tag{20a}$$

and

$$\tan\frac{\Omega}{2} = \frac{\sin\phi_{13}}{-(t_1 t_3)^{-1} + \cos\phi_{13}}. \tag{20b}$$

With identical partial polarizers, $t_1 = t_2 = t_3 = t \neq 1$, say, we obtain the relations (see figure 1d)

$$\phi_{12} = \phi_{23} = \phi = \phi_{13}/2, \tag{21a}$$

$$\Downarrow$$

$$\Omega = 3\phi - 2\pi \mathrm{sgn}\,(\phi) \tag{22b}$$

and

$$t^2 = -(1 + 2\cos\phi), \quad |\phi| \in \left(\frac{2\pi}{3}, \pi\right) \tag{22c}$$

and a universal attenuation factor $\tan^3[(\pi - b)/4]$, b denoting the base of Pancharatnam triangles in figure 1d. The results, obtained here for spin polarized neutrons, can also be tested experimentally for light with identical dichroic polarizers whose axes lie in the plane transverse to the direction of light propagation, the successive axes being misaligned by $\phi/2$.

6. Conclusions

We have elucidated the Pancharatnam triangle geometric phase in the partial projection case. We have derived analytic relations for the effected spin rotations, geometric phase, triangle closure conditions for initial states $|z\rangle$, $|-z\rangle$ or both, as well as the scenario of equivalence of three successive partial projections to an $SU(2)$ rotation.

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