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Characterizing and modelling cyclic behaviour in non-stationary time series through multi-resolution analysis

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Abstract. A method based on wavelet transform is developed to characterize variations at multiple scales in non-stationary time series. We consider two different financial time series, S&P CNX Nifty closing index of the National Stock Exchange (India) and Dow Jones industrial average closing values. These time series are chosen since they are known to comprise of stochastic fluctuations as well as cyclic variations at different scales. The wavelet transform isolates cyclic variations at higher scales when random fluctuations are averaged out; this corroborates correlated behaviour observed earlier in financial time series through random matrix studies. Analysis is carried out through Haar, Daubechies-4 and continuous Morlet wavelets for studying the character of fluctuations at different scales and show that cyclic variations emerge at intermediate time scales. It is found that Daubechies family of wavelets can be effectively used to capture cyclic variations since these are local in nature. To get an insight into the occurrence of cyclic variations, we then proceed to model these wavelet coefficients using genetic programming (GP) approach and using the standard embedding technique in the reconstructed phase space. It is found that the standard methods (GP as well as artificial neural networks) fail to model these variations because of poor convergence. A novel interpolation approach is developed that overcomes this difficulty. The dynamical model equations have, primarily, linear terms with additive Padé-type terms. It is seen that the emergence of cyclic variations is due to an interplay of a few important terms in the model. Very interestingly GP model captures smooth variations as well as bursty behaviour quite nicely.

Keywords. Non-stationary time series; wavelet transform; genetic programming.

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1. Introduction

A number of non-stationary time series are known to comprise of fluctuations having both stochastic and cyclic or periodic components. Isolating fluctuations from these time series, at different scales for the purpose of characterization and modelling is a research area of significant interest [1–3]. Here we explicate a wavelet-based approach for separating structured variations from the stochastic ones in a time series before modelling them through genetic programming (GP) using the embedding technique [4] in the reconstructed phase space. For the purpose of illustration we have chosen two financial time series, since the same is well-known to exhibit random and structured behaviour at different scales [5,6]. The wavelet analysis shows that at small time scales, the fluctuations are primarily stochastic in nature; at higher scales the random part is averaged out and characteristic nature of the variations become transparent. For a reliable analysis one also needs to have a reasonably long dataset for which the financial time series are well-suited. The chosen time series are S&P CNX Nifty closing index of the National Stock Exchange (India) and Dow Jones industrial average closing values, representing two sufficiently different economic climates so as to bring out the efficacy of the present approach.

Stochastic nature of the high frequency fluctuations and presence of structured behaviour have emerged through the study of these time series using random matrix theory. In particular, analysis of the cross-correlations between different stocks [7,8] reveal universal and non-universal phenomena. The latter ones indicate correlated behaviour between stocks of different companies. This behaviour can manifest in the composite stock price indices, where the correlated behaviour of several companies can give rise to structured or cyclic behaviour in appropriate time scales. It may be pointed out that for stochastic systems recent noteworthy work includes a new approach on Langevian processes [9] and on turbulent cascades [10]. Wavelet transform, because of its multi-resolution analysis property is well-suited to isolate fluctuations and variations at different scales [11–13].

The goal of the present article is to demonstrate the usefulness of wavelet transform for isolating structured variations for modelling purpose. Since wavelets form orthogonal complete basis, it may, in some cases, be useful to expand a given signal in the wavelet basis and model the wavelet coefficients to understand the underlying dynamics. The fact that in our analysis the variations were found to be cyclic in nature at intermediate scales, justifies the above approach. We carry out our analysis on two different financial time series, in order to find out the similarities and differences between them, from the perspective of fluctuations. Apart from sharp transients representing sudden variations ascribable to physical causes, the high frequency fluctuations at small scales are primarily random in character. In all the time series, cyclic behaviour emerges at higher scales, when the random fluctuations are averaged out. At lower scales modelling and characterizing have been attempted earlier [14]. These were non-cyclic because of which the corresponding GP equations are found to be non-linear. The physical nature of the cyclic phenomena is substantiated through both discrete and continuous wavelets. In the case of continuous wavelets, the scalogram clearly reveals cyclic behaviour at intermediate scales. For these non-stationary time series, it is found that Daubechies family of wavelets can be effectively used to capture cyclic variations since these are local in nature. We then proceed to model these variations through genetic programming (GP) [15–19]. For that purpose, we smoothen the cyclic behaviour at every scale, corresponding to different levels of wavelet decomposition, through a cubic spline.

It needs to be mentioned that, since the purpose is to capture cyclic behaviour, physically it is meaningful to smoothen the same, before trying to model them [20]. For this purpose, standard technique of time-delayed vectors by carrying out embedding in the reconstructed phase space has been employed. We have shown here that GP approach model produces map equations capturing the dynamics of the system giving qualitatively equivalent fit results and using fewer variables as compared to that given by artificial neural network model. Additionally, we have also found it to be useful to get further insights towards physical interpretations by analysing the map equations.

We study the fluctuation characteristics of two different financial time series, S&P CNX Nifty closing index of the National Stock Exchange (India) and Dow Jones industrial average closing values, through wavelet transforms belonging to both discrete and continuous families. Haar and Daubechies-4 (Db4) from the discrete wavelet family and the continuous Morlet wavelet are used to analyse the time series. As has been observed earlier, at small scales, the fluctuations captured by the wavelet coefficients exhibit self-similar character [21,22]. Clear cyclic behaviour emerges in medium scales and is evident from both discrete and continuous wavelet analysis. It is found that, GP captures the cyclic behaviour at each scale quite well. The dynamical equations are primarily linear with non-linear additive terms of Padé type. These equations are checked for their predictive capabilities by making out-of-sample predictions. One-step out-of-sample predictions are made which use the given time-lagged values successively and predict the next dataset value. It is found that the one-step predictions are very good.

The paper is organized as follows. In §2, we give a brief introduction to wavelets before carrying out wavelet decomposition of both the datasets considered to study the character of the variations at different scales. Cyclic variations at different scales are extracted through Daubechies-4 wavelet and confirmed by continuous Morlet wavelet. We then proceed to model the cyclic phenomenon through GP in §3 and conclude in §4, after pointing out a number of applications and future directions of work through the present method.

2. Wavelet transform

Wavelet transform provides a powerful tool for the analysis of transient and non-stationary data and is particularly useful in picking out characteristic variations at different resolutions or scales. In the context of financial time series, it has found extensive applications. It has been used for the study of commodity prices [23], in measuring correlations [24], in the study of foreign exchange rates [25] and for predicting stock market behaviour [26], to name a few. This linear transform separates a dataset in the form of low-pass or average coefficients, resembling the data itself, and wavelet or high-pass coefficients at different levels, which capture the variations at corresponding scales. Wavelets can be continuous or discrete. In the latter case, the basis elements are strictly finite in size, enabling them to achieve localization, while disentangling characteristic variations at different frequencies.

In discrete wavelet transform (DWT), the construction of basis set starts with the scaling function $\varphi(x)$ (father wavelet) and the mother wavelet $\psi(x)$, whose height and width are arbitrary: $\int \varphi \mathrm{d}x = A$, $\int \psi \mathrm{d}x = 0$, $\int \varphi\psi \mathrm{d}x = 0$, $\int |\varphi|^2 \mathrm{d}x = 1$, $\int |\psi|^2 \mathrm{d}x = 1$, where A is an arbitrary constant. The scaling and wavelet functions, and their scaled translates, known as daughter wavelets, $\psi_{j,k} = 2^{j/2}\psi(2^j x - k)$, are square integrable at different scales. Here, k and j respectively are the translation and scaling parameters, with $-\infty \leq k \leq +\infty$. Although conventionally, one starts with the scale value $j = 0$, one can begin from any finite value j' and increase it by integral units. The original mother wavelet corresponds to $\psi_{0,0}$. The daughter wavelets are of a similar form as the mother wavelet, except that their width and height differ by a factor of 2^j and $2^{j/2}$ respectively, at successive levels. The translation unit $k/2^j$ is commensurate with the thinner size of the daughter wavelet at scale j. In the limit $j \to \infty$, these basis functions form a complete orthonormal set, allowing us to expand a signal $f(t)$ in the form

$$f(t) = \sum_{k=-\infty}^{+\infty} c_{j,k}\varphi_{j,k}(t) + \sum_{k=-\infty}^{+\infty} \sum_{j' \geq j} d_{j',k}\psi_{j',k}(t). \tag{1}$$

Here, $c_{j,k}$'s are the low-pass coefficients and $d_{j,k}$'s are the high-pass or wavelet coefficients. They respectively capture the average part and variations of the signal at scale j and location k. For the discrete wavelets, the property of multi-resolution analysis (MRA) leads to $c_{j,k}=\sum_n h(n-2k)c_{j+1,n}$, $d_{j,k}=\sum_n \tilde{h}(n-2k)c_{j+1,n}$, where $h(n)$ and $\tilde{h}(n)$ are respectively the low-pass (scaling function) and high-pass (wavelet) filter coefficients, which differ for different wavelets. Both low-pass and high-pass coefficients at a scale j can be obtained from the low-pass coefficients at a higher scale ($c_{j+1,n}$). This implies that, starting from the finest resolution of the signal, one can construct both scaling and wavelet coefficients, by convolution with the filter coefficients $h(n)$ and $\tilde{h}(n)$.

For the Haar wavelet: $h(0)=h(1)=1/\sqrt{2}$ and $\tilde{h}(0)=-\tilde{h}(1)=1/\sqrt{2}$. Haar basis is unique, since it is the only wavelet, which is symmetric and compactly supported. In a level one Haar wavelet decomposition, the level-I low-pass (average) and high-pass (wavelet or detail) coefficients are respectively given by the nearest-neighbour averages and differences, with a normalization factor of $1/\sqrt{2}$. In the subsequent step, the average coefficients are divided into two parts, containing level-II high-pass and level-II low-pass coefficients. The high-pass coefficients now represent differences of averaged data points corresponding to a window size of two. Wavelets belonging to Daubechies family are designed such that, the wavelet coefficients are independent of polynomial trends in the data. For example, Daubechies-4 wavelet satisfies, $\int t\psi(t)\mathrm{d}t = 0$, in addition to all other conditions. Because of this the wavelet coefficients here capture fluctuations over and above the linear variations. For Daubechies-4, $h(0)=-\tilde{h}(3)=\frac{1+\sqrt{3}}{4\sqrt{2}}$, $h(1)=\tilde{h}(2)=\frac{3+\sqrt{3}}{4\sqrt{2}}$, $h(2)=-\tilde{h}(1)=\frac{3-\sqrt{3}}{4\sqrt{2}}$, $h(3)=-\tilde{h}(0)=\frac{1-\sqrt{3}}{4\sqrt{2}}$. We have used both Haar and Daubechies-4 wavelets for isolating these fluctuations at different scales and study their character. For continuous wavelet transform (CWT), we have utilized Morlet wavelet, whose analysing function is given by

$$\psi(t) = \pi^{-1/4}\mathrm{e}^{(-i\omega_0 t - t^2/2)}. \tag{2}$$

The corresponding wavelet coefficients are displayed as a function of scale and time in a scalogram.

In DWT, a maximum of N level decompositions can be carried out, when the dataset is of size 2^N. One may choose to have a lesser number of decompositions than N. Often one needs to supplement the data with additional points to carry out an N-level decomposition. Both in DWT and CWT, one encounters boundary artifacts, due to circular or other forms of extensions. In our case, for minimizing these boundary artifacts, we have used symmetric extension, while studying the behaviour of wavelet coefficients. The variations at different scales are characterized by their respective powers, defined as the squared sum of the wavelet coefficients at that level. Since in wavelet transform power is conserved, the squared sum of all the low-pass and high-pass coefficients add up to the squared sum of the data points of the time series, called as the total power or energy. We have used normalized power which is the power at a given level divided by the total power. Periodic extension has been used for analysing the distribution of power at various levels, since this extension conserves power. The power plots depicting high-pass power and low-pass power clearly reveal the character of the fluctuation and the average behaviour.

2.1 *Wavelet analysis of NSE dataset*

We start with the National Stock Exchange (NSE) of India Nifty daily closing index values, consisting of 2048 data points, from 24 December 1997 to 16 February 2006. This daily index is shown in figure 1. We carry out an eight-level decomposition through Haar transform, since the same yields a transparent picture about the nature of the variations. The high-pass coefficients are depicted in figure 2. Transient and cyclic behaviour at different scales are clearly visible. The corresponding low-pass coefficients corresponding to 8th level are depicted in figure 3. As expected, these low-pass coefficients capture the average behaviour of the time series (see figure 1). At finer resolutions corresponding to lower level wavelet coefficients, one can clearly see the primarily random nature of the fluctuations.

The non-random variations significantly increase from 6th level onwards. It is worth mentioning that the 6th level high-pass coefficients correspond to the differences of data points, averaged over a temporal window size 32. A few transient phenomena are also revealed by these coefficients. After sufficient averaging, cyclic behaviour is seen to emerge. The averaged low-pass coefficients reveal a linear trend like the time series in figure 1; this can affect the high-pass coefficients. In order to remove this and capture the characteristic nature of the variations, we have carried out decomposition through Daubechies-4 wavelets. The structured and cyclic behaviour is transparently demonstrated in figure 4. This justifies the use of Daubechies-4 wavelets, which removes linear trend from the high-pass coefficients. CWT through Morlet wavelets also reveals the cyclic behaviour at intermediate scales as seen in figure 5. The scale values indicate the window size of the Morlet wavelet corresponding to the same number of days. We depict in figure 6 the sum of the continuous wavelet coefficients over all scales as a function of time. An approximate periodic behaviour is seen with a period of about 200 trading days. The

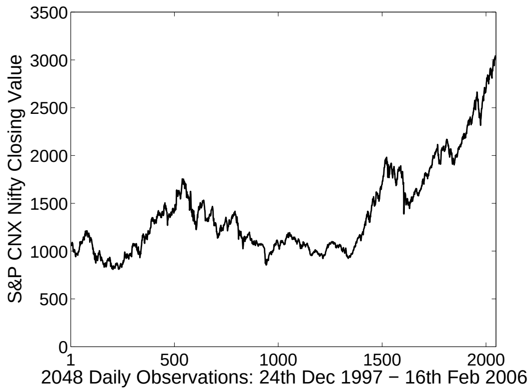

Figure 1. S&P CNX Nifty closing index data having 2048 points covering the daily index lying between 24 December 1997 and 16 February 2006.

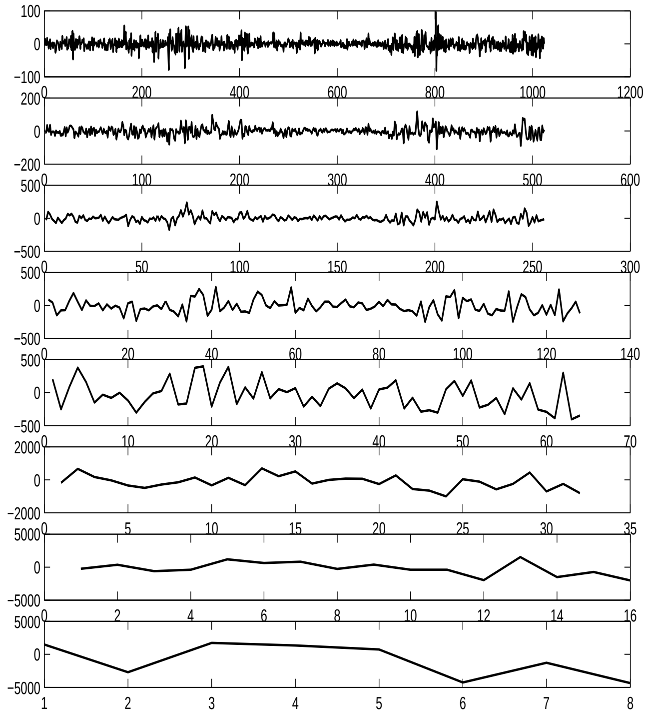

Figure 2. Haar wavelet fluctuation coefficients for levels 1 to 8 for Nifty data. Transient and stochastic behaviour at small scales and ordered variations at higher scales are evident.

fact that wavelet coefficients containing both positive and negative values add up to yield a periodic behaviour indicates the presence of correlated behaviour. As is clear, purely random un-correlated coefficients will not lead to this structure.

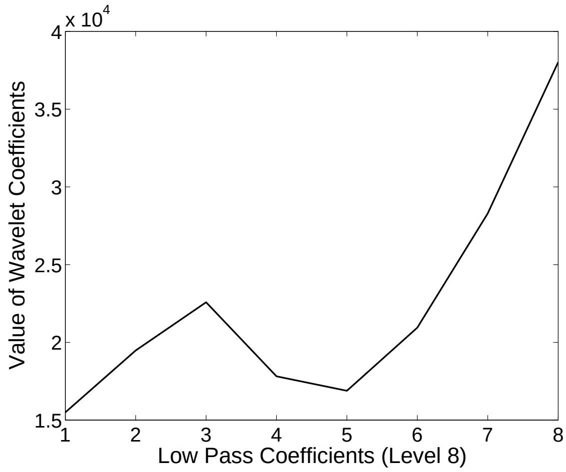

Figure 3. Haar wavelet low-pass coefficients for level 8 for Nifty data. As expected, these coefficients resemble the average behaviour of the time series.

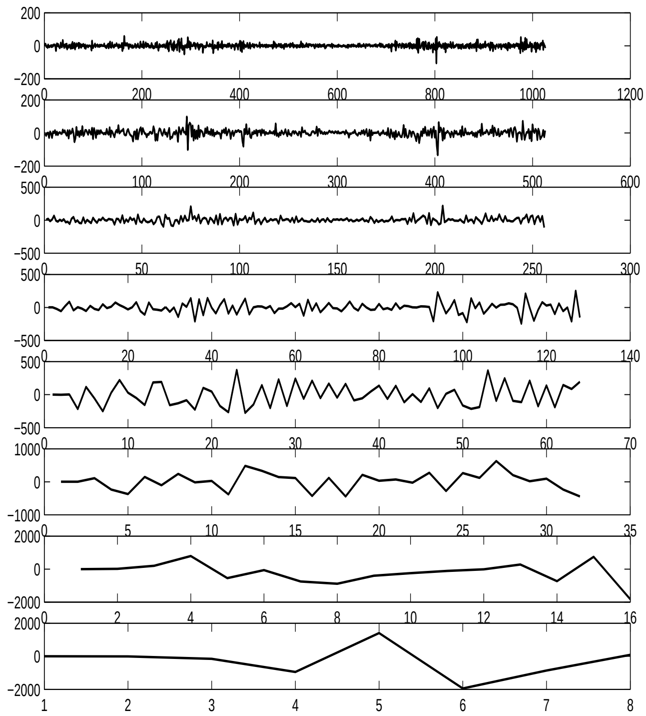

Figure 4. Db4 wavelet coefficients for different levels 1 to 8 for Nifty data. Cyclic behaviour at intermediate scales are well-captured by the wavelet coefficients. This behaviour is present at both local and global levels.

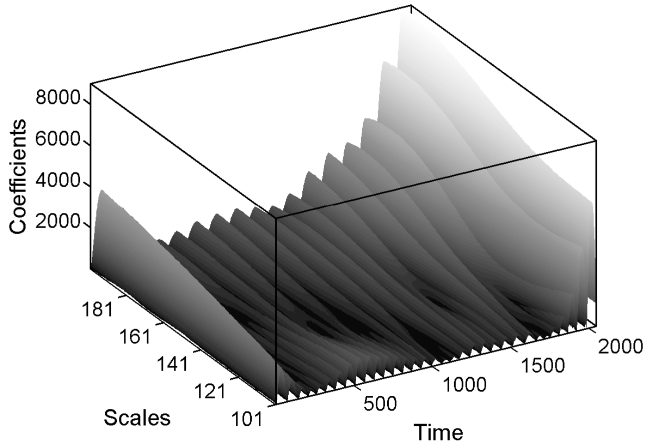

Figure 5. Scalogram of Morlet wavelet coefficients for scales 101–200 of S&P CNX Nifty closing index values.

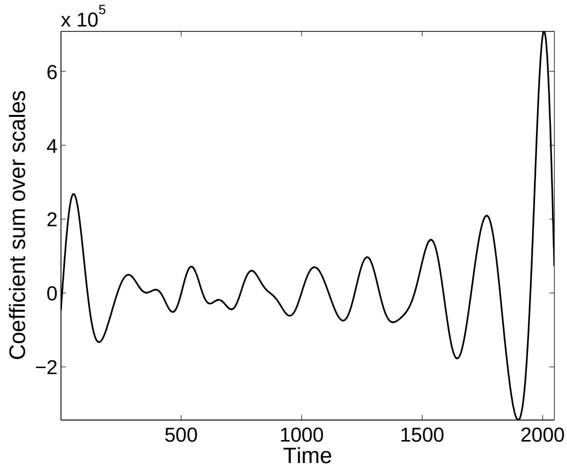

Figure 6. Sum of wavelet coefficients for Nifty data over all scales as a function of time. An approximate periodic behaviour with periodicity of about 200 trading days is evident.

2.2 *Wavelet analysis of Dow Jones industrial average dataset*

We next consider the Dow Jones industrial average (DJIA) closing values, shown in figure 7 having 2048 data points for the duration lying within 22 February 1993 and 25 January 2001. An eight-level decomposition through Haar transform is then carried out to infer about the nature of the variations. The high-pass coefficients are depicted in figure 8. Transient and cyclic behaviour at different scales are clearly visible like the previous case of Nifty data. It is to be noted that the 2nd half of the forward wavelet coefficients for each level is having higher amplitudes as compared

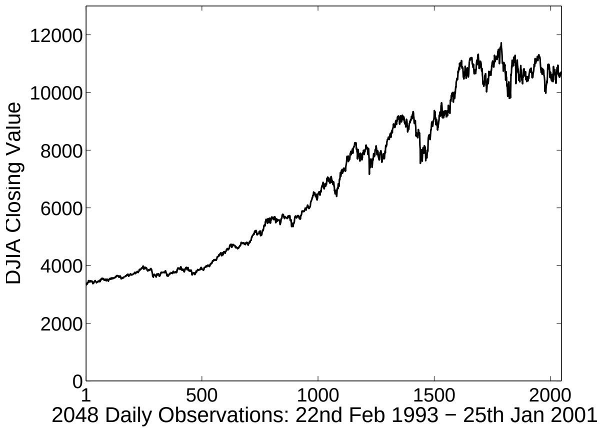

Figure 7. DJIA closing values having 2048 points covering the daily index lying between 22 February 1993 and 25 Jannary 2001.

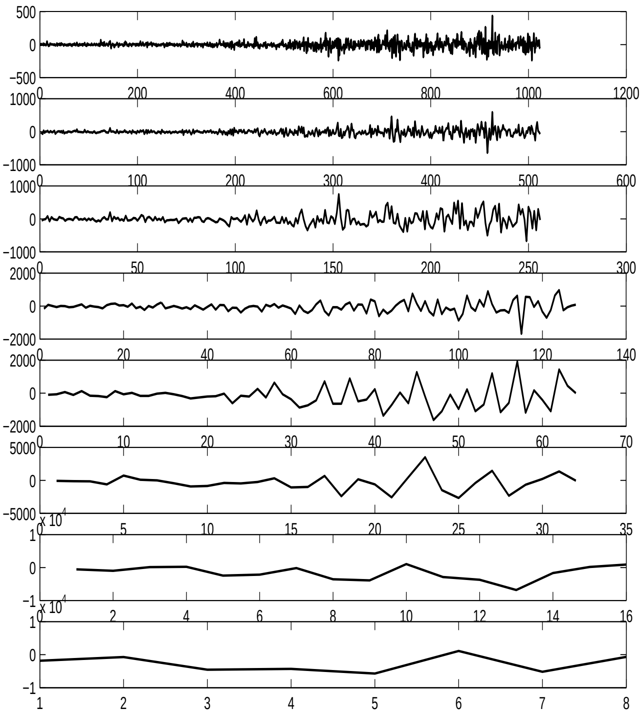

Figure 8. Haar wavelet coefficients for levels 1 to 8 for Dow Jones data.

to the first half of the coefficients. In this respect, variations of wavelet coefficients of Nifty data and DJIA data have different characteristics. The corresponding low-pass coefficients are shown in figure 9. As is evident, these low-pass coefficients show the trend of the time series. In figure 10 we show the Db4 wavelet coefficients

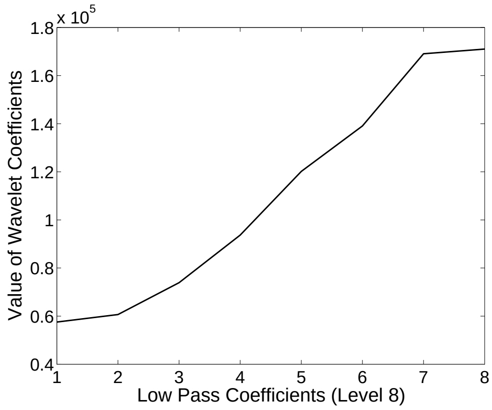

Figure 9. Haar wavelet low-pass coefficients for levels 1 to 8 for Dow Jones data. A linear trend is seen.

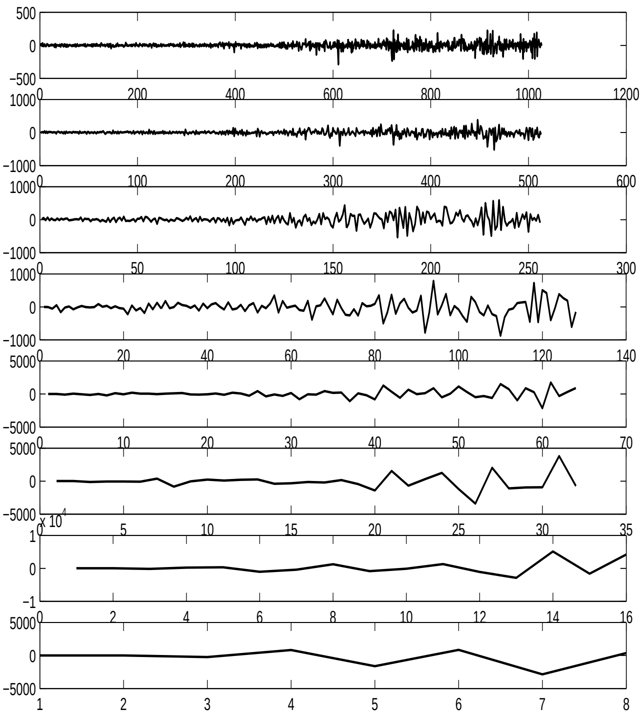

Figure 10. Db4 wavelet coefficients for levels 1 to 8 for Dow Jones data. Significant activity is seen in the second half of the wavelet coefficients.

for levels 1 to 8 for the purpose of comparison with the corresponding Nifty index behaviour.

We have studied the behaviour of DJIA under CWT. The corresponding scalogram is shown in figure 11. Akin to the Nifty case one sees cyclic behaviour in the

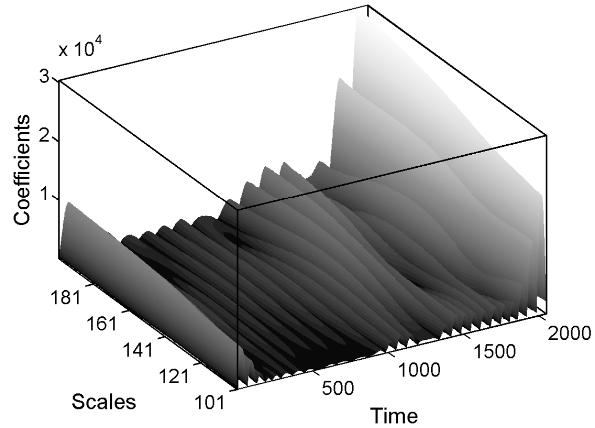

Figure 11. Scalogram of Morlet wavelet coefficients for scales 101–200 of DJIA closing values indicating a cyclic behaviour.

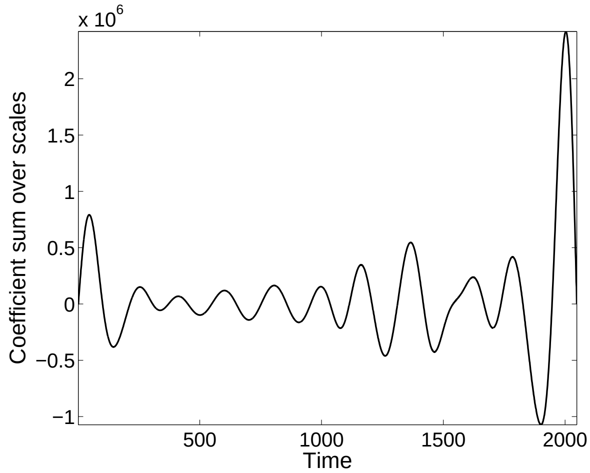

Figure 12. Sum of wavelet coefficients for DJIA over all scales as a function of time. An approximate cyclic behaviour with periodicity of a duration of little less than 200 trading days is evident.

scale range 100 to 200. The sum of the wavelet coefficients at all scales plotted as a function of time (figure 12) reveal a periodic behaviour of little less than 200 trading days. In this context, both financial time series show similar behaviour. However, the DJIA time series is showing a tendency of bursty behaviour which is absent in Nifty case.

Considering the cyclic behaviour of variations at intermediate scales, it is interesting to see how well these wavelet coefficients can be analysed through the techniques of genetic programming in which the model equations are built in the reconstructed phase space. This modelling can reveal characteristic behaviour of fluctuations at different scales.

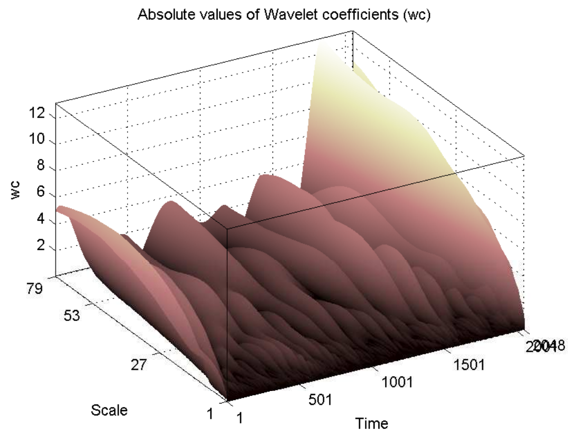

Figure 13. Scalogram of Db4 wavelet coefficients for scales and time of S&P CNX Nifty closing index.

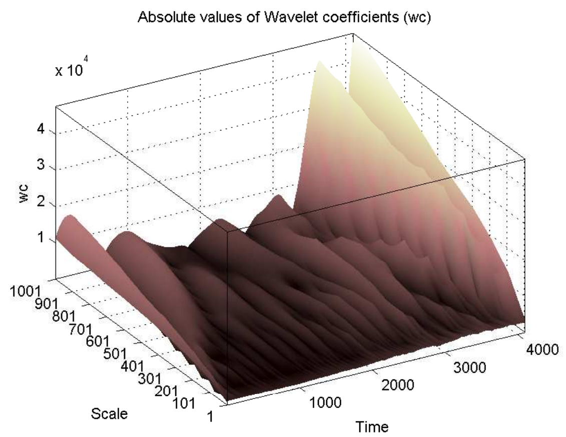

Figure 14. Scalogram of Db4 wavelet coefficients for scales and time of Dow Jones closing index.

It is worthwhile to give the scalogram for Db4 wavelet coefficients in order to explicate the time–frequency localization property of the two datasets considered here, namely S&P CNX Nifty closing index and Dow Jones industrial average closing values. The Db4 wavelet coefficients are further investigated by genetic programming approach later in §3. The scalograms for the above two datasets are given in figures 13 and 14 respectively. As has been pointed out earlier, one observes self-similar behaviour at high frequencies (small scale) and structured variation including periodic ones at moderate to higher scale respectively. The temporal periodic behaviour is more prominent in the plots (figures 15 and 16) where we have depicted the sum of power over scale at each time point.

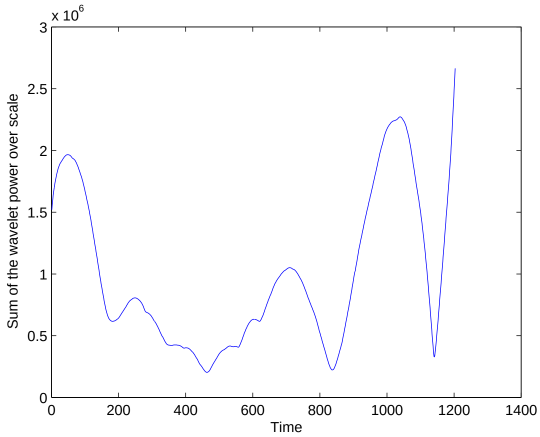

Figure 15. Sum of wavelet coefficients over scales as a function of time for S&P CNX Nifty closing index.

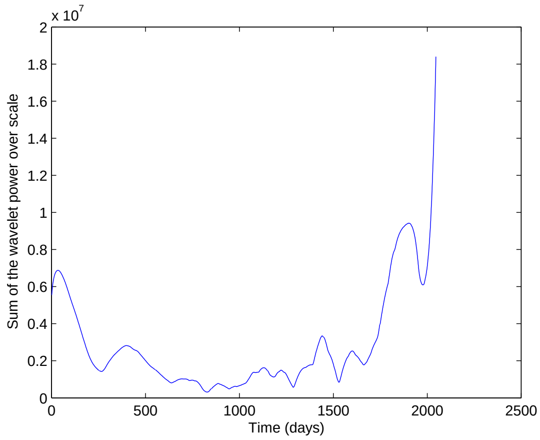

Figure 16. Sum of wavelet coefficients over scales as a function of time for Dow Jones closing index.

3. Modelling cyclic wavelet coefficients through genetic programming

Building dynamical models of time series of real systems is in general useful as it provides a framework for getting insight into the complex dynamics through the analysis of model equations. The model equations of part of the available data can be used to make predictions for the rest of the data thereby testing the efficacy of the numerical approach. In the present work we have used genetic programming (GP) as a tool for building dynamical model equations that map the time series in the reconstructed phase space. The advantage of GP approach stems from the fact that the model equations are evolved (from basic template equations) into quite complex structures involving both linear and non-linear terms. Unlike some other popular approaches (e.g. based on artificial neural network), the GP approach not only attempts to map the given sample data (used for training the model through known input–output pairs of data) but also generates model equations that caricature the inherent dynamics of the system under consideration. Thus we find GP approach more appropriate for the current investigations of cyclic variations in wavelet coefficients. The relevant parts of the basic genetic programming (GP) approach [19] and its minor modifications [20] that has been used here for modelling purpose is described below.

In order to employ genetic programming (GP) modelling of wavelet coefficients one assumes the map equation relating time-lagged variables with the entity X_{t+1} to be of the form

$$X_{t+1} = f(X_t, X_{t-\tau}, X_{t-2\tau}, ..., X_{t-(d-1)\tau}). \tag{3}$$

Here f represents a function involving time series values X_t in the immediate past and arithmetic operators (+, -, $\star$ and $\div$). The numbers appearing in function f are bounded between the range $[-N, N]$, where N is an integer number and we have chosen N to be 10. The numbers within the above range are generated with the precision of 1 digit after decimal point. In the above equation, d represents the number of previous time series values that may appear in the function and τ represents a time delay.

During the GP optimization, one considers a pool of chromosomes, each representing a potential solution. Evolution to successive generations is then carried out stochastically by applying genetic operators, namely copy, cross-over and mutation with a goal to minimize the sum of squared errors defined as

$$\Delta^2 = \sum_{i=1}^{i=N} (X_i^{\text{calc}} - X_i^{\text{given}})^2, \tag{4}$$

where N is the number of X_t values (eq. (3)) fitted with GP optimization.

We use two fitness measures, namely R^2 and r as follows:

$$R^2 = 1 - \frac{\Delta^2}{\sum_{i=1}^{i=N} (X_i^{\text{given}} - \overline{X^{\text{given}}})^2}, \tag{5}$$

$$r = 1 - (1 - R^2)\frac{N-1}{N-k}, \tag{6}$$

where $\overline{X^{\text{given}}}$ represents the average of all X_i (eq. (3)) used during the fitting, N is the number of equations fitted and k is the total number of time-lagged variables of the form X_t, $X_{t-\tau}$, $X_{t-2\tau}$,... (including repetitions) within a chromosome. As shown by Szpiro [19], the modified fitness measure helps producing crisp map equations for chromosomes.

The pool of chromosomes is then iterated using the genetic operators of selection, cross-over and mutation to obtain a convergence for suitable structures in chromosomes that give rise to optimized fit. As per the details given in Ahalpara [20] for some useful improvements in the GP implementations, we have found it useful here also to overcome the problem of GP optimization procedure settling down to persistent solutions (i.e. X_{t+1}=X_t).

We may point out that earlier we have used GP approach for modelling time series of real systems [20] where the entire time series were first smoothened and then modelled using GP approach. However, in the present approach we model wavelet coefficients for variations and not the entire time series dataset directly. Thus, the interest here is to use GP model to see firstly whether a reasonably good model emerges for a given dataset representing cyclic variations and secondly inspect the dynamical equations to possibly find reasons for emergence of cyclic structure in the wavelet coefficients. As will be shown below, modelling the wavelet coefficients brings its own computational problems due to poor convergence, and we have used a novel method to overcome this problem.

On trying to model these wavelet coefficients, it is found that due to sharp variations, the GP optimization does not lead to convergence having good fitness values. We have also confirmed this inability of fitting the wavelet coefficients by applying artificial neural network (ANN) model and have found that it also fails to give good fitness values. For example for CNX Nifty dataset, the fitness value (eq. (12)) for level = 6 is found to be 0.4944 using GP approach, which is rather a poor fitness value. In order to reconfirm this inability for getting a good fitting model, we have used ANN model and again got a poor fitness value, namely 0.43077. We have therefore found it necessary to smoothen these wavelet coefficients using an appropriate method. Cubic spline provides an interesting method to interpolate the given dataset using simple algebraic templates (in the form of piecewise polynomials), and this is found particularly useful in the present context because GP also uses similar structures in the form of algebraic templates (eq. (3)). For all the wavelet coefficients of different levels considered, we have therefore carried out a pre-processing on the datasets by applying cubic spline interpolation on the datasets. The interpolated datasets of wavelet coefficients are generated by incorporating four additional points that are sampled by cubic spline method for each consecutive pairs of points. It is worth emphasizing that this procedure is appropriate since we are modelling cyclic behaviour which are generally smooth in nature as compared to sharp variations of transients.

We now proceed with modelling of cubic spline interpolated Db4 forward wavelet coefficients using GP at level = 6, 7 and 8.

3.1 *Modelling variations in S&P CNX Nifty closing index*

As seen earlier, cyclic and structured variations emerge at relatively higher scales. For modelling purpose, we consider the 6, 7 and 8th level coefficients generated using Db4 forward wavelet transform. These coefficients show considerable cyclic fluctuations both at local and global scales. At 6th level, bursty behaviour is also seen.

We have divided the values of these wavelet fluctuations by 1000 for the sake of computational convenience. The smoothened wavelet coefficients for levels 6, 7 and 8 are shown in figures 17, 18 and 19 along with original wavelet coefficients.

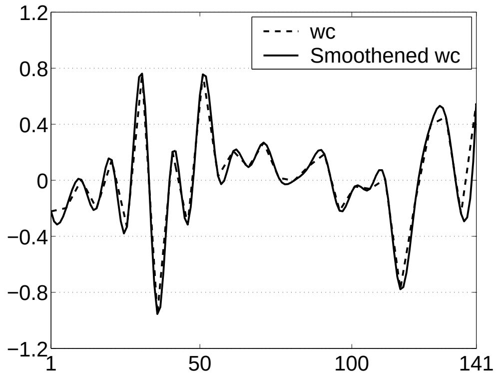

Figure 17. Db4 wavelet coefficients (wc) and spline-interpolated wc for 6th level for Nifty data.

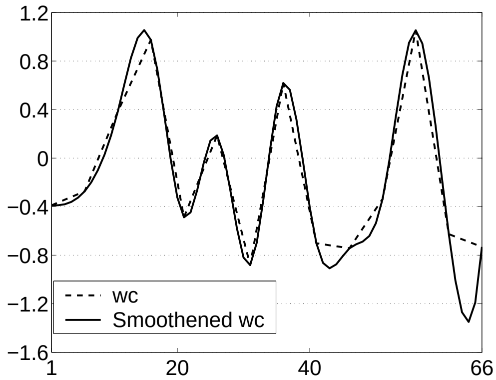

Figure 18. Db4 wavelet coefficients (wc) and spline-interpolated wc for 7th level for Nifty data.

The smoothened wavelet coefficients are then modelled using GP. We have used $d = 5$ and $\tau = 1$ for these fits and the resulting fits are very good having fitness values 0.99499 (level = 6), 0.99498 (level = 7) and 0.997038 (level = 8).

The map equations are shown in eqs (7)–(9)

$$X_{t+1}^{(\text{level}=6)} = 1.9762X_t - 1.5627X_{t-\tau} + 0.3008X_{t-3\tau} + \frac{2.5298X_t}{X_{t-2\tau} + 10.0} + \frac{0.0006023X_{t-2\tau}}{X_{t-4\tau} - 0.0339} \quad (7)$$

$$X_{t+1}^{(\text{level}=7)} = 2.814X_t - 2.9401X_{t-\tau} + 1.1239X_{t-2\tau} - 0.01761X_{t-3\tau} \quad (8)$$

$$X_{t+1}^{(\text{level}=8)} = 1.5X_t - 0.6522X_{t-2\tau} - \frac{0.368(X_{t-\tau} - 2.9X_{t-4\tau} + 1.3103)}{2.4X_t + 3.18}. \quad (9)$$

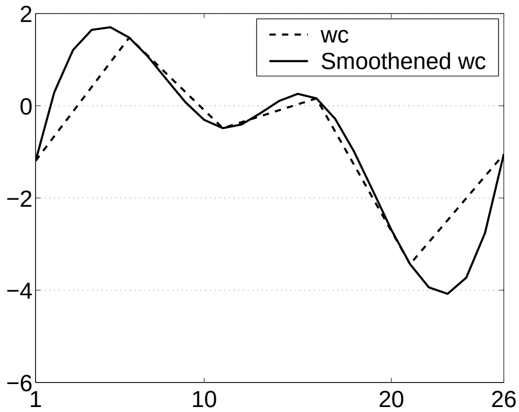

Figure 19. Db4 wavelet coefficients (wc) and spline-interpolated wc for 8th level for Nifty data.

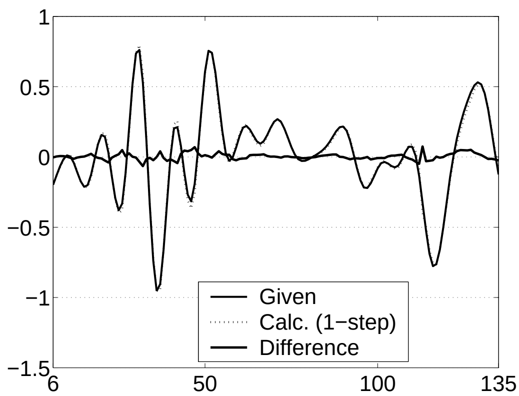

Figure 20. Fit for 130 data points using GP solution for Db4 level 6 wavelet coefficients for Nifty data.

The GP fit obtained by these equations are quite good and these are shown for levels 6, 7 and 8 in figures 20, 21 and 22 respectively. The goodness of the fits is indicated by the small values of the differences between the given and calculated values shown by the line close to 0.0.

It is worth pointing out that the GP map equations representing cyclic variations are primarily of linear type. The non-linearity if any, arising from the Padé-type rational terms, has rather small coefficients as compared to those for the linear terms. It is also noted that the significant departure of the obtained solutions from a persistent solution ($X_{t+1} = X_t$) shows the efficacy of the GP optimization approach. In the context of specific levels, we observe that the 6th level variations show cyclic as well as bursty behaviour. Interestingly, the 7th level coefficients show smooth variations; the corresponding GP equation (eq. (8)) is completely linear. On the contrary, the 8th level variations show non-smooth variations which are not bursty like the 6th level coefficients.

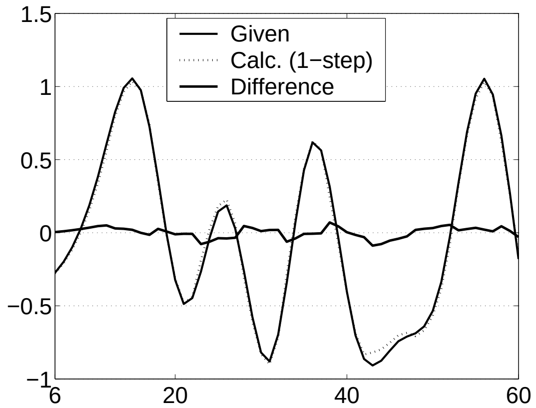

Figure 21. Fit for 55 data points using GP solution for Db4 level 7 wavelet coefficients for Nifty data.

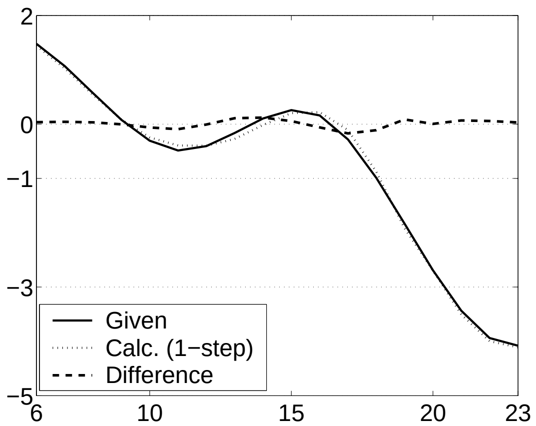

Figure 22. Fit for 18 data points using GP solution for Db4 level 8 wavelet coefficients for Nifty data.

In order to understand the interplay of different terms giving rise to the bursty behaviour, we have computed the contributions arising from each term of eq. (7) individually. The same is shown in figure 23 in which the initial 50 points are considered. It is clearly seen that the first two terms are the dominant ones, which are slightly out of phase with each other. The corresponding cancellation is responsible for the bursty behaviour. The dynamical origin of these terms and their modelling is rather non-trivial, which needs significant investigations.

The map equations will now be used to carry out 1-step out-of-sample predictions beyond the fitted points. The predictions for levels 6, 7 and 8 are shown in figures 24, 25 and 26 respectively with corresponding NMSE values as 0.04923, 0.03907 and 0.03946 respectively. It can be seen that the 1-step predictions are very good and consequently we have a good model for the cyclic variations for CNX Nifty dataset for Db4 smoothened wavelet coefficients with levels 6, 7 and 8.

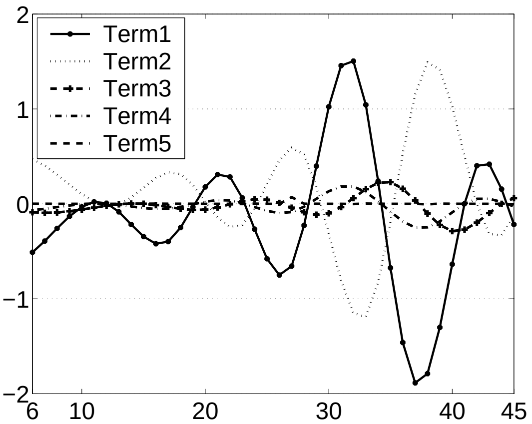

Figure 23. Contributions of individual terms in the right-hand side of eq. (7) for Db4 level-6 wavelet coefficients for Nifty data. It is observed that the first and third terms are the dominant ones which give rise to the bursty behaviour (figure 20) due to their out of phase dynamics.

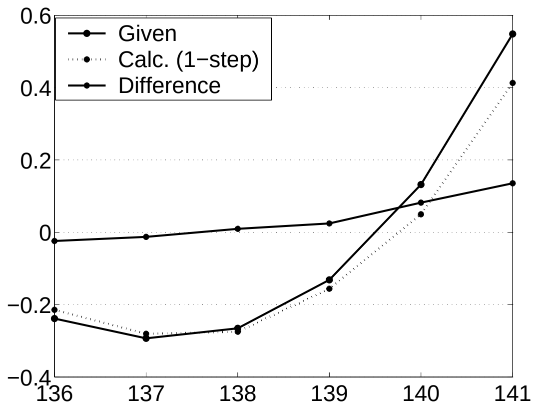

Figure 24. Out-of-sample 1-step predictions using GP solution for Db4 level-6 wavelet coefficients for Nifty data.

In order to further analyse GP map equations, the following calculations have been carried out: (1) effect of successive terms in the model equations on sum of squared errors (eq. (4)), (2) whether the up-down trend of values in the wavelet coefficients of Nifty data are reproduced. First we consider how sum squared errors (eq. (4)) vary when we drop successive individual terms in the GP model equations for Db4 levels 6, 7 and 8 Nifty wavelet coefficients (table 1). In the table, the columns under term 1 to term 5 indicate the rise in sum squared errors as a result of dropping individual terms in the GP map equation. The last column corresponds to retaining all the terms in the map equations. It is evident that the terms that are significant in contributing to the fitness have sharp increase in sum squared errors when the corrsponding terms are dropped. As also noted earlier, the first two terms

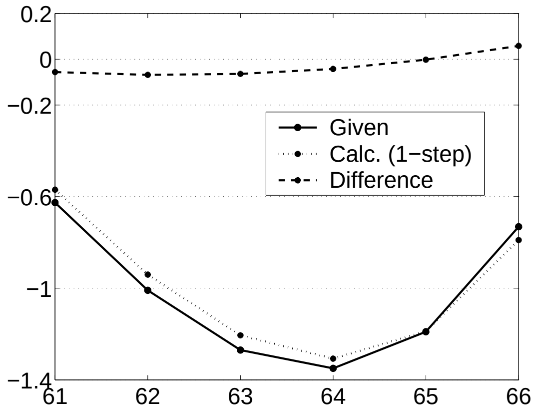

Figure 25. Out-of-sample 1-step predictions using GP solution for Db4 level-7 wavelet coefficients for Nifty data.

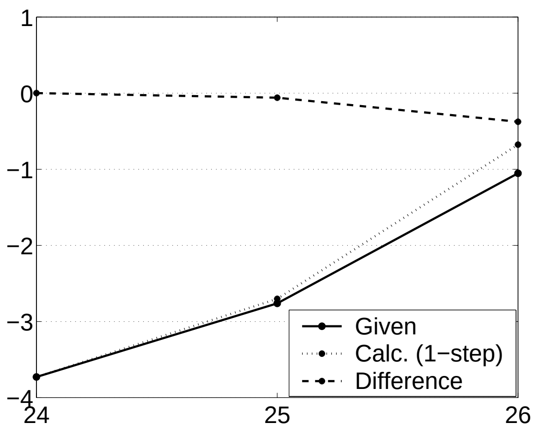

Figure 26. Out-of-sample 1-step predictions using GP solution for Db4 level-8 wavelet coefficients for Nifty data.

are out of phase and they are quite significant in contributing to the overall fitness minimized by GP optimization. In particular it is seen that out of the three equations, the first term is quite prominent for level 8 (eq. (9)) where the sum squared error grows to 102.53 if we drop 1st term (as compared to 0.11 where all terms are considered). Similar effects are seen for level 6 (eq. (7)) and level 7 (eq. (8)) with decreasing intensities. Secondly, we consider effectiveness of fitting up–down trend for the above data series. For each consecutive pair of wavelet coefficients we check whether the trend of rise or fall for the second point with respect to the first point is reproduced by the GP map equation. The resulting up–down trend for the entire series of wavelet coefficients is captured by the percentage of pairs for which the up–down trend is reproduced. The result of such an analysis shows that for level 6 (figure 20), level 7 (figure 21) and level 8 (figure 22) the percentage of correct sign for up–down trend for pairs of consecutive wavelet coefficients are 96.9, 96.3 and

Table 1. Effect on sum squared errors (eq. (4)) by dropping individual terms in the GP model equations for modelling Db4 levels 6, 7 and 8 wavelet coefficients of Nifty data.

	Dropping individual terms from GP map equations					
GP map equation	Term 1	Term 2	Term 3	Term 4	Term 5	Keep all
Level 6 (eq. (7))	52.27	33.02	1.32	0.93	0.08	0.07
Level 7 (eq. (8))	23.10	14.81	0.58	0.40	0.04	0.03
Level 8 (eq. (9))	102.53	10.36	1.11	–	–	0.11

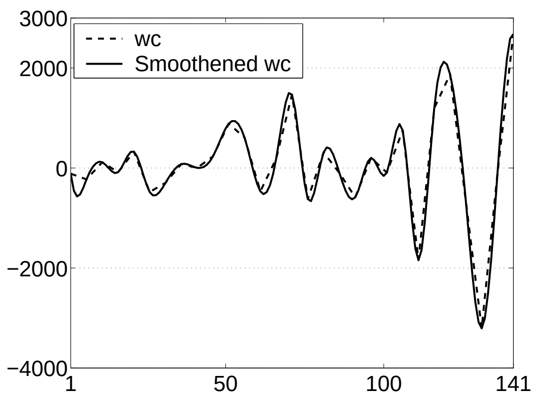

Figure 27. Db4 wavelet coefficients (wc) and spline-interpolated wc for 6th level Dow Jones data.

88.2 respectively (where the total number of pairs are 129, 54 and 17). The rather sharp decrease of percentage of correct sign for level 8 (figure 22) is due to less number of total pairs (17 only) and small fluctuations in wavelet coefficients in the middle of the plot (figure 22); whereas the overall fit is quite good. Thus, on the whole, the GP map equations are found to be quite good in producing reasonable fits.

3.2 *Modelling variations in Dow Jones industrial average closing values*

We next consider modelling of Db4 wavelet coefficients for Dow Jones industrial average (DJIA) closing values. Akin to the GP analysis for Nifty wavelet coefficients, we have found it useful to smoothen the wavelet coefficients using cubic spline for the purpose of GP modelling. This also makes it easier to compare the cyclic variations in the two time series considered.

The comparison of Db4 forward wavelet coefficients for levels 6, 7 and 8 with the cubic spline smoothened wavelet coefficients are shown in figures 27, 28 and 29. The map equations generated by GP for $d = 5$ and $\tau = 1$ are having very good fitness values 0.998013, 0.99721 and 0.99929 and are shown in eqs (10)–(12).

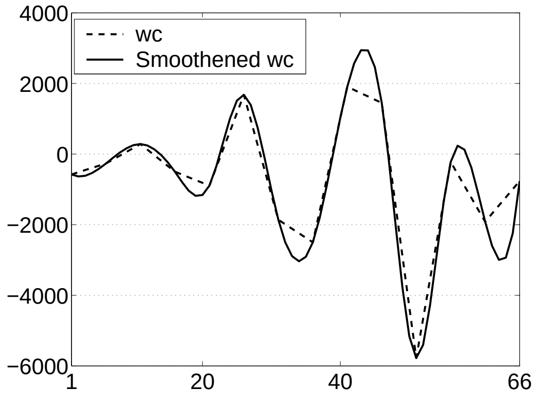

Figure 28. Db4 wavelet coefficients (wc) and spline-interpolated wc for 7th level Dow Jones data.

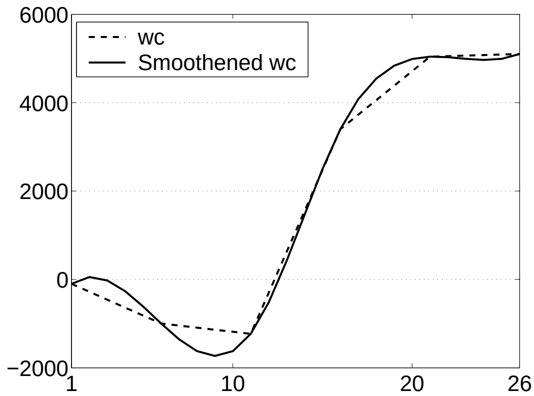

Figure 29. Db4 wavelet coefficients (wc) and spline-interpolated wc for 8th level Dow Jones data.

$$X_{t+1}^{(\mathrm{level}=6)} = 2.4375X_t - 1.8398X_{t-\tau} + 0.12669X_{t-2\tau} + 0.2602X_{t-3\tau} + 0.7002 + \frac{0.0186(X_{t-4\tau} + 248.5)}{X_{t-\tau}} + \frac{0.6841(1.5X_t + 125.85)}{X_{t-4\tau} + 76.3} \tag{10}$$

$$X_{t+1}^{(\mathrm{level}=7)} = 2.2386X_t - 1.5X_{t-\tau} + 0.0947X_{t-2\tau} + 0.1481X_{t-4\tau} - 16.993 - \frac{0.71023(X_{t-\tau} + 540.519)}{X_{t-2\tau} - X_{t-3\tau}} \tag{11}$$

$$X_{t+1}^{(\mathrm{level}=8)} = 2.20989X_t - 1.3999X_{t-\tau} + 0.16484X_{t-3\tau} + 62.9235 - \frac{0.10989X_{t-\tau}}{X_{t-3\tau}}. \tag{12}$$

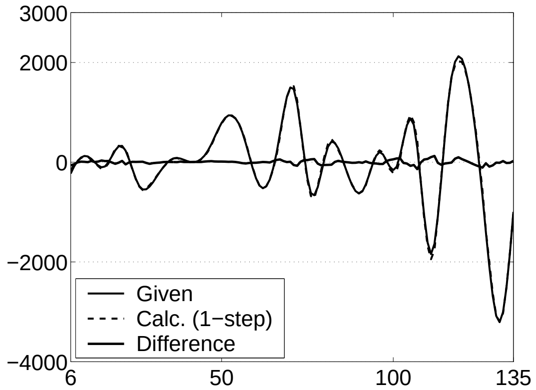

Figure 30. Fit of the GP solution for Db4 level-6 Dow Jones wavelet coefficients. The variations have a bursty character.

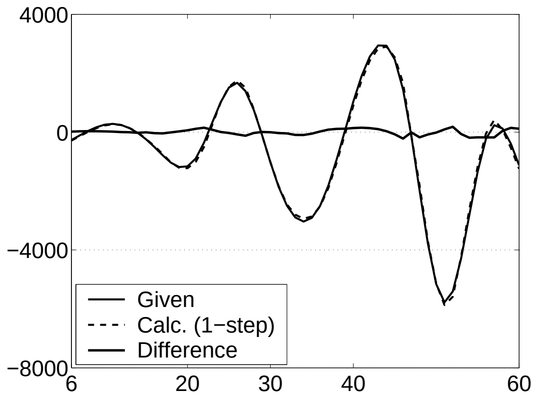

Figure 31. Fit of the GP solution for Db4 level-7 Dow Jones wavelet coefficients. The amplitude of the cyclic variations is seen to increase with time.

As seen in figures 30, 31 and 32 for levels 6, 7 and 8 respectively, the GP fits are quite good.

Similar to the Nifty analysis, the GP equations are primarily of linear type having non-linear terms of Padé type. Equation (10) is primarily linear. The effect of non-linearity as seen from the Padé terms is different from the Nifty behaviour. Equation (11) shows a very interesting behaviour. If difference between two consecutive data points are small, then the Padé term gives a strong contribution, which decreases as the differences increase. Equation (12) representing level 8 coefficients is again mostly linear. We have also carried out analysis, similar to that of Nifty, related to the interplay of different terms in GP equations and have found that the initial few terms are responsible for producing the bursty behaviour in the wavelet coefficients.

We then use these map equations and carry out 1-step out-of-sample predictions beyond the fitted points. These predictions are found to be very good as is reflected

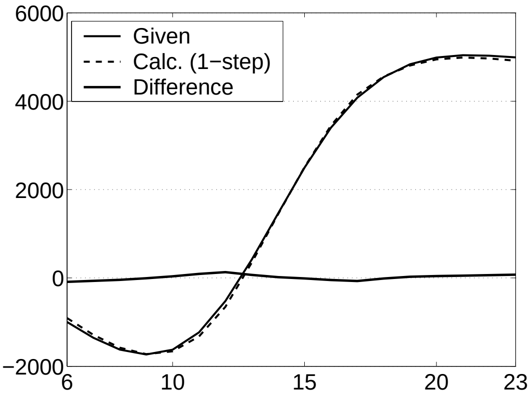

Figure 32. Fit of the GP solution for Db4 level-8 Dow Jones wavelet coefficients. The variations show a step-like behaviour.

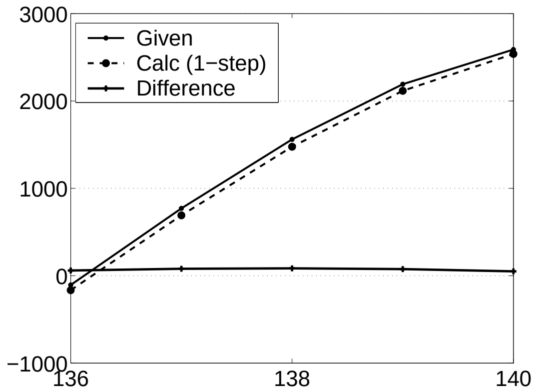

Figure 33. Out-of-sample 1-step predictions using GP solution for Db4 level-6 Dow Jones wavelet coefficients.

from their small NMSE values, 0.001787 (level 6), 0.002379 (level 7) and 0.0004981 (level 8). The predictions are shown for levels 6, 7 and 8 in figures 33, 34 and 35 respectively and are found to be excellent. Compared to CNX Nifty analysis, we have even a better model for the cyclic variations for Dow Jones industrial average closing index for Db4 smoothened wavelet coefficients with levels 6, 7 and 8.

Akin to the analysis made for Nifty data, we have also analysed the effects of successive terms on sum squared errors for the Dow Jones wavelet coefficients and have found that the dominant terms are the initial few ones (that are out of phase with each other) and they are of linear type. The percentage of pairs of consecutive wavelet coefficients that give rise to correct sign of up–down trend is also calculated and it is found that for level 6 (eq. (10)), level 7 (eq. (11)) and level 8 (eq. (12)) the percentages are 98.5, 96.3 and 100 respectively. Thus the overall results for Db4 Dow Jones wavelet coefficients are very good.

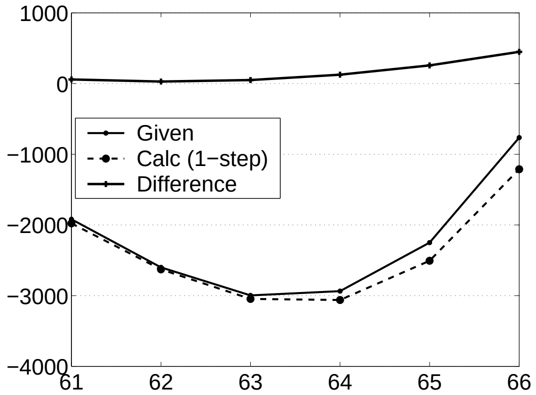

Figure 34. Out-of-sample 1-step predictions using GP solution for Db4 level-7 Dow Jones wavelet coefficients.

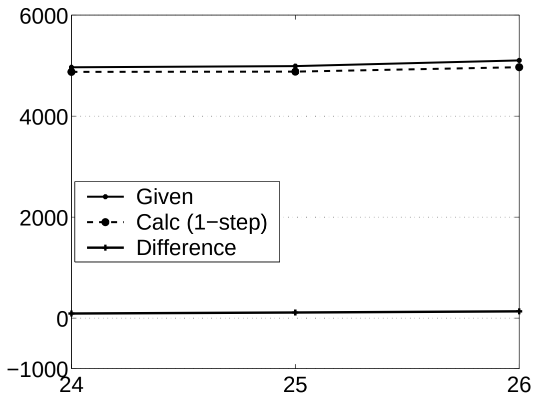

Figure 35. Out-of-sample 1-step predictions using GP solution for Db4 level-8 Dow Jones wavelet coefficients.

4. Conclusion

In conclusion, we have illustrated a wavelet-based approach to separate stochastic and structured variations in non-stationary time series. Modelling different aspects e.g., fluctuations and trend of these time series is a challenging task. It becomes particularly difficult when the fluctuations comprise of random, cyclic and transient variations at multiple scales. The fact that wavelet transform possesses multi-resolution analysis capability, has opened the way to isolate variations at different scales. We have taken advantage of this ability of wavelets to study and model cyclic variations of the financial time series, which are known to be non-stationary. To the best of authors' knowledge, the observation of cyclic behaviour in the wavelet domain for the financial time series studied here, has not been reported earlier. We have found that after pre-processing the wavelet coefficients through cubic spline interpolation, genetic programming models the cyclic behaviour well through crisp dynamical equations. One-step predictions have been carried through and these are found to be quite accurate.

It needs to be pointed out that modelling of variations in general is computationally quite challenging. In the present case, both GP as well as ANN failed to converge to reasonable fits when used directly on the wavelet coefficients. We have overcome this problem by carrying out interpolation on the cyclic variations before modelling them by GP. We have found cubic spline interpolation quite useful in the present context due to algebraic nature of templates considered both in cubic spline as well as in GP approaches. Computationally, the concept used in the present paper is new which opens up the possibility of wavelet domain modelling where one necessarily encounters sharp variations. This also would be useful for many researchers modelling sharp variations in general. The result reported in the present paper is that GP captures the spline interpolated cyclic variations quite well and the resulting model equations are primarily of linear type having additive Padé terms. It is seen that for both GP as well as ANN, interpolation (based on cubic spline method), is indeed required to be carried out before an acceptable model for dynamical equations can be obtained. We would like to add that the method used in the present analysis opens up the possibility, where structured variations of localized character can be combined with high frequency random noise (the high-pass coefficients in the wavelet domain) of appropriate nature for a better understanding and modelling of time series. This is conceptually an alternate and useful approach in addition to modelling the entire time series.

Apart from studying other physical time series, it will be nice to combine the present approach with random matrix-based ones for the purpose of pinpointing emergence of cyclic behaviour. As has been mentioned earlier, random matrix approach has indicated correlation between group of companies in financial time series, which can lead to cyclic or structured variations apparent in the present analysis. Hence, it will be of deep interest to see if these two can be interrelated.

Acknowledgements

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