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Dominance of rare events in some problems in statistical physics

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Abstract. We show how the theory of large deviations in the coin toss experiment can give some insight into nonequilibrium fluctuation theorems and intermittency in turbulence.

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1. Introduction

Rare events in the context of physics can have various connotations. An obvious one is the phenomenon where a barrier exists and one has to hop across the barrier to activate the event. Decay of metastable states and the occurrences of certain chemical reactions are examples of such processes. It is the thermal noise that activates these hopping processes and the rareness of the events is characterized by the large time-scale involved. The relevant time-scales are of the order $e^{\Delta V/K_{\rm B}T}$, where ΔV is the height of the barrier to be overcome, T is the temperature and $K_{\rm B}$ is the Boltzmann's constant. Here, however, our focus will be on rare events which happen with a probability that lies at the tail of the probability distribution in the case of certain problems in statistical physics.

The role of large deviations in physics as a uniform theme of problems in statistical physics was employed by Oono [1] two decades ago. It is in the last decade that the view has become more widespread and study of such systems becomes plentiful [2–5]. One such area is the fluctuation theorems [6–10] which relate to nonequilibrium systems, characterized by irreversible heat losses between the system and its environment; typically a thermal bath. For systems in equilibrium with time reversal symmetry, the probability of absorbing a given amount of heat is equal to the probability of releasing the same amount. This ratio of heat absorbed to heat released is not unity in nonequilibrium situations.

The steady-state nonequilibrium systems are more likely to deliver heat to the surroundings than absorb heat from it. If the system is macroscopic in size, then the probability of heat absorption is insignificant. For small systems (e.g. molecular motors) the probability of absorbing heat can be significant. On an average, heat would be produced but there would be processes that imply occasional absorption of heat. This actually goes back to Loschmidt's objection to Boltzmann's derivation of the second law of thermodynamics from Newton's laws of motion. Since the microscopic laws of motion are invariant under time reversal, Loschmidt argued that there must also be entropy decreasing evolutions which violate the second law of thermodynamics. The fluctuation theorems delineate the occurrence of macroscopic irreversibility from the time reversal invariant microscopic equations of motions. Time reversed trajectories do occur but they become rarer and rarer with increasing size of the system. These are the rare events and their occurrence is a signature of large deviations. A particular form of the fluctuation theorem will be discussed in the next section, where we will show how it relates to the general theory of large deviations which we will discuss in this section.

Probably the earliest that large deviations entered the domain of physics was the 1960's when Kolmogorov and Obukhov reconsidered Kolmogorov's theory of homogeneous isotropic turbulence of 1941 in the light of Landau's objection. In a nonequilibrium steady-state situation for the turbulent velocity field, Kolmogorov had assumed that the energy supplied per unit time at large scales was exactly equal to the energy dissipated at the smallest (molecular) scales. It was assumed that the energy dissipation rate was a constant at all scales and that was contradicted by Landau. Phenomenologically, Kolmogorov and Obukhov [11] introduced fluctuations in the dissipation rate in 1962. Careful experiment revealed the existence of these fluctuations. The fluctuations occurred rarely – these were the rare events of turbulence [12–19]. These rare events constitute one of the most difficult issues to understand in the theory of turbulence. In fourth section we will see how the theory of large deviation, applied in a purely intuitive manner can immediately show a connection between this longstanding problem and the simple coin toss experiment.

While our focus here will be on the fluctuation theorem and turbulence, it should be mentioned that yet another set of problems where the tail of the probability distribution is the key to the problem of persistence [20–24]. In its simplest incarnation, this is the issue of a one-dimensional random walker starting out from the origin at $t = 0$. As it takes random steps to the right and left, there is always the possibility of its crossing the origin in the course of its meandering. We ask what is the probability that after a time t has elapsed, the walker has not returned to the origin even once. This constitutes the rare event. The probability for not returning has to go to zero as $t \to \infty$ and the long time behaviour is characterized by $P(t) \sim 1/t^{\theta}$, where θ is the persistence exponent. Over the last decade, a wide variety of physical systems (e.g. the simple diffusion process, Ising model, Ginzberg–Landau model, model of growth etc.) have been found to exhibit this slow decay in the probability distribution.

2. Theory of large deviation

Here we briefly recall the theory of large deviations [25] by considering the coin toss experiment. If it is a fair coin, then for each toss the probability of obtaining 'heads' is exactly 1/2. If we assign the value 1 to the outcome head and 0 to the 'tail', then the average value M_N for an experiment with N tosses is

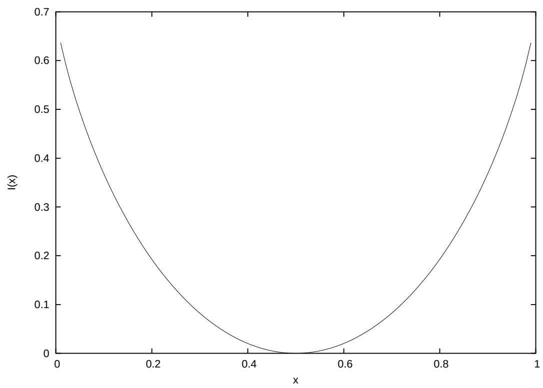

Figure 1. Rate function for a fair coin tossing experiment.

$$M_N = \frac{1}{N}\sum_{i=1}^{N} X_i, \tag{1}$$

where $X_i = 1$ for 'head' and 0 for 'tail'. As N becomes larger and larger, M becomes closer and closer to 1/2. The theory of large deviation has to do with the probability of M being different from 1/2 after N steps. Crammer's theorem is the central result which states that $P(M_N > x > 1/2)$ or $P(M_N < x < 1/2)$ can be written as

$$P \sim \mathrm{e}^{-NI(x)}, \tag{2}$$

where $I(x)$ is called the rate function. It should be noted that the result holds for a sequence of bounded random variables $X_1, X_2, \ldots, X_N$ which are identically distributed and have same mean m. The corresponding probability has to be $P(M_N > x > m)$ or $P(M_N < x < m)$. For the example of coin toss, the function $I(x)$ is exactly known and is given by

$$I(x) = x \ln 2x + (1-x)\ln 2(1-x). \tag{3}$$

Rate function for a fair coin tossing experiment is plotted in figure 1. If the coin is biased and the probability for getting 'heads' is p, then

$$I(x) = x \ln \frac{x}{p} + (1-x)\ln \frac{1-x}{1-p}. \tag{4}$$

Putting $p = 1/2$ in the above equation, we get eq. (3), as expected. For $p = 1/4$, $I(x)$ is shown in figure 2.

We notice that for $p = 1/2$, $I(x) = (x-1/2)^2/(2 \times \frac{1}{4})$ when $x \approx 1/2$ and thus the rate function gives us the central limit theorem, since the mean for the coin toss is 1/2 and the fluctuation is 1/4. The central limit theorem governs the fluctuation near the mean; here the proximity is quantified by $\sigma/\sqrt{N}$ where σ is the deviation. The large deviation theory as examplified by eq. (2) can handle fluctuations which are beyond the range of proximity of the mean. These deviations happen rarely and hence are called rare events. In what follows, we show a simple connection between these rare events and rare events in some physical situations.

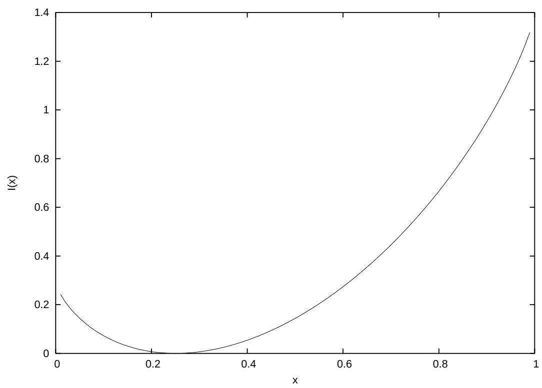

Figure 2. Rate function for a biased coin tossing experiment with $p = 1/4$.

3. Jarzynski equality and the coin toss experiment

In this section, we discuss an example of the importance of rare events which has been an interest over the last decade. It involves the time evolution of a system from $t = 0$ to $t = \tau$ in contact with a heat bath at temperature T, under an explicitly time-dependent force. At $t = 0$, the system is in thermal equilibrium corresponding to the given temperature. It is characterized by the relevant macroscopic variables and there is a definite value for its equilibrium free energy which we denote by F. A time-dependent force $\lambda(t)$ is switched on at $t = 0$ and acts for a period τ. If $t \geq \tau$, the force becomes time independent, and the system would relax to a new equilibrium corresponding to a new thermodynamic free energy $F + \Delta F$. For a general path from $t = 0$ to $t = \tau$, can ΔF be related to the work done W? This question was answered by Jarzynski who first clarified the meaning of work done in the presence of time-dependent force. Jarzynski considered the energy in the form of Hamiltonian and defined W as

$$W = \int_0^\tau \frac{\partial H}{\partial \lambda} \frac{\mathrm{d}\lambda}{\mathrm{d}t} \mathrm{d}t, \tag{5}$$

where $\lambda(t)$ is the time-dependent force appearing in the Hamiltonian. It was established by Jarzynski that

$$\mathrm{e}^{-\Delta F} = \left\langle \mathrm{e}^{-W} \right\rangle, \tag{6}$$

where both ΔF and W are measured in units of $K_{\mathrm{B}}T$ and the average is over, all the initial conditions that constitute large number of microstates corresponding to the macrostate at $t = 0$, and also over all possible paths. Since W can be quite generally be written as $W = \Delta F + W_{\mathrm{diss}}$ we see that from the previous equation we can write

$$\left\langle e^{-W_{\mathrm{diss}}} \right\rangle = 1 \tag{7}$$

where W_{diss} is called dissipative work. Thermodynamics says that $W \geq \Delta F$ which implies $W_{\mathrm{diss}} \geq 0$. However, for the previous equation to hold, paths along which

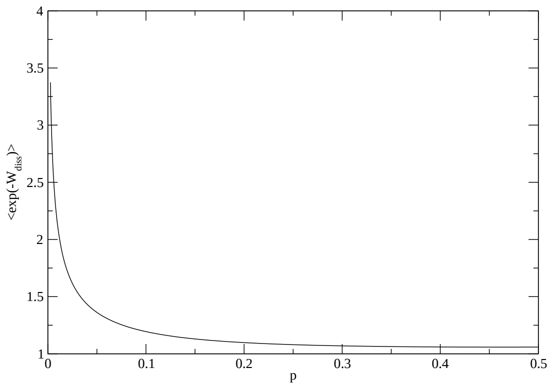

Figure 3. The plot of p and the corresponding integral values for $N = 10$.

$W_{\text{diss}} < 0$ are also probable; though with very low probability. The statement $W_{\text{diss}} \geq 0$ is true only in a statistical sense; it is the importance of the rare events ($W_{\text{diss}} < 0$) which is highlighted by eq. (7). We show below how the theory of large deviations gives an indication of the validity of eq. (7).

Having identified that $W_{\text{diss}} < 0$ values constitute the rare events, we now want to look at the tail of the distribution for W_{diss}. If we define a variable X as

$$X = \frac{1}{2}[1 - \tanh(W_{\text{diss}} + c)] \tag{8}$$

with $c > 0$, then the probability of W_{diss} being negative is the same as X having values between $\frac{1}{2}(1 - \tanh c)$ and 1 while W_{diss} being positive covers the range 0 to $\frac{1}{2}(1 - \tanh c)$. We now imagine doing the evolution from $t = 0$ to $t = \tau$ several times. Denoting the number of times the experiment is done by N, we note that W_{diss} will have different values for different realizations of the experiment. These values constitute the ensemble with a probability distribution. For every value of W_{diss} we have a value of X and thus an ensemble of X values. It is the distribution for X which we relate to the distribution for coin toss. Defining

$$p = \frac{1}{2}(1 - \tanh c), \tag{9}$$

we now return to eq. (2) and assume that X corresponds to the coin toss experiment with a biased coin, where the probability of the value '0' is p lying between 0 and 1/2. The probability of finding a value of X between 0 and 1 is then given by eq. (2) with the rate function given by eq. (4). The biased coin ensures that the negative values of W_{diss} are suppressed. We note from (9) that $\mathrm{e}^{-W_{\text{diss}}} = \mathrm{e}^{c}\sqrt{\frac{X}{1-X}}$ and hence using eq. (7) we get

$$\left\langle \mathrm{e}^{-W_{\text{diss}}} \right\rangle = \mathrm{e}^{c} \frac{\int_0^1 \sqrt{\frac{X}{1-X}} \left(\frac{p}{X}\right)^{NX} \left(\frac{1-p}{1-X}\right)^{N(1-X)} \mathrm{d}X}{\int_0^1 \left(\frac{p}{X}\right)^{NX} \left(\frac{1-p}{1-X}\right)^{N(1-X)} \mathrm{d}X}. \tag{10}$$

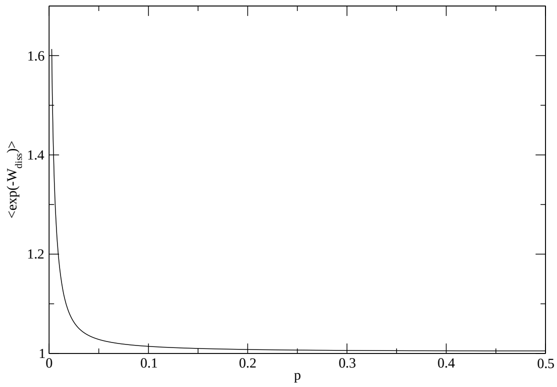

Figure 4. The plot of p and the corresponding integral values for $N = 100$.

Keeping N fixed, we will now numerically evaluate the above integral for different values of $c > 0$ (note that with c, p will also vary between 0 and 1/2 according to the definition of p given by eq. (9)). For $N = 10$ and $N = 100$ we plot p and corresponding values of the integral below. From the plots (figures 3 and 4) it is very clear that, as N becomes larger, the values of the integral for different p converge to 1 faster. Thus we have proved that as $N \to \infty$, $\langle e^{-W_{\text{diss}}} \rangle = 1$ from basic results of the theory of large deviations.

4. Turbulence and coin toss experiment

We look at the 'large deviations' in the problem of turbulence in this section. The velocity field $\vec{v}(\vec{r}, t)$ of a field satisfies the Navier–Stokes equation:

$$\frac{\partial v_\alpha}{\partial t} + v_\beta \partial_\beta v_\alpha = -\partial_\alpha P + \nu \nabla^2 v_\alpha + f_\alpha, \tag{11}$$

where repeated indices are summed over, P is the pressure, ν is the kinematic viscosity and f is an external force. The flow is taken to be incompressible, i.e., $\partial_\alpha v_\alpha = 0$, which makes the pressure term acquire the same structure as the non-linear term. The nature of the flow is determined by the dimensionless number $R = vL/\nu$ (Reynolds number), where v and L are characteristic velocity and length scales. For low values of R (the nonlinear term is negligible), the flow is laminar, while for high values of R (the nonlinear term is dominant), the flow enters a 'turbulent' phase where the velocity field is related to the fact that the solution of the deterministic Navier–Stokes equation will be chaotic for large R and thus show extreme sensitivity to the initial conditions. If we consider an ensemble of varying initial conditions, then there will be a distribution of values of $v_\alpha(\vec{r}, t)$ at any spatial point at a given time and only ensemble averages will make sense. Thus, we need to talk about a probability distribution for the velocity field. It is customary to talk about the probability distribution for $\Delta v_\alpha(\vec{r}, t) \equiv v_\alpha(\vec{x} + \vec{r}, t) - v_\alpha(\vec{x}, t)$, so that mean flow's effects can be diminished. The important experimental fact is that the probability distribution $P(\Delta v_\alpha)$ of $\Delta v_\alpha(\vec{r}, t)$ is universal at large R in the inertial range and thus it makes the calculation greatly interesting. It is the universality of $P(\Delta v)$ (universal implies independent of any kind of fluid, type of energy input and Reynolds numbers). It is of interest to note that the probability distribution is time independent because we are considering a steady-state situation. It we construct $\frac{\partial}{\partial t}\int(\frac{1}{2}v_\alpha v_\alpha \mathrm{d}^3r)$ from eq. (11), we note that the terms $v_\beta\partial_\beta v_\alpha$ and $-\partial_\alpha P$ do not contribute to the rate of change of total energy, while the viscous term's contribution is negative definite. This implies that if there is no external force, the motion would eventually stop. For steady-state turbulence, one needs an external forcing such that the rate of energy input due to the external force exactly equals the rate at which the viscous dissipation takes place.

The time rate of energy injection is taken to be ϵ (this is also the rate of dissipation) and this is a parameter that enters the probability distribution. The universality of $P(\Delta v)$ holds in what is called the inertial range. To understand what is an inertial range, we need to introduce two length scales. One is L the scale at which energy is fed into the fluid. This is a macroscopic length. The other, a microscopic scale l formed from ϵ and ν, is the scale at which the viscous dissipation becomes important. Dimensional analysis shows $l = (v^3/\epsilon)^{1/4}$. Inertial range comprises scale r such that $L \gg r \gg l$. It is obvious that at these scales r, the external forcing which occurs at L and the molecular process which occurs at l do not exert much influence. For the inertial range to exist, the scales L and l must be well-separated. We see that $L/l \sim R^{3/4}$ and hence for large R, the two scales are far apart and a significant inertial range exists.

While it is not obvious how to calculate the probability distribution, Kolmogorov proposed a form for the velocity correlation $\langle[\Delta v(r)]^n\rangle$. To see what the form should be, we note that a typical velocity scale is obtained as $(\epsilon\nu)^{1/4}$ and hence far away from L (i.e., $r \ll L$)

$$\langle[\Delta v(r)]^n\rangle = (\epsilon\nu)^{n/4} f(r/l). \tag{12}$$

In the inertial range the correlation function needs to be independent of ν and assuming $f(r/l) \sim r/l$ for $r \gg l$, we immediately note that $x = n/3$ for the required independence. This leads to the Kolmogorov proposition

$$\langle[\Delta v(r)]^n\rangle = K_n(\epsilon r)^{n/3}, \tag{13}$$

where K_n are constants. For $n = 3$, Kolmogorov found the exact nontrivial result

$$\langle[\Delta v(r)]^3\rangle = -\frac{4}{5}\epsilon r. \tag{14}$$

While the early experiments did seem to conform to eq. (14), careful measurements, particularly for large n, showed very clear deviations from eq. (14). The reason lies in an observation by Landau who noted that the dissipation rate is really a local quantity which can be written down at scale r as

$$\tilde{\epsilon}(r) = \frac{\nu}{2}\int_{\text{Sphere of radius } r \text{ centered at } \vec{x}} \left(\frac{\partial v_\alpha}{\partial x_\beta} + \frac{\partial v_\beta}{\partial x_\alpha}\right)^2 \mathrm{d}^3x. \tag{15}$$

Kolmogorov replaces $\tilde{\epsilon}(r)$ by its average value ϵ. What is the probability of observing fluctuations of $\tilde{\epsilon}(r)$ from ϵ? In general, these fluctuations are small but there are the rare large deviations which could cause a breakdown of eq. (14). In 1962, Kolmogorov and Obukhov independently took account of this possibility by postulating that the probability distribution for the energy $\tilde{\epsilon}(r)$ is log-normal, i.e.,

$$P(\tilde{\epsilon}(r)) \propto \mathrm{e}^{-(\ln\tilde{\epsilon}-m)^2/2\sigma^2}, \tag{16}$$

where $m = \langle \ln\tilde{\epsilon}\rangle$ is the mean of the distribution and the width, conjectured from a perturbative calculation, is written as

$$\sigma^2 = A + 9\delta \ln\left(\frac{L}{r}\right). \tag{17}$$

The average of $[\tilde{\epsilon}(r)]^{n/3}$ can be written as

$$\begin{aligned}\langle[\tilde{\epsilon}(r)]^{n/3}\rangle &= \frac{\int[\tilde{\epsilon}(r)]^{n/3}\mathrm{e}^{-(\ln\tilde{\epsilon}-m)^2/2\sigma^2}\mathrm{d}(\ln\tilde{\epsilon})}{\int \mathrm{e}^{-(\ln\tilde{\epsilon}-m)^2/2\sigma^2}\mathrm{d}(\ln\tilde{\epsilon})} \\ &= \mathrm{e}^{n^2\sigma^2/18}\mathrm{e}^{mn/3}.\end{aligned} \tag{18}$$

For $n = 3$, $\langle[\tilde{\epsilon}(r)]\rangle = \mathrm{e}^{\sigma^2/2}\mathrm{e}^m$, which is required to be some constant times ϵ by eq. (14). This leads to

$$\mathrm{e}^m = \epsilon\mathrm{e}^{-\sigma^2/2}B, \tag{19}$$

where B is a constant. The Kolmogorov law of equation (13) now assumes the form

$$\begin{aligned}\langle[\Delta v(r)]^n\rangle &= K_n\langle\tilde{\epsilon}^{n/3}\rangle r^{n/3} \\ &= \tilde{K}_n\epsilon^{n/3}\mathrm{e}^{\sigma^2/6(n^2/3-n)}r^{n/3} \\ &= C_n\epsilon^{n/3}\left(\frac{L}{r}\right)^{(\delta/2)[n(n-3)]} r^{n/2}.\end{aligned} \tag{20}$$

This is the result obtained by Kolmogorov and Obukhov independently in 1962. The energy dissipation rate can be viewed as $\tilde{\epsilon}(r) = (\Delta v)^2/\tau$ where τ is a time-scale which can be represented as $\tau \sim \Delta v/r$ and thus $\tilde{\epsilon}(r) = (\Delta v)^3/r$. From eq. (20), we thus find $\langle[\tilde{\epsilon}(r)]^2\rangle \sim r^{-9\delta} = r^{-\mu}$, where $\mu = 9\delta$ is generally called the intermittency exponent. The fact that $\mu \neq 0$ is taken to be the signature of intermittency.

In a somewhat different approach to this problem, Stolovitzky and Sreenivasan [26] tried to view turbulence as a general stochastic process. Using global isotropy, they expressed the mean dissipation rate as $\epsilon = 15\nu\langle(\partial u/\partial x)^2\rangle$. Since,

$$\Delta u = \int_x^{x+r}\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x \tag{21}$$

and

$$\epsilon_r = \frac{15\nu}{r}\int_x^{x+r}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2\mathrm{d}x \tag{22}$$

they realized that Δu and $r\epsilon_r$ are related variables, both depending on $\mathrm{d}u/\mathrm{d}x$. Noting that the velocity scale is expressible as $(l\epsilon)^{1/3}$, we can write

$$\frac{\Delta u}{K(l\epsilon)^{1/3}} \approx \sum_{i=1}^{p} X_i = S_p \tag{23}$$

and

$$\frac{r\epsilon_r}{15Kl\epsilon} \approx \sum_{i=1}^{p} X_i^2 = Y_p^2, \tag{24}$$

where $X_i = [l/(l\epsilon)^{1/3}](\mathrm{d}u/\mathrm{d}x)$ and $p = r/Kl$ where Kl is the number of Kolmogorov scales over which smoothness obtains. Modelling X_i as a fractional Brownian motion, the idea was to seek a probability distribution for the variable $\Delta u/(r\epsilon_r)^{1/3}$ for given values of r and $r\epsilon_r$. This allowed the authors to show that in the limit of large R, the parallel distribution for $\Delta u/(r\epsilon_r)^{1/3}$ was indeed universal. While this was a very significant achievement there was a shortcoming in that the distribution was even in the variable which ruled out the existence of the correlations like $\langle(\Delta u)^3\rangle$. The approach of Stolovitz and Sreenivasan allows us to make direct contact with the term of large deviations.

We turn to eq. (24) and note that ϵ_r plays the role of M_N of eq. (1) and it is the deviation from the expected mean value ϵ that we are interested in. As $r \to \infty$, this deviation variable has a distribution according to the rule of eq. (2). We now make a drastic simplification. The variable $\epsilon_r - \epsilon$ can range from large negative to large positive values. We bring the range from 0 to 1 by defining the variable as

$$Z \equiv \frac{1}{2}\left[1 + \tanh\left(\frac{\epsilon_r - \epsilon}{\Delta}\right)\right], \tag{25}$$

where Δ is a constant having dimensions of ϵ. We now make the drastic assumption that since it is the large deviation ϵ_r by ϵ is a rare event, the distribution of Z can be considered similar to the distribution for the coin toss with a fair coin and accordingly we can hypothesize that

$$P(Z) \propto \mathrm{e}^{-rI(Z)}, \tag{26}$$

where r, in dimensionless units, is the number of random variables in eq. (24). The function $I(Z)$ is taken to be that shown in eq. (3). With the distribution, the mean square fluctuation in the dissipation rate can be written as

$$\langle\epsilon_r^2\rangle - \epsilon^2 = \frac{\Delta^2}{4} \frac{\int_0^2 \mathrm{d}y \left[\ln\left(\frac{y}{2}\right) - \ln\left(1 - \frac{y}{2}\right)\right]^2 \left[\left(\frac{1}{y}\right)^y \left(\frac{1}{2-y}\right)^{2-y}\right]}{\int_0^2 \mathrm{d}y \left[\left(\frac{1}{y}\right)^y \left(\frac{1}{2-y}\right)^{2-y}\right]^{r/2}}. \tag{27}$$

This is a slowly decaying function of r as happens in the phenomenon of intermittency (see discussion following eq. (20)). An accurate evaluation of the integral in eq. (27) and the evaluation of $\langle\epsilon_r^{n/3}\rangle$ in general will be reported elsewhere. Here, we simply point out a very crude evaluation of eq. (27). This is done by pointing out that the integrand in the numerator of eq. (27) has a strong contribution where $y \approx 0$ and when $y \approx 2$. Picking up the contribution to the integral from the range only we get

$$\langle \epsilon_r^2 \rangle - \epsilon^2 \approx \frac{\Delta^2}{2} \left\{ \ln \left(\frac{L}{r} \right) \right\}^2, \tag{28}$$

where L is the large distance cut-off (i.e., $L \gg r$). The similarity with eq. (17) is masked. The mean square fluctuation is found to have a logarithmic dependence there as opposed to the square of the logarithm found here. What is noteworthy is the fact that the structure of intermittency arises from a conjectured large deviation – this hypothesis of the large fluctuations of the dissipation rate being a rare event leads to a mean square deviation in reasonable agreement with the involved calculations.

References

[1] Y Oono, *Prog. Theor. Phys. Supp.* **99**, 165 (1989)
[2] J Kurchan, *J. Stat. Mech.* P07005 (2007), doi.10.1088/1742-5468/2007/07/P07005
[3] D J Evans and D J Searles, *Phys. Rev. Lett.* **96**, 120603 (2006)
[4] E M Shevick *et al*, *Ann. Rev. Phys. Chem.* **39**, (2007)
[5] T Bodineau and B Derrida, *Phys. Rev. Lett.* **92**, 180601 (2004)
[6] G Gallavoti and E G D Cohen, *Phys. Rev. Lett.* **74**, 2694 (1995)
[7] D J Evans and D J Searles, *Phys. Rev.* **E50**, 1645 (1994)
[8] C Jarzynski, *Phys. Rev. Lett.* **78**, 2690 (1997)
[9] G E Crooks, *J. Stat. Phys.* **90**, 1481 (1998)
[10] F Douarche *et al*, *Europhys. Lett.* **70**, 593 (2007)
[11] A N Kolmogorov, *J. Fluid Mech.* **12**, 82 (1962)
L D Landau and E M Lifshitz, *Fluid mechanics, course of theoretical physics* (Pergamon, Oxford, 1962) Vol. 6
[12] C Meneveau and K R Sreenivasan, *Phys. Rev. Lett.* **59**, 1424 (1987)
[13] Z She and E Leveque, *Phys. Rev. Lett.* **72**, 336 (1994)
[14] V'-L'vov and I Procaccia, *Phys. Rev.* **E52**, 3840 (1995); *Phys. Rev.* **E54**, 6268 (1996); *Phys. Rev.* **E55**, 7030 (1997)
[15] A Das and J K Bhattacharjee, *Europhys. Lett.* **26**, 527 (1994)
[16] A Sain, Manu and R Pandit, *Phys. Rev. Lett.* **81**, 4377 (1998)
[17] D Mitra and R Pandit, *Phys. Rev. Lett.* **93**, 024501 (2004)
[18] K R Sreenivasan and R A Antonia, *Ann. Rev. Fluid Mech.* **29**, 435 (1997)
[19] U Frisch, *Turbulence* (Cambridge University Press, 1996)
[20] B Derrida, A J Bray and C Godreche, *J. Phys.* **A27**, L357 (1994)
[21] S N Majumdar and C Sire, *Phys. Rev. Lett.* **77**, 1420 (1996)
[22] S N Majumdar, *Curr. Sci.* **77**, 370 (1996)
[23] J Krug *et al*, *Phys. Rev.* **E56**, 2702 (1997)
[24] M Constantin *et al*, *Phys. Rev.* **E69**, 061608 (2004)
[25] J T Lewis and R Russel, *An introduction to large deviations for teletraffic engineeres*
[26] G Stolovitzky and K R Sreenivasan, *Rev. Mod. Phys.* **66**, 229 (1994)