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Realistic searches on stretched exponential networks

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Abstract. We consider navigation or search schemes on networks which have a degree distribution of the form $P(k) \propto \exp(-k^{\gamma})$. In addition, the linking probability is taken to be dependent on social distances and is governed by a parameter λ. The searches are realistic in the sense that not all search chains can be completed. An estimate of $\mu = \rho/s_{\rm d}$, where ρ is the success rate and $s_{\rm d}$ the dynamic path length, shows that for a network of N nodes, $\mu \propto N^{-\delta}$ in general. Dynamic small world effect, i.e., $\delta \simeq 0$ is shown to exist in a restricted region of the λ–γ plane.

Keywords. Small world effect; dynamic paths; social distances.

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The subject of networks has emerged as a multidisciplinary field in which there has been tremendous activity in recent times [1,2]. The interest in networks has grown due to the fact that although networks can be of diverse nature, there are some striking universal properties in their underlying structure. The most important property that appeared to be commonly occurring in networks is the small world property. This means that if any two nodes in the network is separated by an average number of s steps, then $s \propto \ln(N)$, where N is the total number of nodes in the network. In some networks, even slower variation (i.e., sub-logarithmic scaling) has been observed [3].

The first indication that networks have small world behaviour emerged from an experimental study by Milgram *et al* [4], in which it was shown that any two persons (in the USA) can be connected by an average number of six steps. Following the tremendous interest in the study of networks, new experiments have been done to verify this property in real social networks [5,6]. Some studies which involve simulations on real networks [7–9] have been made also. In these studies, one is interested in the length of the dynamic paths, i.e., the number of steps actually taken to transmit a message or signal to another node. In the real experiments, this is done by fixing a target node and randomly selecting source nodes. The source nodes are supplied with some information about the target. The source nodes have to use a strategy to make the signal reach the target through connected nodes. The strategy is based on some partial (usually local) knowledge about the network. This procedure is called 'searching' and when done on a network is also termed as 'navigation on a network'. The question of navigation on small world networks has been addressed theoretically as well in many model networks [10–21].

It must be noted that in navigation or searching on a small world network the dynamic paths $s_{\rm d}$ may not necessarily scale as $\ln(N)$. This is because searching is done using local information only while the average shortest distances are calculated using the global knowledge of the network. This was explicitly shown by Kleinberg [10] in a theoretical study in which nodes were placed on a two-dimensional Euclidean space. Each node here has connections to its nearest neighbours as well as to neighbours at a distance l with probability $P(l) \propto l^{-\alpha}$. Although the network is globally a small world for a range of values of α, navigation on such networks using greedy algorithm showed a small world behaviour (i.e., shortest path scaling as $\ln(N)$) only at $\alpha = 2$.

Conventionally a network is said to be searchable if there exist short dynamic paths scaling as $\ln(N)$. However, search chains have been shown to terminate unsuccessfully in many real experiments. In theoretical studies, this has been considered recently and it has been shown that with the possibility of failures, the scaling of the path lengths alone is not always a good measure of searchability [19]. The ratio of the success rate to the path lengths, on the other hand, gives a reliable measure. Based on this measure, the searchability on scale-free networks and Euclidean networks has been analysed recently [19,20]. The searchability depends on both the network structure and to a large extent on the searching strategy.

In the present work, we have considered a network with a degree distribution which is not scale-free but is given by $P(k) \propto \exp(-k^{\gamma})$, i.e., a stretched exponential behaviour. Here, we have a parameter γ associated with the degree distribution; a large γ value indicates a very fast decay of $P(k)$ such that the probability of having high degree nodes is very small. On the other hand, if γ is small, such a probability may not be negligible even if the degree distribution is not scale-free. Moreover, we consider a characteristic feature attached to each node. The linking probability is taken to be dependent on the social distance or the difference of this characteristic feature between nodes. Thus, this is an attempt to construct a simple model of social network. A similar study was made in [19] where a scale-free degree distribution was used. However, in most social networks, one has a degree distribution with a faster than algebraic decay and therefore we have considered a stretched exponential degree distribution here.

The network is generated by a method described in detail in [19]. In brief, we first assign the degrees to the nodes according to the degree distribution. The characteristic feature called the similarity factor is measured by a variable ξ varying from 0 to 1 and is assigned to each node randomly. The edges are then introduced between pairs of nodes i and j with the probability

$$\mathcal{P}_{i,j} \propto |\xi_i - \xi_j|^{-\lambda}, \tag{1}$$

with $\lambda > 0$.

Thus, we will have a network in which similar nodes will try to link up for $\lambda > 0$. A large value of λ indicates that nodes which are very similar tend to link up while for $\lambda = 0$, there are no correlations among nodes. The minimum and maximum allowed degrees are two and $N^{1/2}$ respectively.

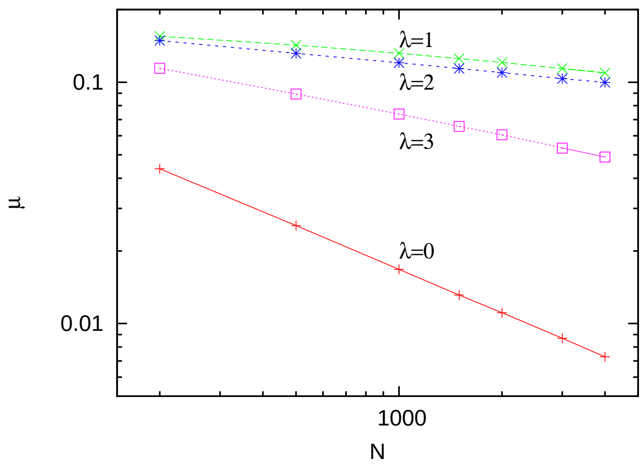

Figure 1. The typical variations of the ratio μ of the success rate to the dynamic path lengths as a function of N are shown for different values of λ for $\gamma = 0.4$. The slopes give the estimate of δ.

We use an algorithm in which each node knows the similarity factor of its immediate neighbours as well as that of the target node. Nodes send messages to a neighbour most similar to the target node. Each node can receive the message only once. In case there is no node to which the message can be forwarded, the search will terminate. Conducting the searches between arbitrary source–target pairs, we estimate the success rates ρ and the average dynamic paths $s_{\rm d}$. The ratio $\mu = \rho/s_{\rm d}$ is then studied as a function of the network size N.

We find that as in [19,20], μ has a power-law decay with N with an exponent δ for all γ and λ in general except for very large values of λ where there seems to be a correction to the power-law scaling. δ lies between zero and one, as found in the previous studies. A smaller value of δ indicates a better searchability and in particular $\delta \simeq 0$ would correspond to a dynamic small world (DSW) effect [19]. For each γ, we estimate δ as a function of λ. We find that for small γ, there is indeed a range of values of λ where δ is very small, close to zero. On the other hand, for $\gamma > \gamma_c$, where γ_c is close to 0.4, there is no such region. In the present study, if $\delta < 0.1$ (which means the decay could as well be logarithmic), we assume that there is a dynamic small world effect.

Figure 1 shows typical variations of μ vs. N for $\gamma = 0.4$ for various values of λ. The slope of these curves gives the exponent δ. In figure 2, δ against λ values are shown for various γ where we observe that for low values of γ, there is a finite region where δ is less than 0.1. This region shrinks as γ is increased. However, even when there is no dynamic small world effect, we find that δ reaches a minimum at $\lambda = \lambda_{\rm min}$ where $\lambda_{\rm min} \simeq 1.25$ independent of γ. Hence, for a stretched exponential network in general, where the linking probability is dependent on the social distances parametrically as in (1), the network is most searchable for a fixed value of the parameter.

We have schematically shown the DSW region in the λ–γ plane in figure 3. We would like to mention here that for the social network considered earlier, with a

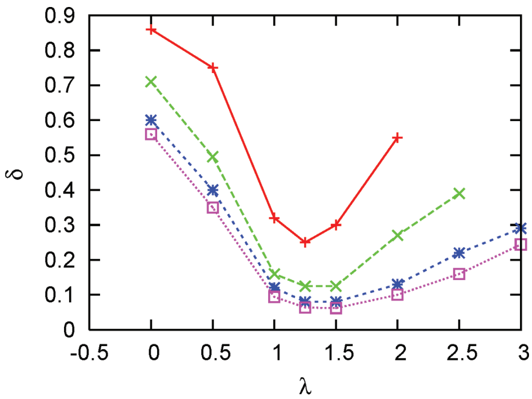

Figure 2. The values of δ against λ show that for low values of γ, a dynamic small world effect can be achieved for a finite interval of λ values. The curves are shown for $\gamma = 0.6, \gamma = 0.5, \gamma = 0.4$ and $\gamma = 0.3$, from top to bottom.

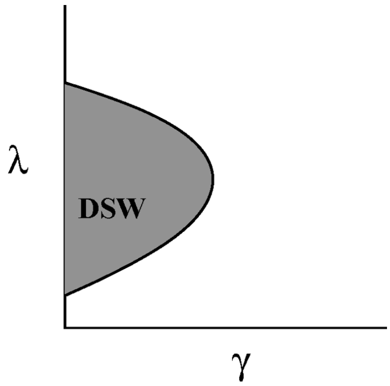

Figure 3. Schematic picture of the λ–γ plane showing the region with the dynamic small world effect.

power-law degree distribution ($P(k) \propto k^{-2}$), DSW had been obtained for a finite range of value of λ as well [19].

Thus we conclude that for realistic searches on social networks, where the degree distribution is given by a stretched exponential function, one can achieve dynamic small world effect as long as the degree distribution does not decay very fast and the network is neither too correlated nor too random as far as the similarity of nodes is concerned. In general, we find that while the quality of the searchability is determined by λ, γ makes it quantitatively different, e.g., at smaller values of γ where there is a larger number of highly connected nodes, the searchability increases.

We also find that in the DSW region, the success rate is almost constant and is a function of N, and the dynamic path lengths increase very slowly indicating a logarithmic increase. The absolute value of path lengths is also 'small', e.g., for a network of size 4000, the dynamic path length is about 5 in the DSW region. This could explain the observed results of searching in real networks. Of course, in a real social search, a much more complicated dynamics is involved, with nodes having more than one characteristic feature and each node using its own searching strategy.

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