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Patterns in the Kardar–Parisi–Zhang equation

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Abstract. We review a recent asymptotic weak noise approach to the Kardar–Parisi–Zhang equation for the kinetic growth of an interface in higher dimensions. The weak noise approach provides a many-body picture of a growing interface in terms of a network of localized growth modes. Scaling in 1d is associated with a gapless domain wall mode. The method also provides an independent argument for the existence of an upper critical dimension.

Keywords. Scaling; weak noise; growth modes; dynamical network; upper critical dimension; nonlinear Schrödinger equation; domain walls; solitons; dispersion; diffusive modes.

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1. Introduction

Nonequilibrium phenomena are on the agenda in modern statistical physics, soft condensed matter and biophysics. Open systems driven far from equilibrium are ubiquitous. A classical case is the driven Navier–Stokes turbulence; other cases are driven lattice gases, growing interfaces, growing fractals, etc. Unlike equilibrium physics where the Boltzmann–Gibbs scheme applies, the ensemble is not known in nonequilibrium. Here the problem is defined in terms of a numerical algorithm, a master equation, or a Langevin equation.

An interesting class of nonequilibrium systems exhibit scale invariance. One example is the diffusion-limited aggregation (DLA) driven by the accretion of random walkers yielding a growing scale invariant fractal with dimension $D \approx 1.7$ in 2d. Another case is a growing interface driven by random deposition or propagating in a random environment. Here the width of the growing front $w(L,t)$ conforms to the dynamical scaling hypothesis $w(L,t) = L^{\zeta} f(t/L^{z})$, where L is the size of the system and ζ and z are the scaling exponents; ζ characterizing the roughness and z describing the dynamical cross-over to the stationary profile [1].

In the present paper we focus on the Kardar–Parisi–Zhang (KPZ) equation which describes an intrinsic nonequilibrium problem and plays the same role as the Ginzburg–Landau functional in equilibrium physics. The KPZ equation was introduced in 1986 in a seminal paper by Kardar, Parisi and Zhang [2]. It has the form

$$\frac{\partial h}{\partial t} = \nu\nabla^2 h + \frac{\lambda}{2}\vec{\nabla}h\vec{\nabla}h - F + \eta, \quad \langle\eta\eta\rangle(\vec{r},t) = \Delta\delta(\vec{r})\delta(t), \tag{1}$$

and purports to describe nonequilibrium aspects of a growing interface (see refs [1,3]). Here $h(\vec{r},t)$ is the height of an interface at position $\vec{r}$ and time t, the linear diffusion term $\nu\nabla^2 h$, characterized by the diffusion coefficient ν, represents surface tension, the nonlinear growth term $(\lambda/2)\vec{\nabla}h\vec{\nabla}h$, characterized by λ, is required to account for the lateral growth, F is an imposed constant drift, and the random aspects, i.e., the random deposition of material or the random character of the medium, are encoded in the noise $\eta(\vec{r},t)$. The noise is assumed to be locally correlated in space, and time, its strength characterized by Δ.

Despite its simple form the KPZ equation is difficult to analyse and many aspects remain poorly understood [1–3]. Apart from its intrinsic interest the KPZ equation is also related to fundamental issues in turbulence and disorder. Introducing the local slope field $\vec{u} = \vec{\nabla}h$ the KPZ equation takes the form of a Burgers equation driven by conserved noise,

$$\frac{\partial\vec{u}}{\partial t} - \lambda(\vec{u}\cdot\vec{\nabla})\vec{u} = \nu\nabla^2\vec{u} + \vec{\nabla}\eta. \tag{2}$$

In the noiseless case setting $\lambda = -1$ and regarding $\vec{u}$ as a velocity field, the nonlinear term appears as the convective term in the Navier–Stokes equation and eq. (2) has been used to model aspects of turbulence. In the 1d case the relaxation of the velocity field takes place subject to a transient pattern formation composed of domain walls and ramps with superimposed diffusive modes. In the driven case, eq. (2) was studied earlier in the context of long time tails in hydrodynamics [4]. On the other hand, applying the nonlinear Cole–Hopf transformation the KPZ equation maps to the Cole–Hopf equation (CH)

$$\frac{\partial w}{\partial t} = \nu\nabla^2 w - \frac{\lambda}{2\nu}wF + \frac{\lambda}{2\nu}w\eta, \quad h = \frac{2\nu}{\lambda}\ln w, \tag{3}$$

a linear diffusion equation driven by multiplicative noise. The CH equation has a formal path integral solution [2,3] which can be interpreted as an equilibrium system of directed polymers (DP) with line tension $1/4\nu$ in a quenched random potential η; a model system in the theory of disorder which has been studied using replica techniques [3,5].

The KPZ equation lives at a critical point and conforms to the dynamical scaling hypothesis. For the height correlations we have

$$\langle hh\rangle(\vec{r},t) = r^{2\zeta}F(t/r^z). \tag{4}$$

Here ζ, z, and F are the roughness exponent, dynamic exponent, and scaling function, respectively. To extract scaling properties the initial analysis of the KPZ equation was based on the dynamic renormalization group (DRG) method, previously applied to dynamical critical phenomena and noise-driven hydrodynamics [4]. An expansion in powers of λ in combination with a momentum shell integration

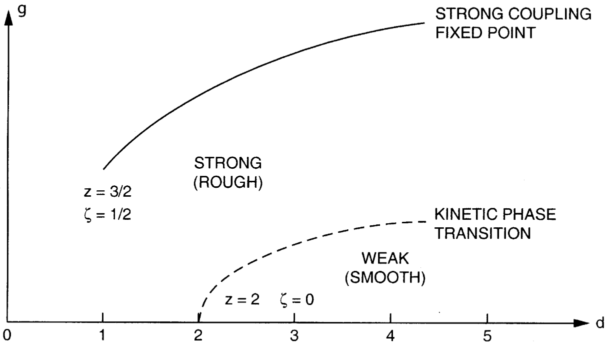

Figure 1. DRG phase diagram for the KPZ equation to leading loop order in $d-2$. In $d=1$ the DRG flow is towards the strong coupling fixed point $\zeta=1/2, z=3/2$. Above the lower critical dimension $d=2$ there is an unstable kinetic transition line, separating a rough phase from a smooth phase.

yield to leading order in $d-2$ the DRG equation $\mathrm{d}g/\mathrm{d}l=\beta(g)$, with beta-function $\beta(g)=(2-d)g+\text{const.}g^4$. Here $g=\Delta\lambda^2/\nu^3$ is the effective coupling strength and l is the logarithmic scale parameter [1,2]. The emerging DRG phase diagram is depicted in figure 1.

Before discussing the phase diagram we note two further properties of the KPZ equation. First, subject to a Galilean transformation the equation is invariant provided we add a constant slope to h and adjust the drift F, i.e.,

$$\vec{r}\to\vec{r}-\lambda\vec{u}^0t,\quad h\to h+\vec{u}^0\cdot\vec{r},\quad F\to F+(\lambda/2)\vec{u}^0\cdot\vec{u}^0. \tag{5}$$

Note that the slope field $\vec{u}$ and the diffusive field w transform like $\vec{u}\to\vec{u}+\vec{u}^0$ and $w\to w\exp[(\lambda/2\nu)\vec{u}^0\cdot\vec{r}]$, respectively. The Galilean invariance implies the scaling law

$$\zeta+z=2, \tag{6}$$

relating ζ and z [2]; the Galilean invariance is a fundamental dynamical symmetry specific to the KPZ equation, delimiting the universality class. Second, a fluctuation–dissipation theorem is operational in 1d since the stationary Fokker–Planck equation admits the explicit solution [6]

$$P_0(h)\propto\exp\left[-(\nu/\Delta)\int\mathrm{d}x(\nabla h)^2\right]. \tag{7}$$

This distribution shows that the slope $u=\nabla h$ fluctuations are uncorrelated and that the height field $h=\int^x u\,\mathrm{d}x'$ performs a random walk. Consequently, from eq. (4) we infer the roughness exponent $\zeta=1/2$ and from the scaling law (6) the dynamic exponent $z=3/2$. In other words, the scaling exponents associated with the strong coupling fixed point are exactly known in 1d (see figure 1); moreover, results for the scaling function can be obtained by loop expansions [6].

The lower critical dimension is $d=2$. Below $d=2$ the DRG flow is towards the strong coupling fixed point. Above $d=2$ the DRG equation yields an unstable kinetic phase transition line as indicated in figure 1. For λ below a critical coupling strength λ_c the DRG flow is towards $\lambda = 0$, corresponding to the linear Edwards–Wilkinson (EW) equation, the KPZ equation for $\lambda = 0$, yielding $z = 2$ and $\zeta = (2-d)/2$, note that the scaling law is not operational for $\lambda = 0$; for $\lambda > \lambda_c$ the DRG flow is towards a nonperturbative strong coupling fixed point.

A DRG analysis to all orders in $d - 2$ yields $z = 2$ and $\zeta = 0$ on the transition line and a singularity at the upper critical dimension $d_{\text{upper}} = 4$. Mode coupling techniques also give $d_{\text{upper}} = 4$, whereas a directed polymer analysis yields $d_{\text{upper}} \approx 2.5$ [6].

In order to disentangle the properties of the KPZ equation, both regarding scaling and otherwise, there is a need for alternative methods. In the present paper we summarize a nonperturbative weak noise approach which we have pursued in recent years [7]. The working hypothesis here is to focus on the noise strength Δ as the determining parameter rather that the nonlinearity λ and ensuing DRG analysis or mapping to DP. For $\Delta = 0$ the interface relaxes subject to a transient pattern formation; for $\Delta \approx 0$ the interfaces initially decays but is eventually driven into a stationary fluctuating state; the cross-over time diverging for $\Delta \to 0$. In the next section we summarize the scheme that will allow us to access the weak noise regime in a nonperturbative fashion.

2. Weak noise scheme

The weak noise scheme takes as its starting point a generic Langevin equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -F(x) + \eta, \quad \langle\eta\eta\rangle(t) = \Delta\delta(t), \tag{8}$$

determining the problem we wish to analyse. Here x is a multi-dimensional random variable, $F(x)$ a general nonlinear drift, and η is the additive white noise correlated with strength Δ. In order to implement a weak noise approximation we consider the equivalent Fokker–Planck equation for the distribution $P(x,t)$

$$\Delta\frac{\partial P}{\partial t} = \frac{1}{2}\Delta^2\frac{\partial^2 P}{\partial x^2} + \Delta\frac{\partial}{\partial x}(FP). \tag{9}$$

Interpreting $\Delta\partial/\partial x$ as a momentum operator, P as an effective wave function, and Δ as an effective Planck constant, eq. (9) has the form of an imaginary time Schrödinger equation and it is natural to introduce the well-known WKB or eikonal approximation

$$P \propto \exp[-S/\Delta]. \tag{10}$$

To leading order in Δ the action then obeys a principle of least action $\delta S = 0$ as expressed by the Hamilton–Jacobi equation $\partial S/\partial t + H(x,p) = 0$ with canonical momentum $p = \partial S/\partial x$. The Hamiltonian (energy) takes the form

$$H = \frac{1}{2}p^2 - pF(x) = \frac{1}{2}p[p - 2F(x)], \tag{11}$$

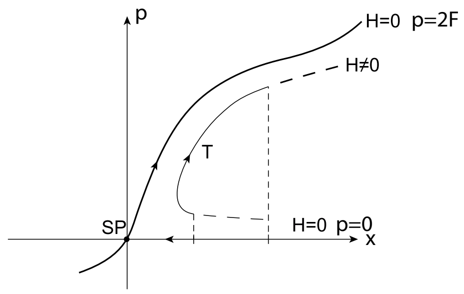

Figure 2. Canonical phase space plot. The finite-time orbit from the initial x_1 to the final x in transition time T lies on the energy manifold $H \neq 0$. In the long time limit the orbit migrates to the zero-energy manifold composed of a transient submanifold for $p = 0$, corresponding to the noiseless case, and a stationary manifold for $p = 2F$, corresponding to the noisy case. The submanifolds intersect in the saddle point (SP), determining the Markovian behaviour.

yielding the coupled equations of motion

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -F + p, \quad \frac{\mathrm{d}p}{\mathrm{d}t} = p\frac{\mathrm{d}F}{\mathrm{d}x}. \tag{12}$$

The action associated with an orbit from x_1 to x in time T is

$$S(x_1 \to x, T) = \int_{x_1,0}^{x,T} \mathrm{d}t \left[p\frac{\mathrm{d}x}{\mathrm{d}t} - H\right] = \frac{1}{2}\int_{x_1,0}^{x,T} \mathrm{d}t p(t)^2. \tag{13}$$

The weak noise recipe is clear. We solve the equations of motion (12) and identify an orbit from x_1 to x in time T with p as an 'adjusted' variable. The orbits lie on constant H manifolds. Evaluating the action S for a specific orbit the WKB approximation (10) yields the transition probability from x_1 to x in time T. Assuming $F(x) \propto x$ for small x the phase space has the generic structure depicted in figure 2.

The present variationally based weak noise scheme dates back to Onsager. In more recent formulations it corresponds to the saddle point contribution (optimal path) in the functional Martin–Siggia–Rose scheme (see refs [7,8]).

3. Growth modes

The weak noise scheme applies directly to the KPZ equation in the Cole–Hopf formulation (3). Extending the scheme in order to incorporate multiplicative noise (see refs [7]), the WKB scheme yields the equations of motion, action, and distribution

$$\frac{\partial w}{\partial t} = \nu[\nabla^2 w - k^2 w] + k_0^2 w^2 p, \quad \frac{\partial p}{\partial t} = -\nu[\nabla^2 w - k^2 w] - k_0^2 p w^2, \tag{14}$$

$$S(w,T) = (k_0^2/2)\int^{w,T} \mathrm{d}\vec{r}\mathrm{d}t(wp)^2, \quad P(w,T) \propto \exp[-S(w,T)/\Delta], \tag{15}$$

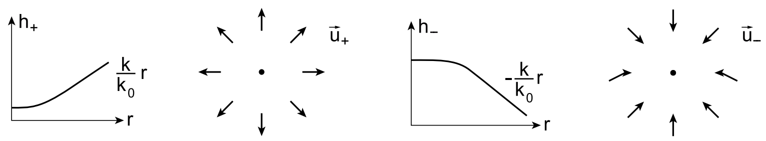

Figure 3. The static growth modes in the height and slope fields in 2d.

with parameters $k^2 = \lambda F/2\nu$ and $k_0 = \lambda/2\nu$. On the transient and stationary manifolds $p = 0$ and $p \propto w$ the equations of motion reduce in the static case to the diffusion and nonlinear Schrödinger equations

$$\nabla^2 w = k^2 w, \tag{16}$$

$$\nabla^2 w = k^2 w - k_0^2 w^3, \tag{17}$$

admitting localized spherically symmetric solutions $w_+(r) \propto \exp(kr)$ and $w_-(r) \propto \exp(-kr)$ for large r, respectively. In terms of the height field $h = k_0^{-1} \ln w$ and slope field $\vec{u} = \vec{\nabla} h$ we obtain the fundamental localized static growth modes for large r

$$h_\pm(r) \propto \pm \frac{k}{k_0} r \quad \text{and} \quad \vec{u}_\pm(r) \propto \pm \frac{k}{k_0} \frac{\vec{r}}{r}. \tag{18}$$

The growth modes are characterized by the amplitude or charge $k \propto F^{1/2}$, determined by the imposed drift F in the KPZ equation. The growth mode with positive charge $h \propto \exp(kr/k_0)$, $\vec{u} \propto \vec{r}/r$ lives on the transient 'noiseless' $p = 0$ manifold and corresponds to a cone or dip in h and a constant positive monopole in $\vec{u}$. The mode carries zero action, zero energy $H = -S/T = 0$, and zero momentum $\vec{\Pi} = \int \mathrm{d}\vec{r} w \vec{\nabla} p = 0$; the distribution $P \propto \exp(-S/\Delta)$ associated with the mode is of $O(1)$. The growth mode with negative charge $h \propto \exp(-kr/k_0)$, $\vec{u} \propto -\vec{r}/r$ is associated with the stationary 'noisy' manifold $p \propto w$ and corresponds to an inverted cone or tip in h and a constant negative monopole in $\vec{u}$. This mode carries a finite action

$$S \propto T(\nu/k_0)^2 k^{4-d}, \tag{19}$$

yielding the distribution $P \propto \exp[-S/\Delta]$.

In the 1d case the growth modes correspond to right-hand and left-hand domain walls, $u \propto \pm \tanh(kx)$. The right-hand domain wall is the viscosity broadened shock wave in the deterministic Burgers equation; the left-hand domain wall is 'noise induced', in the present scheme characterized by a finite p (see refs [7]). In figure 3 we have depicted the static growth modes in the height and slope fields.

Applying the Galilean transformation (5) we can boost the static modes and obtain the propagating growth modes

$$h_\pm(r,t) = \pm \frac{k}{k_0} |\vec{r} + \lambda \vec{u}_0 t| + \vec{u}_0 \cdot \vec{r}, \quad u_\pm(r,t) = \pm \frac{k}{k_0} \frac{\vec{r} + \lambda \vec{u}_0 t}{|\vec{r} + \lambda \vec{u}_0 t|} + \vec{u}_0. \tag{20}$$

The moving localized growth modes are the fundamental elementary excitations incorporating the nonlinear aspects of the KPZ equation. In a quantum field theory context the growth modes correspond to instantons or solitons.

4. Stochastic pattern formation

By means of the propagating localized growth modes (20) we construct a global solution of the field equations (14). The Galilean invariance (5) determines the matching of the modes and we obtain the dilute network solution

$$h(\vec{r},t) = k_0^{-1}\sum_i k_i|\vec{r}-\vec{r}_i(t)|, \quad \vec{u}(\vec{r},t) = k_0^{-1}\sum_i k_i\frac{\vec{r}-\vec{r}_i(t)}{|\vec{r}-\vec{r}_i(t)|}, \tag{21}$$

$$\vec{v}_i(t) = -2\nu\sum_{l\neq i} k_l\frac{\vec{r}_i(t)-\vec{r}_l(t)}{|\vec{r}_i(t)-\vec{r}_l(t)|}, \quad \vec{r}_i(t) = \int_0^t \vec{v}_i(t')\mathrm{d}t' + \vec{r}_i(0). \tag{22}$$

Here k_i is the assignment of charges and $\vec{r}_i(0)$ the initial positions. This network construction corresponds to a multi-instanton solution in quantum field theory. As time evolves the modes propagate and the velocities adjust to constant values given by the self-consistent equation

$$\vec{v}_i = -2\nu\sum_{l\neq i} k_l\frac{\vec{v}_i-\vec{v}_l}{|\vec{v}_i-\vec{v}_l|}. \tag{23}$$

Superimposed on the propagating network of growth modes is a spectrum of linear extended diffusive modes with dispersion $\omega = \nu k^2$, following from a linear analysis of the field equations (14).

For large $\vec{r}$ the slope field $\vec{u} \sim (\vec{r})/r\sum_i k_i$. In order to ensure the boundary condition of a flat interface at large distances we impose the neutrality condition $\sum_i k_i = 0$; note that this boundary condition still allows for a local offset of h and the propagation of facets or steps. During time evolution the dynamical network propagates across the system. Imposing periodic or bouncing boundary conditions increments are added to h and the interface grows (see also ref. [7]).

From an analysis of eqs (21) and (22) it follows that the growth modes with positive charge exerts an attraction of the other modes, whereas the negatively charged growth modes repel the other modes. Although our analysis so far only applies to a dilute network, it follows tentatively from a numerical simulation of eqs (21) and (22) that the modes form dipoles and that the long time stable network configuration is composed of a gas of propagating dipoles, i.e. the pairing of monopole growth modes with opposite charges.

In terms of the slope field a dipole mode with charges k and $-k$ has the form $u_{\mathrm{dip}} \sim (k/k_0)[(\vec{r}-\vec{v}t-\vec{r}_1)/|\vec{r}-\vec{v}t-\vec{r}_1|-(\vec{r}-\vec{v}t-\vec{r}_2)/|\vec{r}-\vec{v}t-\vec{r}_2|]$, $\vec{v} = \lambda(k/k_0)(\vec{r}_2-\vec{r}_1)/|\vec{r}_2-\vec{r}_1|$. The mode propagates with velocity v and carries energy, momentum and action, $E_{\mathrm{dip}} \propto (\nu^2/k_0^2)k^{4-d}$, $\Pi_{\mathrm{dip}} \propto (\nu/k_0^2)k^{3-d}$, and $S_{\mathrm{dip}} \propto (\nu^2/k_0^2)k^{4-d}T$. In terms of the height field we have $h_{\mathrm{dip}} \sim (k/k_0)[|\vec{r}-\vec{v}t-\vec{r}_1|-|\vec{r}-\vec{v}t-\vec{r}_2|]$; the mode is depicted in figure 4. Asymptotically the height field is flat corresponding to a

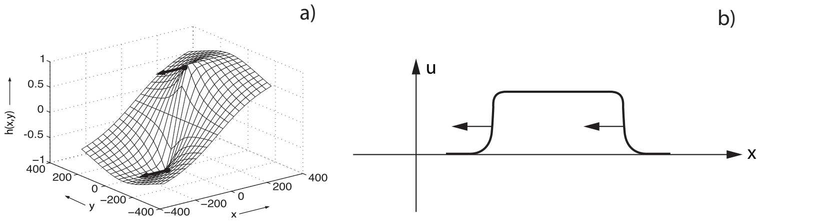

Figure 4. We show a dipole configuration composed of two paired monopoles of opposite charges. In (**a**) we depict the dipole mode for the height field in 2d; in (**b**) we show the two-domain wall configuration for the slope field in 1d.

vanishing slope field. The dipole mode corresponds to a propagating local defect or deformation. In 1d the dipole mode corresponds in the slope field to a matched right-hand and left-hand domain wall propagating across the system (see figure 4).

5. Scaling and upper critical dimension

The scaling issues for the KPZ equation remain unsettled except in 1d where the fluctuation–dissipation theorem (7) and the scaling law (6) give access to the strong coupling fixed point with scaling exponents $\zeta = 1/2$ and $= 3/2$ (see figure 1). In higher d there has been many attempts to access the strong coupling fixed point both on the basis of DRG, mode coupling, directed polymers (DP), and numerically; however, the strong coupling features remain elusive [6].

The present weak noise method is not a scaling approach but rather a many-body description of a growing interface. Nevertheless, the method allows a discussion of some of the scaling features. Since the scheme is consistently Galilean invariant, the scaling law (6) is automatically obeyed. The roughness exponent ζ is associated with the static correlations $\langle hh\rangle(\vec{r}) = \int\prod \mathrm{d}h h(\vec{r})h(0)P_0(h) \propto r^{\zeta}$ and requires the static distribution $P_0(h)$, $P_0(h) \propto \exp[-S_0(h)/\Delta]$, $S_0(h) = \lim_{T\to\infty} S(h_1 \to h, T)$. The stationary distribution is associated with the zero-energy stationary manifold which in general is difficult to identify for a system with many degrees of freedom.

In 1d in terms of the slope field $u = \nabla h$ and associated noise field p the Hamiltonian takes the form $H = \int \mathrm{d}x\, p[\nu\nabla^2 u + \lambda u\nabla u - (1/2)\nabla^2 p]$ and it follows that the stationary zero-energy manifold is given by $p = 2\nu u$, yielding a Hamiltonian density as a total differential. Correspondingly, the stationary action $S_0 \propto \int \mathrm{d}x\mathrm{d}tp\partial u/\partial t = \nu\int \mathrm{d}x(\nabla h)^2$ and the distribution $P_0(u) \propto \exp[-S_0/\Delta]$ in accordance with (7), implying $\zeta = 1/2$. Since Galilean invariance is built in the exponent $z = 3/2$ follows automatically. However, we can also infer $z = 3/2$ from an independent argument based on the dispersion law for the low-lying gapless excitations. The elementary excitations are the right-hand and left-hand domain walls. A composite quasi-particle or dipole mode satisfying the boundary condition of vanishing slope can be constructed by paring two domain walls. The dipole mode propagates with energy $E \propto u^3$ and momentum $\Pi \propto u^2$, where u is the dipole amplitude. Using the analogy between the stochastic formulation and quantum mechanics, i.e., the canonical quantization of the weak noise scheme with Δ as an effective Planck constant, and using the spectral representation $\langle uu\rangle(x,t) = \int \mathrm{d}\Pi F(\Pi)\exp(-Et/\Delta + i\Pi x/\Delta)$, $F(\Pi)$ is a form factor, we note that the bottom of the gapless dipole energy spectrum $E \propto \Pi^{3/2}$ implies $\langle uu\rangle(x,t) \propto G(t/x^z)$, where $z = 3/2$.

In higher d we have not been able to identify the stationary zero-energy manifold and the exponent ζ. Also, in order to determine z, for example from the form of the time-dependent correlations $\langle hh\rangle(\vec{r},t)$, we need both the transition probabilities and the stationary distribution, i.e., by definition $\langle hh\rangle(\vec{r},t) = \int \prod \mathrm{d}h_1 \mathrm{d}h_2 h_1(\vec{r})h_2(0)P(h_1 \to h_2, t)P_0(h_1)$. In the weak noise scheme $P(h_1 \to h_2, t) \propto \exp[-S(h_1 \to h_2, t)/\Delta]$ which requires a detailed analysis of the dynamical network and the associated action.

In the dipole sector we can, however, present some preliminary scaling results. Since the propagating dipole mode according to (19) carries action $S \propto k^{4-d}T$ and propagates with velocity $v \propto k$ the mode moves the distance $L = vT$ in time T. Expressing S in the form $S \propto L^{4-d}/T^{3-d}$ we infer the single dipole distribution $P \propto \exp[-\mathrm{const}.L^{4-d}/T^{3-d}]$ and, correspondingly, the dipole mean square displacement $\langle \delta L^2\rangle \propto T^{2H}$ with Hurst exponent $H = (3-d)/(4-d)$. Note that the dynamic exponent $z = H^{-1} = (4-d)/(3-d)$. In the stochastic representation the dipole mode thus performs anomalous diffusion. In $d = 0$ we have $H = 3/4$, $z = 4/3$, in agreement with a formal DP result [3,6]. In $d = 1$ we obtain $H = 2/3$ and the exact result $z = 3/2$, i.e., the dipole modes exhaust the spectrum. In $d = 2$ we have $H = 1/2$ and $z = 2$, i.e., ordinary diffusion. In $d = 3$ we obtain $H = 0$ and $z = \infty$, the dipole mean square displacement falls off logarithmically $\langle \delta L^2\rangle \propto \ln T$, however, z diverges at variance with accepted DRG and DP results. In $d = 4$ we have $H = -\infty$ and the mean square displacement is arrested. Below $d = 2$ (the lower critical dimension) the dipole modes superdiffuse, above $d = 2$ we have subdiffusion. These scaling results only refer to the dipole sector.

The last issue is the much discussed upper critical dimension for the KPZ equation [6]. The weak noise approach allows a nonscaling argument for the existence of a critical dimension. Above $d = 4$ the negative growth mode as a bound state solution to the nonlinear Schrödinger equation (17) ceases to exist. This implies that the dynamical network representation of a growing interface ceases to be valid. This result follows from a numerical analysis of the nonlinear Schrödinger equation (17) but can also be inferred by an algebraic proof based on Derrick's theorem. First, introducing $K = (1/2)\int \mathrm{d}^d x(\nabla w)^2$, $N = \int \mathrm{d}^d x w^2$, and $I = \int \mathrm{d}^d x w^4$ we infer from (17) the identity $-2K = k^2 N - k_0^2 I$. Second, deducing (17) from a variational principle $\delta F/\delta w = 0$ with $F = K + (1/2)k^2 N - (k_0^2/4)I$ and performing a constrained minimization $w(\vec{r}) \to w(\mu\vec{r})$, $K \to \mu^{d-2}K$, $N \to \mu^d N$, $I \to \mu^d I$ and $\delta F/\delta\mu|_{\mu=1} = 0$ we have the second identity $(d-2)K + (k^2/2)\mathrm{d}N - (k_0^2/4)\mathrm{d}I = 0$. Finally, requiring $N, I > 0$ the identities imply $d < 4$.

6. Summary and conclusion

In this paper we have presented a short review of a recently developed asymptotic weak noise approach to the Kardar–Parisi–Zhang equation. The scheme provides a many-body description of a growing interface in terms of a dynamical network of growth modes. The growth modes are the elementary building blocks and their propagation accounts for the kinetic growth. Kinetic transitions are determined by an associated dynamical action, replacing the customary free energy landscape. Superimposed on the network is a gas of diffusive modes. In 1d the dispersion laws delimit the universality classes: In the KPZ case the gapless domain wall modes yield $z = 3/2$, the diffusive modes being subdominant; in the EW case the domain walls are absent and the gapless diffusive modes yield $z = 2$. In higher d the scaling results based on the weak noise method are still subject to scrutiny. Finally, we mention that the weak noise method also has been applied to the noise-driven Ginzburg–Landau equation, a finite-time-singularity model, and DNA bubble dynamics [9].

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