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Synchronization of networks

R E AMRITKAR
Physical Research Laboratory, Navrangapura, Ahmedabad 380 009, India
E-mail: amritkar@prl.ernet.in

Abstract. We study the synchronization of coupled dynamical systems on networks. The dynamics is governed by a local nonlinear oscillator for each node of the network and interactions connecting different nodes via the links of the network. We consider existence and stability conditions for both single- and multi-cluster synchronization. For networks with time-varying topology we compare the synchronization properties of these networks with the corresponding time-average network. We find that if the different coupling matrices corresponding to the time-varying networks commute with each other then the stability of the synchronized state for both the time-varying and the time-average topologies are approximately the same. On the other hand, for non-commuting coupling matrices the stability of the synchronized state for the time-varying topology is in general better than the time-average topology.

Keywords. Synchronization; nonlinear dynamical systems; networks.

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1. Introduction

Several complex systems have underlying structures that are described by networks or graphs [1,2]. Recent interest in networks is due to the discovery that several naturally occurring networks come under some universal classes and they can be simulated with simple mathematical models, viz. small-world networks [3], scale-free networks [4] etc. These models are based on simple physical considerations and have attracted a lot of attention from physics community as they give simple algorithms to generate graphs which resemble several actual networks found in many diverse systems [2].

Several networks in the real world consist of dynamical elements interacting with each other and the interactions define the links of the network. Several of these networks have a large number of degrees of freedom and it is important to understand their dynamical behaviour. Here, we study the synchronization and cluster formation in networks consisting of interacting dynamical elements. A general model of coupled dynamical systems on networks will consist of the following three elements: (a) The evolution of uncoupled elements, (b) the nature of couplings and (c) the topology of the network.

Synchronization of coupled dynamical systems [5–7] is manifested by the appearance of some relation between the functionals of different dynamical variables and is known as generalized synchronization. The exact synchronization corresponds to the situation where the dynamical variables for different nodes have identical values. The phase synchronization corresponds to the situation where the dynamical variables for different nodes have some definite relation between their phases [8]. In this paper we will consider only exact synchronization. However, several results are also valid for other forms of synchronization.

Depending on the network topology and the coupling, we can get both single-cluster or multi-cluster synchronization. These conditions are discussed in the next section. Several natural networks are not static in time and the structure of nodes and links changes with time. We will discuss some interesting aspects of the synchronization of such time-varying networks.

2. Coupled dynamical systems and synchronized clusters

Let us first consider the conditions for the occurrence and stability of the single cluster synchronization where all the nodes synchronize together. We denote an n-cluster synchronization by nCS and thus a single-cluster synchronization by 1CS. Consider a network of N nodes of interacting dynamical systems or oscillators. Let $\mathbf{x}^i(t) \in R^m$ be the m-dimensional variable of the ith node. Let the uncoupled dynamics of each node be defined by the function $\mathbf{f}(\mathbf{x}^i(t))$ and the coupling by the function $\mathbf{u}$: $R^m \to R^m$. Let G be the $N \times N$ coupling matrix of the network. We allow the possibility of directed networks and also links with different weights. The dynamics of ith node is given by

$$\dot{\mathbf{x}}^i(t) = \mathbf{f}(\mathbf{x}^i(t)) + \sum_j G_{ij}\mathbf{u}(\mathbf{x}^j(t)). \tag{1}$$

A single-cluster synchronization (1CS) is defined by $\mathbf{x}^1 = \cdots = \mathbf{x}^N = \mathbf{x}$. The 1CS is a solution of eq. (1) provided the coupling matrix satisfies the condition that

$$\sum_j G_{ij} = g, \ \forall i, \tag{2}$$

where g is a constant independent of i. We note that this condition is a generalization of the synchronization condition which is normally used and is the most general one possible [9]. The synchronized state is a solution of

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + g\mathbf{u}(\mathbf{x}). \tag{3}$$

If $g = 0$ (as in ref. [10]) then the synchronized state is a solution of the uncoupled dynamics, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$.

The condition (2) ensures that G has one eigenvector of the type $e_1^R = (1, \ldots, 1)^T$ with eigenvalue $\gamma_1 = g$. This eigenvector defines the synchronization manifold and it has the dimension m. All the remaining eigenvectors belong to the transverse

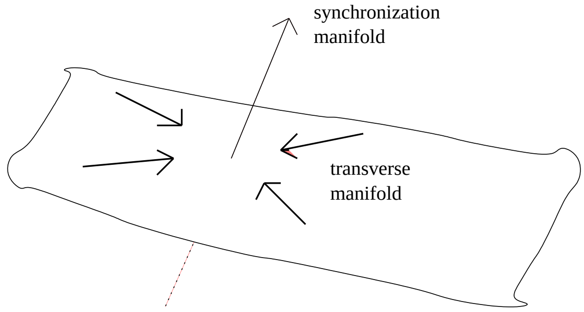

Figure 1. A schematic diagram of synchronization and transverse manifolds. For the synchronized state to be stable, all Lyapunov exponents in the transverse manifold must be negative, i.e. all the neighbouring trajectories in the transverse directions must converge to the synchronized state. There is no restriction on the Lyapunov exponents in the synchronization manifold.

manifold. Accordingly, the Lyapunov exponents can be separated into two sets, corresponding to the synchronization and transverse manifolds. The 1CS is stable provided all the transverse Lyapunov exponents are negative. Figure 1 shows a schematic diagram of the stability condition for synchronization.

It is possible to cast the stability conditions into a master stability equation as [10]

$$\dot{\phi} = [D\mathbf{f} + \alpha D\mathbf{u}]\phi, \tag{4}$$

where α is a parameter and ϕ is an m-dimensional vector. We can determine the master stability function $\lambda_{\max}$, which is the largest Lyapunov exponent for eq. (4), as a surface over the complex plane defined by α. The 1CS is stable if the master stability function is negative at each of the transverse eigenvalues of the coupling matrix G with $\alpha = \gamma_k$ $(k \neq 1)$ [10].

Let us now consider the conditions for the existence of the 2CS. The results can be easily extended to the n-cluster synchronization (nCS). In general, we assume that the nodes of the two clusters are governed by different dynamical systems and denote the variables by $\mathbf{x}^i,\ i = 1, \ldots, N_1$ and $\mathbf{y}^j,\ j = 1, \ldots, N_2$ of dimension m and N_1 and $N_2 = N - N_1$ are the number of nodes of the two clusters. Thus, the dynamics can be written as

$$\dot{\mathbf{x}}^i = \mathbf{f}_1(\mathbf{x}^i) + \sum_{l=1}^{N_1} A_{il}\mathbf{g}_1(\mathbf{x}^l) + \sum_{n=1}^{N_2} B_{in}\mathbf{g}_2(\mathbf{y}^n), \tag{5a}$$

$$\dot{\mathbf{y}}^j = \mathbf{f}_2(\mathbf{y}^j) + \sum_{l=1}^{N_1} C_{jl}\mathbf{g}_1(\mathbf{x}^l) + \sum_{n=1}^{N_2} D_{jn}\mathbf{g}_2(\mathbf{y}^n), \tag{5b}$$

where the coupling matrix G is split into four blocks A, B, C, D.

We define the 2CS state by $\mathbf{x}^1 = \cdots = \mathbf{x}^{N_1} = \mathbf{x}$ and $\mathbf{y}^1 = \cdots = \mathbf{y}^{N_2} = \mathbf{y}$. Existence of the 2CS requires that it should be a solution of the dynamics (eq. (5)). This implies that G should have eigenvectors of the form $e^R = (\mu, \ldots, \mu, \nu, \ldots, \nu)^T$.

There will be two such linearly independent eigenvectors and they lead to the synchronization manifold. Using these considerations it is easy to show that

$$\sum_j A_{ij} = a, \quad \sum_j B_{ij} = b, \quad \sum_j C_{ij} = c, \quad \sum_j D_{ij} = d, \ \forall i, \tag{6}$$

where a, b, c, d are constants. The condition (2) for the existence of 1CS will be satisfied if $a + b = c + d$. We note that the synchronization manifold for 2CS has dimension $2m$ while the transverse manifold has dimension $(N - 2)m$. The 2CS is stable provided all the transverse Lyapunov exponents are negative (figure 1).

Using eqs (5) and (6), the synchronized variables of 2CS are seen to satisfy the equations

$$\dot{\mathbf{x}} = \mathbf{f}_1(\mathbf{x}) + a\mathbf{g}_1(\mathbf{x}) + b\mathbf{g}_2(\mathbf{y}), \tag{7a}$$

$$\dot{\mathbf{y}} = \mathbf{f}_1(\mathbf{y}) + c\mathbf{g}_1(\mathbf{x}) + d\mathbf{g}_2(\mathbf{y}). \tag{7b}$$

A master stability analysis for the multi-cluster synchronization is possible for the separable case where the transverse eigenvectors split into two independent subspaces corresponding to the two clusters. The complete bipartite network is an example [11,12]. For this separable class of networks, we get a set of two master stability equations [9]

$$\dot{\phi} = [D\mathbf{f}_1 + \alpha D\mathbf{g}_1]\,\phi, \tag{8a}$$

$$\dot{\psi} = [D\mathbf{f}_2 + \alpha D\mathbf{g}_2]\,\psi. \tag{8b}$$

These are $2m$ equations and can be solved using the 2CS solution obtained from eqs (7). Thus, we can determine the cluster master stability functions (CMSFs) which are given by the largest Lyapunov exponents for the two equations in (8), as two surfaces over the complex plane defined by α. The stability condition for 2CS is that at the transverse eigenvalues the CMSF for the respective clusters must be negative [12].

3. Time-varying networks

Several natural networks are not static in time and their topology changes with time. Both the number of nodes and the edges connecting the nodes can vary with time. Such a time-varying topology can occur in social networks, computer networks, WWW, biological systems, spread of epidemics etc. Here, we investigate the synchronization properties of networks with time-varying structure and compare it with the synchronization in static time-average networks.

We now consider the time-varying topology where the network periodically switches between networks with coupling matrices $G_1, G_2, \ldots, G_g$ with periods $\tau_1, \tau_2, \ldots, \tau_g$ respectively and the total period $T = \sum_i \tau_i$. Thus,

$$G(t) = \sum_{i=1}^{g} G_i \chi_{[t_{i-1},t_i]}, \tag{9}$$

where $\chi_{[t_{i-1},t_i]}$ is the indicator function with support $[t_{i-1}, t_i)$ and $t_i = t_{i-1} + \tau_i$. The time-averaged $G(t)$ is

$$\bar{G} = \frac{1}{T} \sum_{i=1}^{g} G_i \tau_i. \tag{10}$$

In ref. [13] it is shown that if the network synchronizes for the static time-average of the topology, i.e. with $\bar{G}$, then the network will synchronize with the time-varying topology if the time-variation is done sufficiently fast.

3.1 *Commuting matrices*

Let us first consider the case when the different coupling matrices commute with each other, i.e.

$$[G_i, G_j] = 0, \quad \text{for } i, j = 1, \ldots, g. \tag{11}$$

Thus all the G_i and $\bar{G}$ will have the same eigenvectors, though different eigenvalues. Hence, the expanding and contracting directions in the synchronization and transverse manifolds will not change with time. Hence, to a first approximation the time-varying and time-average evolutions will be similar and we can write a relation between the time-varying and time-average Lyapunov exponents as [14]

$$\bar{\lambda}_k \approx \frac{1}{T} \sum_{i=1}^{g} \lambda_k^i \tau_i. \tag{12}$$

We thus see that the stability range for the time-varying and commuting case should be approximately the same as that for the time-average case.

3.2 *Non-commuting matrices*

We now consider the case when the different time-varying coupling matrices do not satisfy condition (11). In this case the eigenvectors corresponding to the different coupling matrices are in general not the same. Note that the largest eigenvalue ($\gamma_0 = 0$) and the corresponding eigenvector $(1, \ldots, 1)$ which define the synchronization manifold are the same for all the coupling matrices. However, in the transverse manifold the action of different coupling matrices G_i is to cause a rotation of the different eigenvectors and an evolution of the rotated eigenvectors. The effect of this periodic rotation is to take projections of different expansions and contractions along different directions. This has the effect of reducing the spread of the transverse Lyapunov exponents and in particular the larger exponents will decrease. This should in general enhance the stability of the synchronized state. Hence for non-commuting matrices, the time-varying case will have in general a better stability than a time-average case. We demonstrate this by using an example of coupled Rössler systems.

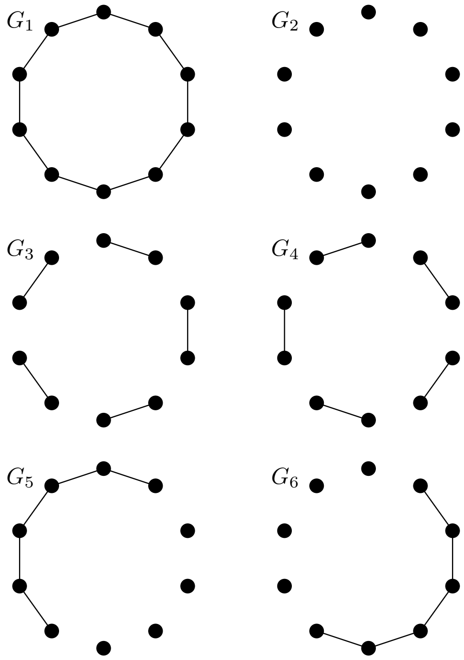

Figure 2. The figure shows the six different networks used to demonstrate the synchronization in time-varying networks.

3.3 *Illustration*

As an illustration we consider a system of coupled Rössler oscillators.

$$\begin{aligned}
\dot{x}_i(t) &= -y_i(t) - z_i(t) - \sigma \sum_{j=1}^{N} (G(t))_{ij} x_j(t), \\
\dot{y}_i(t) &= x_i(t) + a y_i(t), \\
\dot{z}_i(t) &= b + z_i(t)(x_i(t) - c),
\end{aligned} \tag{13}$$

where $G(t)$ is given by eq. (9), $i = 1, \ldots, N$, $a = 0.2$, $b = 0.2$, $c = 7.0$. We consider several networks with $N = 10$ as shown in figure 2. For simplicity we report here the results for the combination of two graphs each ($g = 2$).

(a) The combination (G_1, G_2) represents commuting coupling matrices. Both the time-varying and time-average cases show a stable synchronized state in the range $\sigma \in (0.75, 2.30)$.

(b) The combination (G_3, G_4) represents non-commuting coupling matrices. The time-varying case is stable in the range $\sigma \in (0.70, 2.30)$ while the time-average case is stable in the range $\sigma \in (0.75, 2.30)$. Thus the lower limit which corresponds to the long-wavelength instability [10] gets extended for the time-varying case.

(c) The combination (G_5, G_6) again represents non-commuting coupling matrices. The time-varying case is stable in the range $\sigma \in (0.75, 2.45)$ while the time-average case is stable in the range $\sigma \in (0.75, 2.30)$. Thus the upper limit which corresponds to the short-wavelength instability gets extended for the time-varying case.

4. Conclusion

In this paper we have discussed the existence and stability conditions for single- and multi-cluster synchronization. We find that the coupling matrices must satisfy certain conditions (eqs (6)) in order to obtain multi-cluster synchronization.

For networks with time-varying topology we find that for commuting coupling matrices, the stability of the synchronized state for the time-varying case is mostly unaffected and is almost the same as that for the average case. On the other hand, the non-commuting coupling matrices, in general, lead to better stability of the synchronized state for the time-varying case than the time-average case.

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