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Bifurcation methods of dynamical systems for generalized Kadomtsov–Petviashvili–Benjamin–Bona–Mahony equation

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Abstract. By applying the bifurcation theory of dynamical system to the generalized KP–BBM equation, the phase portraits of the travelling wave system are obtained. It can be shown that singular straight line in the travelling wave system is the reason why smooth periodic waves converge to periodic cusp waves. Under different parametric conditions, various sufficient conditions to guarantee the existence of the above solutions are given. Some exact explicit parametric representations of the above waves are obtained.

Keywords. Compacton solution; periodic wave solution; periodic cusp wave solution; generalized KP–BBM equation.

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1. Introduction

The Benjamin–Bona–Mahony (BBM) equation [1]

$$u_t + u_x + uu_x - u_{xxt} = 0 \tag{1}$$

has been proposed as the model for propagation of long waves where nonlinear dispersion is incorporated.

The balance between the nonlinear convection term uu_x and the dispersion effect term u_{xxx} in the spatially one-dimensional KdV equation

$$u_t + a(u^2)_x + u_{xxx} = 0 \tag{2}$$

gives rise to solitons. The KdV equation is a model that governs the one-dimensional propagation of small amplitude, weakly dispersive waves. The well-known two-dimensional generalization of the KdV equations, namely the Kadomtsov–Petviashvili (KP) equation [2] is given by

$$(u_t + auu_x + u_{xxx})_x + u_{yy} = 0. \tag{3}$$

Both BBM and KdV equations cover the following cases: surface waves of long wavelength in liquids, acoustic-gravity waves in compressible fluids, hydromagnetic waves in cold plasma and acoustic waves in anharmonic crystals. In [3], Wazwaz considered the nonlinear dispersive generalized KP-BBM equation

$$(u_t + u_x - a(u^n)_x - b(u^n)_{xxt})_x + ku_{yy} = 0. \tag{4}$$

The author obtained some soliton solutions and periodic solutions of the above equation using extended tanh method. But the author did not study the bifurcation behaviour of the travelling wave solutions of the corresponding travelling wave equations in its parameter space. It is very important to understand the dynamical behaviour of solutions of the travelling wave equation (4). To answer this question, we shall consider the bifurcations of travelling wave solutions of (4) using the bifurcation method of dynamical system.

To find the travelling wave solutions of eq. (4), we make the transformation

$$u(x,y,t) = \phi(\xi), \quad \xi = x + y - ct, \tag{5}$$

where c is the speed of wave. Substituting (5) into (4) yields

$$(-c\phi' + \phi' - a(\phi^n)' + bc(\phi^n)''')' + k\phi'' = 0. \tag{6}$$

Integrating the equation twice and neglecting the constant of integration we find

$$(1 + k - c)\phi - a\phi^n + bc(n\phi^{n-1}\phi'' + n(n-1)\phi^{n-2}(\phi')^2) = 0, \tag{7}$$

where prime is the derivative with respect to ξ. Clearly, eq. (7) is equivalent to the following two-dimensional system:

$$\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = y, \quad \frac{\mathrm{d}y}{\mathrm{d}\xi} = \frac{e + a\phi^{n-1} - bcn(n-1)\phi^{n-3}y^2}{bcn\phi^{n-2}}, \tag{8}$$

where $e = c - k - 1$. System (8) has the first integral

$$H(\phi, y) = \phi^{2(n-1)}y^2 - \frac{2}{bcn}\left(\frac{e}{n+1}\phi^{n+1} + \frac{a}{2n}\phi^{2n}\right) = h. \tag{9}$$

From the theory of dynamic systems, the smooth travelling wave solutions of eq. (4) are given by smooth orbits of eq. (8): solitary wave solutions correspond to homoclinic orbits at a single equilibrium point; periodic waves come from periodic orbits; while heteroclinic orbit connecting two equilibrium points yields kink (or anti-kink) solutions. Thus, to investigate all possible bifurcations of solitary waves and periodic waves of (4), we need to find all periodic annuli and homoclinic orbits of (8), which depend on the system parameters. The bifurcation theory of dynamical systems plays an important role in our study [4,5].

2. Bifurcations of phase portraits of system (8)

In this section, we will study bifurcation of phase portraits of system (8) in its parameter space. Throughout we assume that $bc > 0$, otherwise, we can make a transformation to reduce (8) to this case.

Case I. $n = 2$

Let $\mathrm{d}\xi = 2bc\phi\,\mathrm{d}\tau$. Then, except on the line $\phi = 0$, the system (8) has the same topological phase portraits as the following system:

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = 2bc\phi y, \quad \frac{\mathrm{d}y}{\mathrm{d}\tau} = e\phi + a\phi^2 - 2bcy^2. \tag{10}$$

It has two equilibrium points at $E_1(0,0)$ and $E_2(-\frac{e}{a},0)$.

Case II. $n = 3$

Let $\mathrm{d}\xi = 3bc\phi\,\mathrm{d}\tau$. Then, except on the line $\phi = 0$, the system (8) has the same topological phase portraits as the following system:

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = 3bc\phi y, \quad \frac{\mathrm{d}y}{\mathrm{d}\tau} = e + a\phi^2 - 6bcy^2. \tag{11}$$

When $a > 0, e > 0, bc > 0$, system (11) has two equilibrium points at $E_{1,2}(0, y_\pm)$, where $y_\pm = \pm\sqrt{\frac{e}{6bc}}$; when $a < 0, e > 0, bc > 0$, it has four equilibrium points at $E_{1,2}(0, y_\pm)$ and $E_{3,4}(\phi_\pm, 0)$, respectively, where $\phi_\pm = \pm\sqrt{-\frac{e}{a}}$; when $a > 0, e < 0, bc > 0$, it has two equilibrium points at $E_{3,4}(\phi_\pm, 0)$.

Case III. $n > 3$ $(n \in Z^+)$

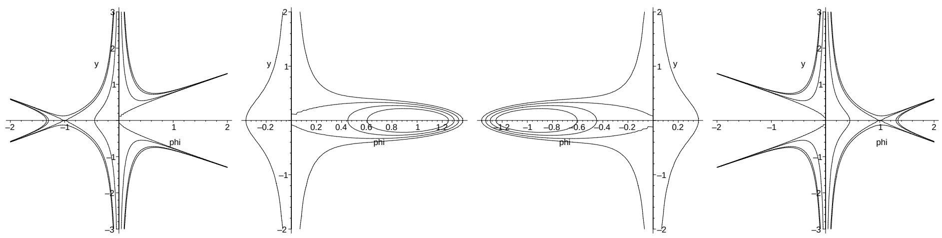
Let $\mathrm{d}\xi = bcn\phi^{n-2}\,\mathrm{d}\tau$. Then, except on the line $\phi = 0$, the system (8) has the same topological phase portraits as the following system:

$$\frac{\mathrm{d}\phi}{\mathrm{d}\tau} = bcn\phi^{n-2}y, \quad \frac{\mathrm{d}y}{\mathrm{d}\tau} = e + a\phi^{n-1} - bcn(n-1)\phi^{n-3}y^2. \tag{12}$$

When $n = 2m$ $(m \geq 2, m \in Z^+)$, system (12) has one equilibrium point at $E_1((-\frac{e}{a})^{\frac{1}{2m-1}}, 0)$; when $n = 2m+1$ $(m \geq 2, m \in Z^+)$, it has two equilibrium points at $E_{1,2}(\phi_\pm, 0)$, where $\phi_\pm = \pm(-\frac{e}{a})^{\frac{1}{2m}}$.

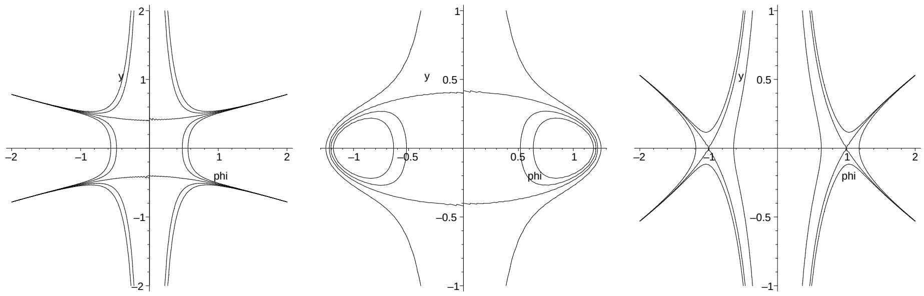
Let $M(\phi_e, y_e)$ be the coefficient matrix of the linearized system (8) at an equilibrium point (ϕ_e, y_e) and $J(\phi_e, y_e) = \det(M(\phi_e, y_e))$. By the theory of planar dynamical systems, we know that for an equilibrium point (ϕ_e, y_e) of a planar integrable system, if $J < 0$ then the equilibrium point is a saddle point; if $J > 0$ and $\mathrm{Trace}(M(\phi_e, y_e)) = 0$ then it is a center point; if $J > 0$ and $(\mathrm{Trace}(M(\phi_e, y_e)))^2 - 4J(\phi_e, y_e) > 0$ then it is a node; if $J = 0$ and the index of the equilibrium point is zero then it is a cusp; if $J = 0$ and the index of the equilibrium point is not zero then it is a high-order equilibrium point.

When $n = 2$, we have $J(0,0) = 0, J(-\frac{e}{a}, 0) = -\frac{2bce^2}{a}$; when $n = 3$, then $J(0, y_\pm) = -6bce, J(\phi_\pm, 0) = 6bce$, where $y_\pm = \pm\sqrt{\frac{e}{6bc}}, \phi_\pm = \pm\sqrt{-\frac{e}{a}}$; when $n = 2m$ $(m \geq 2, m \in Z^+)$, we have $J((-\frac{e}{a})^{\frac{1}{2m-1}}, 0) = -2abcm(2m-1)(-\frac{e}{a})^{\frac{4m-4}{2m-1}}$;



(1-1) (1-2) (1-3) (1-4)

Figure 1. The phase portraits of system (8) for $n = 2, bc > 0$. (**1-1**) $a > 0, e > 0$; (**1-2**) $a < 0, e > 0$; (**1-3**) $a < 0, e < 0$; (**1-4**) $a > 0, e < 0$.



(2-1) (2-2) (2-3)

Figure 2. The phase portraits of system (8) for $n = 3, bc > 0$. (**2-1**) $a > 0, e > 0$; (**2-2**) $a < 0, e > 0$; (**2-3**) $a > 0, e < 0$.

when $n = 2m+1$ $(m \geq 2, m \in Z^+)$, we have $J(\phi_\pm, 0) = -2abcm(2m+1)(-\frac{e}{a})^{\frac{4m-2}{2m}}$, where $\phi_\pm = \pm(-\frac{e}{a})^{\frac{1}{2m}}$.

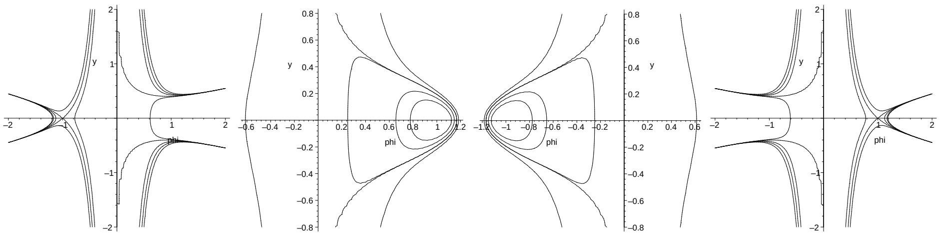
Remark 2.1

(a) When $n = 2, a < 0, bc > 0$, there is a periodic annulus surrounded by the ellipse $12bcy^2 - 4e\phi - 3a\phi^2 = 0$, which forms a degenerate homoclinic orbit (see figure 1(1-2,1-3)).

(b) When $n = 3, a < 0, e > 0, bc > 0$, there are two periodic annuli surrounded by three boundary curves consisting of a segment of $\phi = 0$ and two arcs of the ellipse $18bcy^2 - 2a\phi^2 = 3e$, which form three heteroclinic orbits (see figure 2(2-2)).

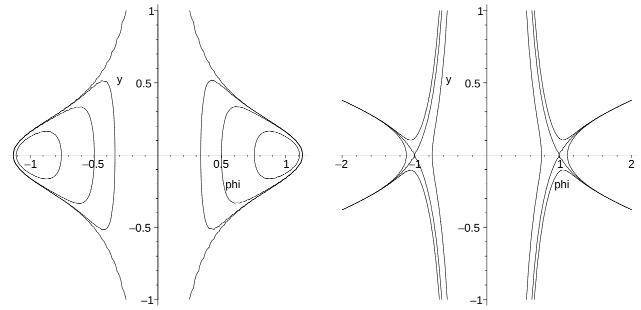
(c) When $n = 2m$ $(m \geq 2, m \in Z^+), a < 0, bc > 0$, there is a family of periodic orbits enclosing the center point $((-\frac{e}{a})^{\frac{1}{2m-1}}, 0)$ (see figure 3(3-2,3-3)).

(d) When $n = 2m + 1$ $(m \geq 2, m \in Z^+), a < 0, e > 0, bc > 0$, there are two families of periodic orbits enclosing the center points $(\phi_\pm, 0)$, respectively, where $\phi_\pm = \pm(-\frac{e}{a})^{\frac{1}{2m}}$ (see figure 4(4-1)).



(3-1) (3-2) (3-3) (3-4)

Figure 3. The phase portraits of system (8) for $n = 2m$ $(m \geq 2), bc > 0$. (**3-1**) $a > 0, e > 0$; (**3-2**) $a < 0, e > 0$; (**3-3**) $a < 0, e < 0$; (**3-4**) $a > 0, e < 0$.



(4-1) (4-2)

Figure 4. The phase portraits of system (8) for $n = 2m+1$ $(m \geq 2), bc > 0$. (**4-1**) $a < 0, e > 0$; (**4-2**) $a > 0, e < 0$.

3. The existence of smooth and non-smooth travelling wave solutions of system (8)

In this section, we shall point out that the existence of straight line $\phi = 0$ in the (ϕ, y)-phase plane of the system (8) is the original reason for the appearance of non-smooth travelling wave solutions in our travelling wave models. To discuss the existence of cusp waves, we need to use the following lemma relating to the straight line [5].

Lemma 3.1. *The boundary curves of a periodic annulus are the limit curves of closed orbits inside the annulus. If these boundary curves contain a segment of the singular line $\phi = 0$ of (8), then along this segment and near this segment, in very short time interval $y = \phi_\xi$ jumps rapidly.*

Based on Remark 2.1 and Lemma 3.1, we have the following result.

Theorem 3.1.

(a) *When $n = 2, a < 0, bc > 0$, corresponding to the curve $H(\phi, y) = 0$ defined by (9), eq. (4) has a compacton solution. Corresponding to the family of curves $H(\phi, y) = h(h \in (H(-\frac{e}{a}, 0), 0))$ defined by (9), eq. (4) has a family of smooth periodic travelling wave solutions.*

(b) *When* $n = 3, a < 0, e > 0, bc > 0$, *corresponding to two arch curves* $H(\phi, y) = 0$ *defined by* (9), *eq.* (4) *has two periodic cusp wave solutions. Corresponding to the curves* $H(\phi, y) = h(h \in (H(\phi_\pm, 0), 0))$ *defined by* (9), *eq.* (4) *has two families of periodic travelling wave solutions, where* $\phi_\pm = \pm\sqrt{-\frac{e}{a}}$. *When* h *varies from* $H(\phi_\pm, 0)$ *to* 0, *these periodic travelling waves will gradually lose their smoothness, and evolve from smooth periodic travelling waves to periodic cusp waves, finally approach a periodic cusp wave of valley-type and a periodic cusp wave of peak-type defined by* $H(\phi, y) = 0$ *of* (9).

(c) *When* $n = 2m$ $(m \geq 2, m \in Z^+), a < 0, bc > 0$, *corresponding to the curves* $H(\phi, y) = h(h \in (H((-\frac{e}{a})^{\frac{1}{2m-1}}, 0), 0)$ *defined by* (9), *eq.* (4) *has a family of periodic travelling wave solutions. When* h *varies from* $H((-\frac{e}{a})^{\frac{1}{2m-1}}, 0), 0)$ *to* 0, *these periodic travelling waves will gradually lose their smoothness, and evolve from smooth periodic travelling waves to periodic cusp waves.*

(d) *When* $n = 2m + 1$ $(m \geq 2, m \in Z^+), a < 0, e > 0, bc > 0$, *corresponding to the curves* $H(\phi, y) = h(h \in (H(\phi_\pm, 0), 0))$ *defined by* (9), *eq.* (4) *has two families of periodic travelling wave solutions, where* $\phi_\pm = \pm(-\frac{e}{a})^{\frac{1}{2m}}$. *When* h *varies from* $H(\phi_\pm, 0)$ *to* 0, *these periodic travelling waves will gradually lose their smoothness, and evolve from smooth periodic travelling waves to periodic cusp waves.*

We next give some exact travelling wave solutions.

1. When $n = 2, a < 0, bc > 0$, the curve $H(\phi, y) = 0$ defined by (9) has the algebraic equation

$$y^2 = \frac{e}{3bc}\phi + \frac{a}{4bc}\phi^2. \tag{13}$$

Thus, by using the first equation of (8) and eq. (13), we obtain the parametric representation of the compacton solution as follows:

$$\phi(\xi) = -\frac{2e}{3a} + \frac{2e}{3a}\cos\left(\sqrt{-\frac{a}{4bc}}\xi\right). \tag{14}$$

2. When $n = 3, a < 0, e > 0, bc > 0$, the curve $H(\phi, y) = 0$ defined by (9) has the algebraic equation

$$y^2 = \frac{e}{6bc} + \frac{a}{9bc}\phi^2. \tag{15}$$

Thus, by using the first equation of (8) and eq. (15), we obtain the parametric representation of the periodic cusp wave solution as follows:

$$\phi(\xi) = \pm\sqrt{-\frac{3e}{2a}}\cos\left(\sqrt{-\frac{a}{9bc}}\xi\right), \tag{16}$$

where

$$\xi \in \left(\frac{3(2p-1)\pi}{2}\sqrt{-\frac{bc}{a}}, \frac{3(2p+1)\pi}{2}\sqrt{-\frac{bc}{a}}\right), \quad p = 2q, q \in Z.$$

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