PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 71, No. 1
July 2008
pp. 49–55

Travelling wave-like solutions of the Zakharov–Kuznetsov equation with variable coefficients

YAN-ZE PENG[1,*], E V KRISHNAN[2] and HUI FENG[1]
[1]School of Mathematics and Statistics, Wuhan University, Wuhan 430072, People's Republic of China
[2]Department of Mathematics and Statistics, Sultan Qaboos University, P.O. Box 36, Al-Khod 123, Muscat, Sultanate of Oman
*Corresponding author
E-mail: yanzepeng@163.com; krish@squ.edu.om

MS received 3 December 2007; revised 20 February 2008; accepted 4 March 2008

Abstract. Travelling wave-like solutions of the Zakharov–Kuznetsov equation with variable coefficients are studied using the solutions of Raccati equation. The solitary wave-like solution, the trigonometric periodic wave solution and the rational wave solution are obtained with a constraint between coefficients. The property of the solutions is numerically investigated. It is shown that the coefficients of the equation do not change the wave amplitude, but may change the wave velocity.

Keywords. Zakharov–Kuznetsov equation; travelling wave solution; Raccati equation.

PACS Nos 05.45.Yv; 02.30.Ik; 02.30.Jr

1. Introduction

It is very important to search for exact solutions of a nonlinear partial differential equation (PDE) because many physical phenomena are described by PDEs. Particularly, PDEs with variable coefficients can reflect the real thing even more than those with constant coefficients. So many methods are available for constructing exact solutions of nonlinear partial differential equations. Some of the most important methods are the inverse scattering transformation [1], the bilinear method [2], symmetry reductions [3,4], Bäcklund and Darboux transformations [5], the singular manifold method [6], and so on. Travelling waves, whether their solution expressions are in explicit or implicit forms, are very interesting from the point of view of applications. These types of waves will not change their shapes during propagation and are thus easy to detect. Of particular interest are three types of travelling waves: the solitary waves, which are localized travelling waves, asymptotically zero at large distances, the periodic waves, and the kink waves, which rise or descend from one asymptotic state to another. The exact methods like inverse scattering, bilinearization etc. exist only for integrable system which is a small subclass of nonlinear PDEs. In general, there is no standard method for solving nonlinear PDEs and more so for PDEs in 2+1 dimensions. Therefore, exact solutions of such a PDE with time-dependent coefficients should be important. An algebraic method is developed in this paper to look for exact travelling wave-like solutions of the PDE in (2+1) dimension with variable coefficients.

The nonlinear development of ion-acoustic waves in a magnetized plasma under the restrictions of small wave amplitude, weak dispersion, and strong magnetic fields is described by the Zakharov–Kuznetsov (ZK) equation [7–11]

$$u_t + \alpha(t)uu_x + \beta(t)u_{xxx} + \gamma(t)u_{xyy} = 0. \tag{1}$$

Shivamoggi [12] gives only four polynomial conservation laws of the ZK equation with constant coefficients. Moussa [13] and Qu [14] obtain some similarity solutions of eq. (1) with $\alpha(t) = \beta(t) = \gamma(t) = 1$ by using symmetry group method. The importance of eq. (1) motivates this study. The plan of paper is as follows: Section 2 is devoted to obtaining exact solutions of eq. (1) using the solutions of the Raccati equation. In §3, the property of the solutions is numerically investigated. Discussion and conclusions are given in the last section.

2. Exact solutions of eq. (1)

In general, one should expand the solution of PDE with variable coefficients into the form $u(x,y,t) = \sum_{i=1}^{n} A_i(t)w^i(\xi)$, where n is fixed by balancing the linear term of the highest-order derivative with nonlinear term in the equation under consideration. For our eq. (1), it is easily found that $n = 2$. So, we seek for the solutions of eq. (1) of the form

$$u(x,y,t) = A_0(t) + A_1(t)w(\xi) + A_2(t)w^2(\xi), \tag{2}$$

where $w(\xi)$ satisfies the Riccati equation

$$w' = w^2 + pw + q. \tag{3}$$

Here the prime denotes the derivation with respect to ξ (throughout the paper), p, q are parameters, $\xi = f(t)x + g(t)y + h(t)$, $f(t)$, $g(t)$ and $h(t)$ are functions of t to be determined. Substituting eq. (2) with eq. (3) into eq. (1) and equating the coefficients of like powers of w, one gets

$$\begin{aligned}
&24\beta f^3 A_2 + 24\gamma f g^2 A_2 + 2\alpha f A_2^2 = 0,\\
&6\beta f^3 A_1 + 6\gamma f g^2 A_1 + 54p\beta f^3 A_2\\
&\quad +54p\gamma f g^2 A_2 + 3\alpha f A_1 A_2 + 2p\alpha f A_2^2 = 0,\\
&12p\beta f^3 A_1 + 12p\gamma f g^2 A_1 + \alpha f A_1^2 + 38p^2\beta f^3 A_2\\
&\quad +40q\beta f^3 A_2 + 38p^2\gamma f g^2 A_2 + 40q\gamma f g^2 A_2 + 2\alpha f A_0 A_2\\
&\quad +3p\alpha f A_1 A_2 + 2q\alpha f A_2^2 + 2(f'x + g'y + h')A_2 = 0,
\end{aligned}$$

$$
\begin{aligned}
&7p^2\beta f^3A_1 + 8q\beta f^3A_1 + 7p^2\gamma fg^2A_1 + 8q\gamma fg^2A_1 \\
&\quad +\alpha fA_0A_1 + p\alpha fA_1^2 + 8p^3\beta f^3A_2 + 52pq\beta f^3A_2 \\
&\quad +8p^3\gamma fg^2A_2 + 52pq\gamma fg^2A_2 + 2p\alpha fA_0A_2 + 3q\alpha fA_1A_2 \\
&\quad +(f'x+g'y+h')A_1 + 2p(f'x+g'y+h')A_2 + A_2' = 0, \\
&p^3\beta f^3A_1 + 8pq\beta f^3A_1 + p^3\gamma fg^2A_1 + 8pq\gamma fg^2A_1 \\
&\quad +p\alpha fA_0A_1 + q\alpha fA_1^2 + 14p^2q\beta f^3A_2 + 16q^2\beta f^3A_2 \\
&\quad +14p^2q\gamma fg^2A_2 + 16q^2\gamma fg^2A_2 + 2q\alpha fA_0A_2 \\
&\quad +p(f'x+g'y+h')A_1 + 2q(f'x+g'y+h')A_2 + A_1' = 0, \\
&p^2q\beta f^3A_1 + 2q^2\beta f^3A_1 + p^2q\gamma fg^2A_1 + 2q^2\gamma fg^2A_1 \\
&\quad +q\alpha fA_0A_1 + 6pq^2\beta f^3A_2 + 6pq^2\gamma fg^2A_2 \\
&\quad +q(f'x+g'y+h')A_1 + A_0' = 0. \qquad (4)
\end{aligned}
$$

It follows from here that

$$
\begin{aligned}
&A_0 = -\frac{(p^2+8q)f(\beta f^2+\gamma g^2)+h'}{\alpha f}, \\
&A_1 = -\frac{12p(\beta f^2+\gamma g^2)}{\alpha}, \quad A_2 = -\frac{12(\beta f^2+\gamma g^2)}{\alpha}, \\
&f = k, \quad g = l, \quad h = c_1\int \alpha(t)\mathrm{d}t + c_2, \quad \alpha = c(\beta f^2+\gamma g^2), \qquad (5)
\end{aligned}
$$

where c, c_i, k and l are the constants of integration. So, we obtain the exact solution of eq. (1) as follows:

$$
\begin{aligned}
u = &-\frac{(p^2+8q)k(k^2\beta+l^2\gamma)+c_1\alpha}{k\alpha} \\
&-\frac{12p(k^2\beta+l^2\gamma)}{\alpha}w - \frac{12(k^2\beta+l^2\gamma)}{\alpha}w^2, \qquad (6)
\end{aligned}
$$

where w satisfies eq. (3) with $p,\ q,\ c,\ c_i,\ k$ and l being arbitrary constants, $\xi = kx + ly + c_1\int \alpha(t)\mathrm{d}t + c_2$, and a restriction or reduction of the given parameters of the system, $\alpha(t) = c[k^2\beta(t) + l^2\gamma(t)]$, under which only the exact solution is possible to obtain. It is easy to see from eq. (6) that the coefficients of the equation do not change the wave amplitude, but may change the wave velocity. In what follows, we have several types of exact solutions of eq. (1).

Case 1. $q = 0,\ p \neq 0$. Equation (3) has the solution

$$
w = \frac{p}{c_0\exp(-p\xi)-1}, \qquad (7)
$$

where c_0 is an integrating constant (throughout the paper). So, the corresponding solution of eq. (1) reads

$$u = \begin{cases} -\dfrac{p^2k(k^2\beta + l^2\gamma) + c_1\alpha}{k\alpha} \\ +\dfrac{3p^2(k^2\beta + l^2\gamma)}{\alpha}\mathrm{sech}^2\left[\dfrac{1}{2}(-p\xi + \ln|c_0|)\right], \quad \text{for} \quad c_0 < 0, \\ \\ -\dfrac{p^2k(k^2\beta + l^2\gamma) + c_1\alpha}{k\alpha} \\ -\dfrac{3p^2(k^2\beta + l^2\gamma)}{\alpha}\mathrm{csch}^2\left[\dfrac{1}{2}(-p\xi + \ln c_0)\right], \quad \text{for} \quad c_0 > 0. \end{cases} \tag{8}$$

Case 2. $q \neq 0,\ p = 0$. According to the sign of the constant q, one has subcases to discuss.

As $q < 0$, eq. (1) has the solitary wave-like solution

$$u = \begin{cases} -\dfrac{8qk(k^2\beta + l^2\gamma) + c_1\alpha}{k\alpha} \\ +\dfrac{12q(k^2\beta + l^2\gamma)}{\alpha}\tanh^2\left[\sqrt{-q}\xi + \dfrac{1}{2}\ln|c_0|\right], \quad \text{for} \quad c_0 < 0, \\ \\ -\dfrac{8qk(k^2\beta + l^2\gamma) + c_1\alpha}{k\alpha} \\ +\dfrac{12q(k^2\beta + l^2\gamma)}{\alpha}\coth^2\left[\sqrt{-q}\xi + \dfrac{1}{2}\ln c_0\right], \quad \text{for} \quad c_0 > 0. \end{cases} \tag{9}$$

As $q > 0$, we have the trigonometric periodic wave solution

$$u = \begin{cases} -\dfrac{8qk(k^2\beta + l^2\gamma) + c_1\alpha}{k\alpha} \\ -\dfrac{12q(k^2\beta + l^2\gamma)}{\alpha}\tan^2[\sqrt{q}\xi + c_0], \\ \\ -\dfrac{8qk(k^2\beta + l^2\gamma) + c_1\alpha}{k\alpha} \\ -\dfrac{12q(k^2\beta + l^2\gamma)}{\alpha}\cot^2[-\sqrt{q}\xi + c_0]. \end{cases} \tag{10}$$

Case 3. $q = 0,\ p = 0$. One has the rational-type travelling wave-like solution

$$u = -\frac{12(k^2\beta + l^2\gamma)}{\alpha(kx + ly + c_1\int\alpha(t)\mathrm{d}t + c_0)^2}. \tag{11}$$

Case 4. $q \neq 0,\ p \neq 0$. In this case, the solution of eq. (3) reads

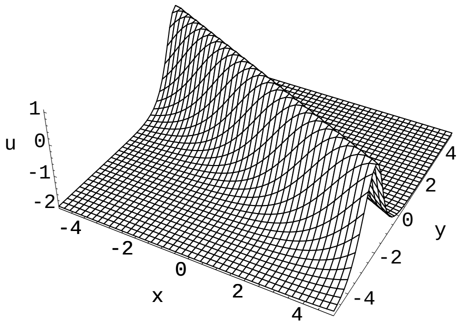

Figure 1. The spatial structure graph of eq. (8) as $\gamma(t) = t$.

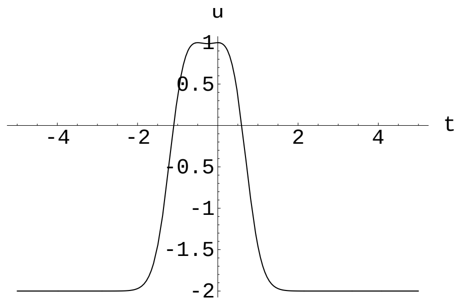

Figure 2. The temporal structure graph of eq. (8) as $\gamma(t) = t$.

$$w = \frac{w_1 - w_2 c_0 \exp[(w_1 - w_2)\xi]}{1 - c_0 \exp[(w_1 - w_2)\xi]}, \tag{12}$$

where w_1 and w_2 are two roots of the equation $w^2 + pw + q = 0$. From eq. (6) we obtain the solitary wave-like solution of eq. (1)

$$u = \begin{cases} -\dfrac{(p^2+8q)k(k^2\beta+l^2\gamma)+c_1\alpha}{k\alpha} + \dfrac{12pq(k^2\beta+l^2\gamma)}{\alpha} \\ +\dfrac{3(k^2\beta+l^2\gamma)}{\alpha}(w_1-w_2)^2\operatorname{sech}^2\dfrac{1}{2}[(w_1-w_2)\xi+\ln|c_0|], \\ \text{for} \quad c_0<0, \\ -\dfrac{(p^2+8q)k(k^2\beta+l^2\gamma)+c_1\alpha}{k\alpha} + \dfrac{12pq(k^2\beta+l^2\gamma)}{\alpha} \\ -\dfrac{3(k^2\beta+l^2\gamma)}{\alpha}(w_1-w_2)^2\operatorname{csch}^2\dfrac{1}{2}[(w_1-w_2)\xi+\ln c_0], \\ \text{for} \quad c_0>0. \end{cases} \tag{13}$$

3. The property of solutions

In this section, we discuss the property of the solution, taking the first of eq. (8) as an example. Fixing $k = 1,\ l = 2,\ c = 1,\ c_0 = -1,\ c_1 = 1,\ c_2 = 0,\ p = -1$ and $\beta(t) = 1$, let us see how the solution changes with the model parameter $\gamma(t)$. As $\gamma(t) = t$, its spatial structure at $t = 0$ is shown in figure 1, and its temporal structure at $x = 0,\ y = 0$ is depicted in figure 2. When $\gamma(t) = \mathrm{e}^{-t},\ \mathrm{e}^{t},\ \sin t$ and $\tanh t$, their temporal structures are illustrated in figures 3–6, respectively, while their spatial ones are similar to figure 1. From these figures, one can easily see that the coefficients of the equation do not change the wave amplitude, but may change the wave velocity.

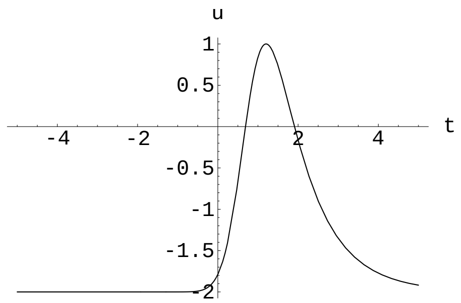

Figure 3. The temporal structure graph of eq. (8) as $\gamma(t) = \mathrm{e}^{-t}$.

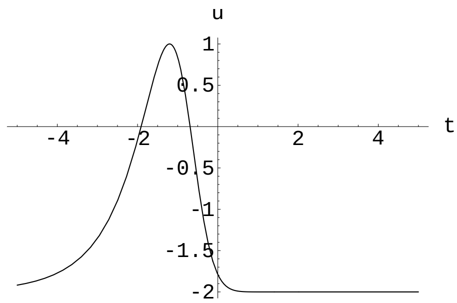

Figure 4. The temporal structure graph of eq. (8) as $\gamma(t) = \mathrm{e}^{t}$.

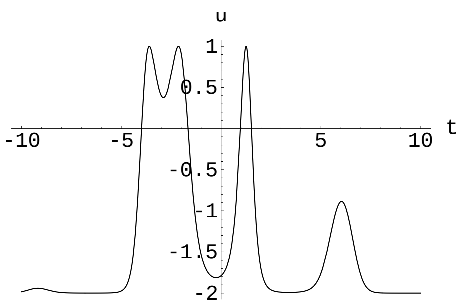

Figure 5. The temporal structure graph of eq. (8) as $\gamma(t) = \sin t$.

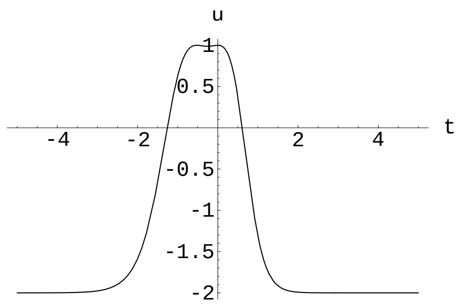

Figure 6. The temporal structure graph of eq. (8) as $\gamma(t) = \tanh t$.

4. Discussion and conclusion

Travelling wave-like solutions of the Zakharov–Kuznetsov equation with variable coefficients have been studied using the solutions of Raccati equation. Under a constraint between coefficients, the solitary wave-like solution, the trigonometric periodic wave solution and the rational wave solution are obtained. The property of the solutions is numerically investigated and it is found that the coefficients of the equation do not change the wave amplitude, but may change the wave velocity. So, the speeds of solitary waves may be modulated by adjusting the model parameters.

For PDEs with constant coefficients, many authors obtain the exact solutions by using Riccati equation. However, there are few authors to solve PDEs with variable coefficients by it. This paper is a successful experiment.

It is interesting to see that some of the solutions obtained in this paper develop singularity at a finite point, i.e. for any fixed $t = t_0$, there exists x_0 at which the solutions blow up. There is much interest in the formation of the so-called hot spots or blow up of solutions [15,16]. It appears that these singular solutions will model this physical phenomena. It can be easily seen that the method used in this paper is suitable to solve more nonlinear partial differential equations.

Acknowledgement

One of the authors (Hui Feng) is supported by NCET of China and the work of Yan-ze Peng is supported by Research Fund for the Doctoral Program of Higher Education of China (No. 20070486094). The authors would like to acknowledge the anonymous referee for his/her valuable comments.

References

[1] C S Gardner, J M Green, M D Kruskal and R M Miura, *Phys. Rev. Lett.* **19**, 1095 (1967)
[2] R Hirota, *Phys. Rev. Lett.* **27**, 1192 (1971)
[3] P J Olver, *Application of Lie group to differential equation* (Springer, New York, 1986)
[4] P A Clarkson and M D Kruskal, *J. Math. Phys.* **30**, 2201 (1989)
[5] V B Matveev and M A Salle, *Darboux transformations and solitons* (Springer, Berlin, 1991)
[6] J Weiss, M Tabor and G Carnevale, *J. Math. Phys.* **24**, 522 (1983)
[7] V E Zakharov and E M Kuznetsov, *Sov. Phys. JETP* **39**, 285 (1974)
[8] E W Laedke and K H Spatschek, *Phys. Rev. Lett.* **47**, 719 (1981)
[9] E W Laedke and K H Spatschek, *Phys. Fluids* **25**, 985 (1982)
[10] E W Laedke and K H Spatschek, *J. Plasma Phys.* **28**, 469 (1982)
[11] E Infeld, *J. Plasma Phys.* **33**, 171 (1985)
[12] B K Shivamoggi, *J. Plasma Phys.* **41**, 83 (1989)
[13] M H M Moussa, *Int. J. Eng. Sci.* **39**, 1565 (2001)
[14] C Z Qu, *Int. J. Theor. Phys.* **34**, 99 (1995)
[15] N A Kudryashov and D Zargaryan, *J. Phys.* **A29**, 8067 (1996)
[16] P A Clarkson and E L Mansfield, *Physica* **D70**, 250 (1993)