Message transfer in a communication network

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Abstract. We study message transfer in a 2-d communication network of regular nodes and randomly distributed hubs. We study both single message transfer and multiple message transfer on the lattice. The average travel time for single messages travelling between source and target pairs of fixed separations shows q-exponential behaviour as a function of hub density with a characteristic power-law tail, indicating a rapid drop in the average travel time as a function of hub density. This power-law tail arises as a consequence of the log-normal distribution of travel times seen at high hub densities. When many messages travel on the lattice, a congestion–decongestion transition can be seen. The waiting times of messages in the congested phase show a Gaussian distribution, whereas the decongested phase shows a log-normal distribution. Thus the waiting time distributions carry the signature of congested or decongested behaviour.

Keywords. Gradient mechanism; coefficient of betweenness centrality; waiting time.

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1. Introduction

Transport processes on networks have been a topic of intensive research in recent years. Examples of transport processes on networks include the traffic of information packets [1–4], transport processes on biological networks [5,6], and road traffic. The structure and topology of the network, as well as the mechanism of transport, have been seen to play crucial roles in the optimisation of the efficiency of the transport process [7]. It is therefore important to study this interplay in the context of realistic networks so that their performance can be optimised.

The gradient mechanism of transport has been shown to be very efficient for message communication. The topology of the substrate network on which the gradient is set up, has proved to be extremely important in the context of the jamming and congestion of messages. The gradient congests very rapidly on a random network substrate, but is much less prone to congestion on a scale-free substrate. Here, we study the gradient mechanism on a two-dimensional substrate network of nodes and hubs which incorporates local clustering and geographic separations. We study single message transport as well as multiple message transport for this system. In the case of single message transfer, we study the dependence of average travel times on the hub density, and find that travel times fall off as a

q-exponential with a power-law tail at high hub densities. The distribution of travel times is used to analyse this behaviour. In the case of multiple message transfer, the waiting time distribution of the messages has the characteristic signature of the phase, being normal in the congested phase and log-normal in the decongested phase.

2. Model of communication network

The substrate model on which message communication takes place is shown in figure 1a. This is a regular two-dimensional lattice with two types of nodes, the regular nodes, connected to their nearest neighbours (e.g. node X in figure 1), and hubs at randomly selected locations which are connected to all nodes in their area of influence, a square of side a (e.g. node Y in the same figure).

To set up the gradient mechanism, we need to assign a capacity to each hub, the hub capacity being defined as the number of messages the hub can process simultaneously. Here, each hub is randomly assigned some message capacity between one and $C_{\max}$. A gradient flow is assigned from each hub to all the hubs with the maximum capacity ($C_{\max}$). Thus, the hubs with lower capacities are connected to the hubs with highest capacity $C_{\max}$ by the gradient mechanism (see figure 1 [8]).

Two nodes with coordinates (is, js) and (it, jt) separated by a fixed distance D_{st} $= |is - it| + |js - jt|$ are chosen from a lattice of a given size L^2, and assigned to be the source and target. We set free boundary conditions. If a message is routed from a source S to a target T on this lattice through the baseline mechanism, it takes the path S–1–2–3–⋯–7–P–8–⋯–11–Q–12–⋯–T as in figure 1. After the implementation of the gradient mechanism, the distance between A and B is covered in one step as shown by the link g and a message is routed along the path S–1–2–3–G–g–F–4–5–6–T.

3. Mechanism of single message transfer

Consider a message that starts from the source S and travels towards a target T. Any node which holds the message at a given time, transfers the message to the node nearest to it, in the direction which minimises the distance between the current message holder and the target. If a constituent node is the current message holder, it sends the message directly to its own hub. When the hub becomes the current message holder, the message is sent to the constituent node within the square region, the choice of the constituent node being made by minimising the distance to the target. When a hub in the lattice becomes the current message holder, the message is transferred to the hub connected to the current message holder by the gradient mechanism, if the new hub is in the direction of the target, otherwise it is transferred to the constituent nodes of the current hub. The constituent node is chosen such that the distance from the target is minimised.

The average travel time for a message for a fixed source–target distance, is a good measure of the efficiency of the network. It was observed earlier [8] that if the hubs in the lattice are connected by the gradient mechanism, the average travel
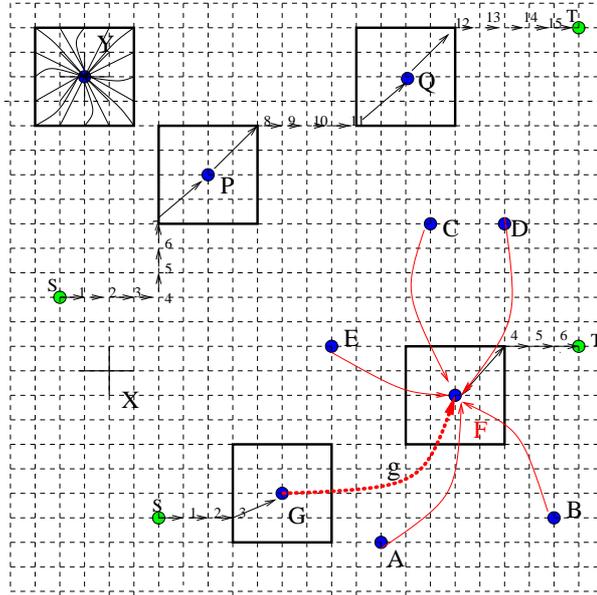
Figure 1. A regular two-dimensional lattice. X is an ordinary node with nearest-neighbour connections. Each hub has a square influence region (as shown for the hub Y). A typical path from the source S to the target T is shown with labelled sites. The path S–1–2–3–⋯–7–P–8–⋯–11–Q–12–⋯–T passes through the hubs P and Q. Hubs A–G are distributed randomly in the lattice and each hub is assigned with some message capacity between 1 and 10. In the figure F has maximum capacity 10. The hubs are connected by the gradient mechanism as shown by one way arrows. After the implementation of the gradient mechanism the distance between G and F is covered in one step. The gradient path is given by S–1–2–3–G–g–F–4–5–6–T.

time as a function of the number of hubs shows a q-exponential behaviour (figure 2a) of the form $f(x) = A(1 - (1-q)\frac{x}{x_0})^{\frac{1}{(1-q)}}$ with the parameter $q = 3.51$, with a power-law tail for a higher number of hubs.

The power-law tail at high hub densities can be understood by looking at the travel time distribution at high hub densities. The travel time distribution at a hub density of 4% is plotted in figure 2b for lattice size $L = 100$. The travel time distribution shows log-normal behaviour [8] of the form

$$f(t) = \frac{1}{t\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln t - \mu)^2}{2\sigma^2}\right), \tag{1}$$

where μ and σ are the mean and standard deviation of the travel time distribution. The mean travel time for this log-normal distribution $\langle t \rangle$ is given by $\exp(\mu + \sigma^2/2)$ [9]. We plot the quantity $\exp(\mu + \sigma^2/2)$ as a function of N, where N is the number of hubs in figure 3. It is clear that the plot of $\exp(\mu + \sigma^2/2)$ as a function of N can be fitted by a power-law $f(x) = Bx^{-\beta}$ at higher hub densities with the exponent $\beta = 0.36 \pm 0.008$. Since this quantity is equal to the average travel time

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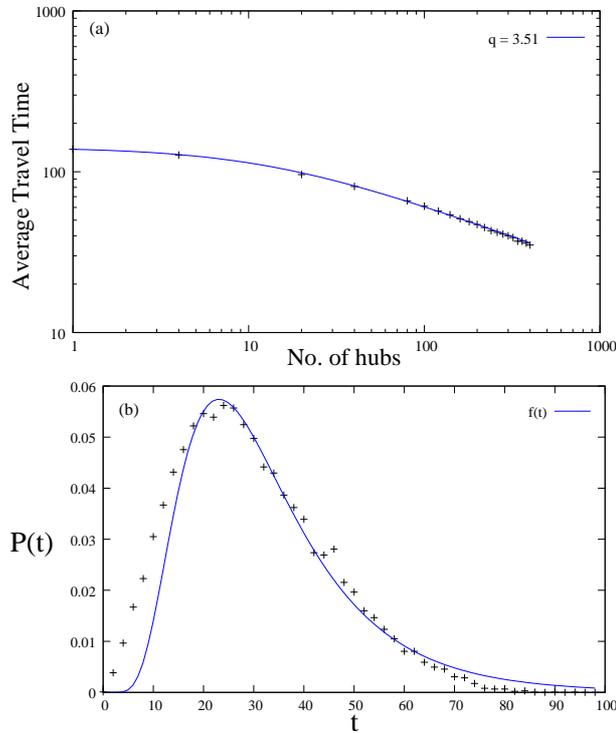
Figure 2. (**a**) The average travel time as a function of hub density follows a q-exponential behaviour if the hubs are connected by the gradient mechanism. Here $q = 3.51$ for the gradient mechanism for a 100×100 lattice with 4% hub density and $D_{st} = 142$. (**b**) The travel time distribution for the gradient mechanism. The data is fitted by a log-normal distribution (eq. (1)) where $\mu = 7.76$, $\sigma = 1.24$.
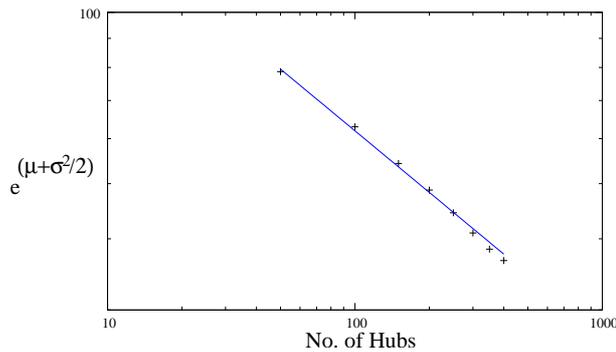
Figure 3. Plot of $\exp(\mu + \sigma^2/2)$ as a function of N shows a power-law behaviour $(f(x) = Bx^{-\beta})$ for the gradient mechanism. Here $\beta = 0.36 \pm 0.008$ for 100×100 lattice and $D_{st} = 142$.
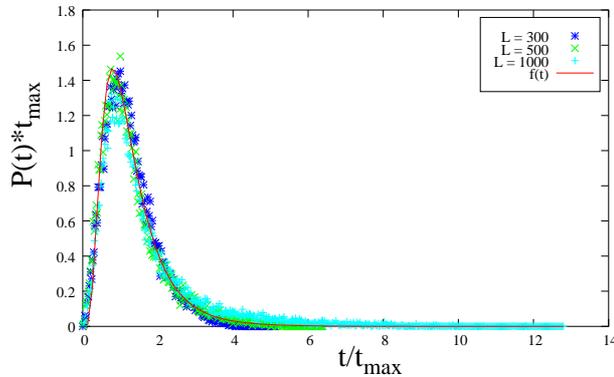
Figure 4. The scaled travel time distribution for the gradient mechanism. Different symbols represent lattices of different sizes. The hub density is 0.5% for all the lattice sizes. Here $f(t)$ is as in eq. (1).

$\langle t \rangle$, the average travel time at high hub density is also expected to show power-law behaviour with a similar exponent. This explains the power-law tail of the average travel time for single message transfer plotted in figure 2a where the exponent $\beta = 0.36 \pm 0.006$ [8] is seen.

Finite size scaling is observed for the average travel time as a function of hub density. For details, see ref. [8]. The scaled behaviour of travel time distribution fits into a log-normal distribution for 0.5% hub density (figure 4) where $\mu = -1.44$, $\sigma = 1.47$. Here t_{max} is the value of t at which $P(t)$ is maximum for the unscaled travel time distributions. See ref. [8] for details.

4. Congestion and decongestion

In this section we consider a large number of messages which are created at the same time and travel towards their destinations simultaneously. The hubs on the lattice, and the manner in which they are connected, are crucial elements which control the efficiency of message transfer. On the one hand, it is clear that the hubs provide short paths through the lattice. On the other hand, when many messages travel simultaneously on the network, the finite capacity of the hubs can lead to the trapping of messages in their neighbourhoods, and a consequent congestion or jamming of the network. A crucial quantity which identifies these hubs is called the coefficient of betweenness centrality (CBC) [10], defined to be the ratio of the number of messages N_k which pass through a given hub k to the total number of messages which run simultaneously, i.e. $CBC = N_k/N$. Hubs with higher CBC are more prone to congestion. Here we set up a gradient by identifying the top five hubs ranked by their CBC values and connecting them by the gradient mechanism [8]. Here, the capacities of the top five hubs are enhanced proportional to their CBC values and a gradient is set up between them.

We choose N source–target pairs randomly, separated by a fixed distance D_{st} on the lattice. All sources send messages simultaneously to their respective targets

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at an initial time $t = 0$. The messages are transmitted by a routing mechanism similar to that for single messages, except when the next node or hub on the route is occupied. We carry out parallel updates of nodes.

If the would be recipient node is occupied, then the message waits for a unit time step at the current message holder. If the desired node is still occupied after the waiting time is over, the current node selects any unoccupied node from its remaining neighbours and hands over the message. If all the neighbouring nodes are occupied, the message waits at the current node until one of them is free. If the current message holder is the constituent node of a hub which is occupied, the message waits at the constituent node until the hub is free.

We plot the waiting time distributions for the gradient mechanism mentioned above for lower and higher hub densities. We calculate the total waiting time for all the nodes where the messages wait to be delivered to adjacent nodes along the path of their journey to respective targets. It was observed earlier that for low hub density messages get trapped [8] at some hubs which lead to congestion in the network. The network is decongested for higher hub densities [8,10].

We allow 2000 messages to run simultaneously in a 100×100 lattice and $D_{\text{st}} = 142$, for 50 hubs and run time of $30D_{\text{st}}$. During this phase, the messages remain undelivered in the lattice and the system is in the congested regime. The waiting time distribution fits into a Gaussian of the form

$$f(w) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(w-a)^2}{2\sigma^2}\right). \tag{2a}$$

For 400 hubs and $4D_{\text{st}}$ all the messages get delivered to their targets and the data for waiting time distribution fits into a log-normal behaviour of the form

$$f(w) = \frac{1}{w\sigma\sqrt{2\pi}}\exp\left(-\frac{(\ln w - \mu)^2}{2\sigma^2}\right). \tag{2b}$$

Thus it is evident that during the congested phase the waiting time distribution for messages travelling simultaneously in the network, fits into a Gaussian as compared to the log-normal fit during the decongested phase. In the decongested phase, messages are not trapped in the lattice and all the messages reach their respective destinations [11]. This congestion–decongestion transition is encrypted in the cross-over from Gaussian to log-normal behaviour in the waiting time distribution for messages travelling in the system.

To summarize, in this paper, we have studied single message and multiple message transport in a gradient network based on a substrate lattice of nodes and hubs. The average travel time on this lattice shows q-exponential behaviour as a function of hub density. The power-law tail of this behaviour can be explained in terms of the log-normal distribution of travel times seen at high hub densities.

Congestion effects are observed when many messages run simultaneously on the base network. At low hub densities the waiting time distribution in this regime fits a Gaussian. At higher hub densities the network is in the decongested phase and the waiting time distribution shows log-normal behaviour. We note that networks which incorporate geographic clustering and encounter congestion problems arise in many practical situations, e.g. cellular networks [12] and air traffic networks [13]. Our results may have relevance in these contexts.
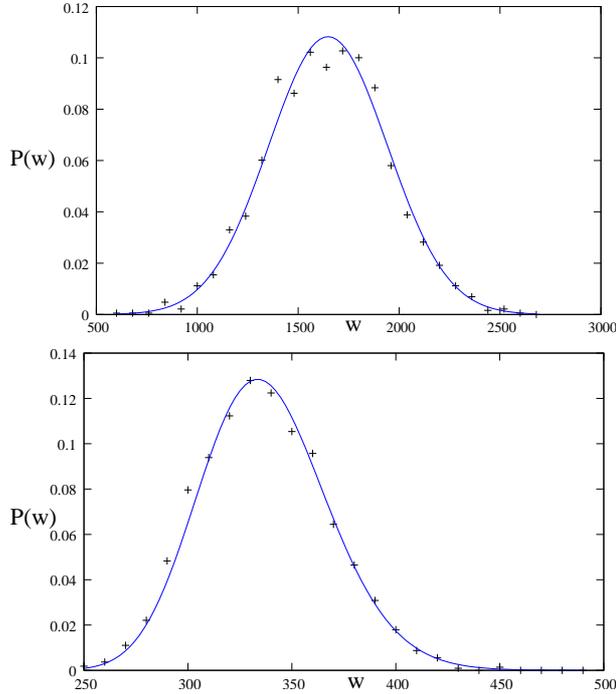
Figure 5. Waiting time distribution for 2000 messages running simultaneously in the lattice for (**a**) 50 hubs and run time of $30D_{st}$ and (**b**) 400 hubs and run time of $4D_{st}$ when various decongestion strategies are applied. In (a) the waiting time distribution for messages fits into a Gaussian of the form $\frac{1}{\sigma\sqrt{2\pi}}\exp(-\frac{(w-a)^2}{2\sigma^2})$. The system is congested due to the formation of transport traps. The standard deviation $\sigma = 294.88$. In (b) all the messages reach their respective destinations. In this case the waiting time distribution fits into a log-normal behaviour of the form $\frac{1}{w\sigma\sqrt{2\pi}}\exp(-\frac{(\ln w - \mu)^2}{2\sigma^2})$. The standard deviation $\sigma = 0.091$. The averaging is done over 200 hub configurations.

Acknowledgments

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[11] As observed in figure 5a, the standard deviation is quite large as compared to that in figure 5b. This is due to the fact that for low hub density (0.5% as in figure 5a) messages get trapped leading to large fluctuations. The fluctuations are reduced once the messages get cleared for higher hub density (4.0% as in figure 5b)

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