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Solutions of two-mode Jaynes–Cummings models

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Abstract. A simple procedure to solve two fully quantized non-linear Jaynes–Cummings models is presented, one in which an atom interacts with a two-mode radiation field in a Raman-type process and the other involving multiphoton interaction between the two-mode field and the atom. Effect of intensity-dependent coupling between the field and the atom in both the above-mentioned cases has also been investigated. The unitary transformation method presented here not only solves the time-dependent problem but also permits a determination of the eigensolutions of the interacting Hamiltonian at the same time. Graphical features of the time dependence of the population inversion have been analysed when one of the field modes is prepared initially in a coherent state while the other one in a vacuum state.

Keywords. Jaynes–Cummings (J–C) model; rotating wave approximation (RWA); unitary operator $\hat{T}$; two-mode Raman model; two-mode k-photon interaction ($2k$ interaction) model; intensity-dependent coupling; atomic population inversion $W(t)$.

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1. Introduction

The system consisting of a two-level atom coupled to a single quantized mode of the radiation field under the rotating-wave approximation is known as the Jaynes–Cummings model [1]. The model has been used to obtain exact solutions in a wide variety of problems [2–8] in quantum optics and it predicts many new features in the limit of strong coupling g of the atom and the cavity mode.

Over the last three decades, there has been intensive study [9–11] on the solvable Jaynes–Cummings (J–C) model and its various extensions. In the existing theoretical works, much attention has been paid to the detailed discussion on single-mode two-photon process [12–16]. However, there have been few publications concerning the nondegenerate case. Gou [17] has studied the dynamical behavior of a two-level atom interacting with two modes of light in a cavity and has obtained some results such as the population probability, photon distribution etc. Using time-dependent Schrödinger equation, Cardimona *et al* [18] have investigated the atomic dynamics for revivals and secondary revivals of the initial radiation field when a multilevel atom interacts with two-mode quantized radiation field in a Raman type process.

In the present paper the eigenfunctions and eigenvalues of the Hamiltonian of the interacting system have been obtained in two Jaynes–Cummings models separately, one in which the transition are mediated by two different modes of photon and the other involving two-mode k-photon interaction between the field and the atom. Investigations have also been done for the intensity-dependent coupling between the field and the atom in both the above-mentioned cases. Graphical features of the time dependence of the population inversion have been analysed when one of the field modes is prepared initially in a coherent state while the other one in a vacuum state.

The paper aims to show that a straightforward method, that of unitary transformation in quantum mechanics by Sudha Singh and Roy [19,20] can be conveniently used to obtain the eigenfunctions and eigenvalues of the Hamiltonian of the interacting system. The method, besides being general, is mathematically simpler.

2. Two-mode Raman model

In dealing with two-mode Raman-type processes, a three-level system of energies E_1, E_2 and E_3 in the Λ configuration is considered to be interacting with a pump mode ω_1 and a Stoke mode ω_2 [18,21,22].

The Hamiltonian of the system is written as [21–24]

$$\hat{H} = \sum_{i=1}^{3} E_i\sigma_{ii} + \hbar\omega_1\hat{a}_1^\dagger\hat{a}_1 + \hbar\omega_2\hat{a}_2^\dagger\hat{a}_2 + \hbar g_1(\hat{a}_1\hat{\sigma}_{31} + \hat{a}_1^\dagger\sigma_{13}) + \hbar g_2(\hat{a}_2\hat{\sigma}_{32} + \hat{a}_2^\dagger\hat{\sigma}_{23}), \tag{1}$$

where symbols $\hat{a}_j$ and $\hat{a}_j^\dagger$ $(j = 1, 2)$ represent the field annihilation and creation operators for modes 1 and 2, $\hat{\sigma}_{ii} = |i\rangle\langle i|$ are the level occupation number and $\hat{\sigma}_{ij} = |i\rangle\langle j|(i \neq j)$ are the transition operators from level j to i. Levels 3 and 1(2) are coupled by a dipole-coupling constant $g_1(g_2)$. There is no direct coupling between levels 1 and 2. The quantities Δ_1 and Δ_2 denote de-tuning given by $\Delta_j = (E_3 - E_j)/\hbar - \omega_j, j = 1, 2$. It has been shown by Wu [23] that this three-level problem can be exactly transformed into a two-level problem, regardless of whether the de-tuning is small or large. The corresponding two-level Hamiltonian under the rotating wave approximation (RWA) reads as [18,21–24]

$$\hat{H} = \frac{1}{2}\hbar\omega_0\hat{\sigma}_3 + \hbar\omega_1\hat{a}_1^\dagger\hat{a}_1 + \hbar\omega_2\hat{a}_2^\dagger\hat{a}_2 + \hbar g(\hat{a}_2^\dagger\hat{a}_1\sigma_+ + \hat{a}_1^\dagger\hat{a}_2\sigma_-). \tag{2}$$

The pump radiation mode ω_1 and the Stokes radiation mode ω_2 are in two-mode resonance with the $|1\rangle$ to $|2\rangle$ transition such that the frequency difference $\omega_1 - \omega_2$ is roughly equal to the transition frequency ω_0 of the two atomic levels. This Hamiltonian ignores Stark shifts due to coupling through virtual $|j\rangle$ levels. The $\hat{\sigma}$'s are 2×2 Pauli matrices

$$\hat{\sigma}_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \hat{\sigma}_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \hat{\sigma}_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The $\hat{\sigma}$'s and the $\hat{a}$'s obey the following commutation relations:

$$[\hat{\sigma}_3, \hat{\sigma}_\pm] = \pm 2\hat{\sigma}_\pm, \quad [\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_3, \quad [\hat{a}, \hat{a}^\dagger] = 1.$$

Denoting by $|a\rangle$ and $|b\rangle$ respectively the higher and lower eigenkets of the isolated atom and by $|n_1\rangle$ and $|n_2\rangle$ the eigenstate of the free field with frequency ω_1 and ω_2, the basis eigenkets of the interacting system can be designated by $|a, n_1, n_2\rangle$ and $|b, n_1, n_2\rangle$ and the state vector for the system at any time t is then

$$|\psi(t)\rangle = \sum_{n_1,n_2} a_{n_1,n_2}(t)|a, n_1, n_2\rangle + \sum_{n_1,n_2} b_{n_1,n_2}(t)|b, n_1, n_2\rangle. \tag{3}$$

The state vector $|\psi(t)\rangle$ develops from the state vector $|\psi(0)\rangle$ at $t = 0$ according to

$$|\psi(t)\rangle = \hat{T}(t)|\psi(0)\rangle, \tag{4}$$

where the unitary operator $\hat{T}(t)$ satisfies

$$i\hbar \frac{\mathrm{d}\hat{T}(t)}{\mathrm{d}t} = \hat{H}\hat{T}(t). \tag{5}$$

In the Heisenberg representation, at two-photon resonance in which case $\omega_1 - \omega_2 \approx \omega_0$, the time dependence of $\hat{H}$ given by eq. (2) drops out. A direct integration of eq. (5) with $\hat{H}$ given by eq. (2) then yields

$$\hat{T} = \exp\left[-it\left\{\frac{\omega_0}{2}\hat{\sigma}_3 + \omega_1\hat{a}_1^\dagger\hat{a}_1 + \omega_2\hat{a}_2^\dagger\hat{a}_2 + g(D_+ + D_-)\right\}\right], \tag{6}$$

where

$$D_+ = \hat{a}_2^\dagger\hat{a}_1\sigma_+ \quad \text{and} \quad D_- = \hat{a}_1^\dagger\hat{a}_2\sigma_-. \tag{7}$$

Considering the initial state of the system as $|a, n_1, n_2\rangle$ we get from (4)

$$|\psi(t)\rangle = \hat{T}(t)|a, n_1, n_2\rangle. \tag{8}$$

The right-hand side of eq. (8) can be obtained by expanding $\hat{T}(t)$ given by eq. (6) and then operating $|a, n_1, n_2\rangle$ by each term of the expansion. When this is done, eq. (8) becomes

$$\begin{aligned} |\psi(t)\rangle = \Bigg[& 1 - it\left(\omega_1 n_1 + \omega_2 n_2 + \frac{\omega_0}{2}\right) \\ & -\frac{t^2}{2!}\left\{\left(\omega_1 n_1 + \omega_2 n_2 + \frac{\omega_0}{2}\right)^2 + g^2 n_2(n_1+1)\right\} \\ & +\frac{it^3}{3!}\left\{\left(\omega_1 n_1 + \omega_2 n_2 + \frac{\omega_0}{2}\right)^3 + g^2(n_1+1)n_2 \right. \end{aligned}$$

$$\times\left(2\omega_1 n_1+\omega_1(n_1+1)+2\omega_2 n_2+\frac{\omega_0}{2}+\omega_2(n_2-1)\right)\Bigg\}+\cdots\Bigg]|a,n_1,n_2\rangle$$
$$+\Bigg[-igt\sqrt{(n_1+1)}\sqrt{n_2}-g\left(\frac{t^2}{2!}\right)\{(\sqrt{(n_1+1)}\sqrt{n_2})(\omega_1(n_1+1)$$
$$+\omega_1 n_1+\omega_2(n_2-1)+\omega_2 n_2)\}+\frac{igt^3}{3!}\Bigg\{(\sqrt{(n_1+1)}\sqrt{n_2})(g^2(n_1+1)n_2$$
$$+\omega_2^2(n_2-1)^2+\omega_2^2 n_2(n_2-1)+\omega_2^2 n_2^2+\left(\frac{\omega_0}{2}\right)^2$$
$$+\omega_1^2(n_1+1)^2+\omega_1^2 n_1(n_1+1)+\omega_1^2 n_1^2-\frac{\omega_0}{2}\omega_1$$
$$+2\omega_1\omega_2(n_1+1)(n_2-1)+\frac{\omega_0}{2}\omega_2+2\omega_1 n_1\omega_2 n_2+\omega_1(n_1+1)\omega_2 n_2$$
$$+\omega_2(n_2-1)\omega_1 n_1)\Bigg\}+\cdots\Bigg]|b,n_1+1,n_2-1\rangle. \tag{9}$$

At resonance in which case $\omega_1-\omega_2\approx\omega_0$, the operators D_+ and D_- become time-dependent in the Heisenberg picture and the following properties that have been used in obtaining the above follow from the definitions of the operators

$$\begin{aligned}
&\hat{a}_1^\dagger\hat{a}_1|a,n_1,n_2\rangle=n_1|a,n_1,n_2\rangle;\\
&\hat{a}_2^\dagger\hat{a}_2|a,n_1,n_2\rangle=n_2|a,n_1,n_2\rangle,\\
&D_+|a,n_1,n_2\rangle=\hat{a}_1\hat{a}_2^\dagger\sigma_+|a,n_1,n_2\rangle=0;\\
&D_-|b,n_1,n_2\rangle=\hat{a}_1^\dagger\hat{a}_2\sigma_-|b,n_1,n_2\rangle=0,\\
&D_+|b,n_1,n_2\rangle=\sqrt{n_1}\sqrt{n_2+1}|a,n_1-1,n_2+1\rangle;\\
&D_-|a,n_1,n_2\rangle=\sqrt{n_1+1}\sqrt{n_2}|b,n_1+1,n_2-1\rangle,\\
&D_+D_-|a,n_1,n_2\rangle=n_2(n_1+1)|a,n_1,n_2\rangle;\\
&D_-D_+|b,n_1,n_2\rangle=n_1(n_2+1)|b,n_1,n_2\rangle.
\end{aligned}\tag{10}$$

In the light of eqs (3) and (8) it is seen that the coefficient of $|a,n_1,n_2\rangle$ and $|b,n_1+1,n_2-1\rangle$ on the right-hand side of eq. (9) gives the probability amplitude $a_{n_1,n_2}(t)$ and $b_{n_1+1,n_2-1}(t)$. Collecting the coefficient of $|b,n_1+1,n_2-1\rangle$ on the right-hand side of eq. (9) and making algebraic simplifications we obtain the transition probability for transition of the atom from the higher to the lower state as

$$|b_{n_1+1,n_2-1}|^2=\frac{4g^2(n_1+1)n_2}{[\Delta^2+4g^2(n_1+1)n_2]}\left[\sin^2\frac{t}{2}\sqrt{\Delta^2+4g^2(n_1+1)n_2}\right], \tag{11}$$

where $\Delta=\{(\omega_1-\omega_2)-\omega_0\}$.

Considering the initial state of the system as $|b,n_1,n_2\rangle$ the transition probability for transition of the atom from the lower to the higher state is similarly obtained as

$$|a_{n_1-1,n_2+1}|^2 = \frac{4g^2(n_2+1)n_1}{[\Delta^2+4g^2(n_2+1)n_1]} \times \left[\sin^2 \frac{t}{2}\sqrt{\Delta^2+4g^2(n_2+1)n_1}\right]. \tag{12}$$

At resonance, $\omega_1 - \omega_2 \approx \omega_0$ and eqs (11) and (12) become

$$|b_{n_1+1,n_2-1}|^2 = \sin^2\left[gt\sqrt{(n_1+1)n_2}\right], \tag{13}$$

$$|a_{n_1-1,n_2+1}|^2 = \sin^2\left[gt\sqrt{(n_2+1)n_1}\right]. \tag{14}$$

It follows from eq. (9) that the only eigenstates of the interacting system are $|a, n_1, n_2\rangle$ and $|b, n_1+1, n_2-1\rangle$. The occurrence of the transformed state $|a, n_1, n_2\rangle$ corresponds to no interaction whereas the occurrence of the state $|b, n_1+1, n_2-1\rangle$ admits interaction in which one photon of mode ω_1 is emitted whereas another photon of mode ω_2 is absorbed resulting in two-mode process. It is easy to see from eq. (13) that the transition probability and hence the energy emitted depends upon $\sqrt{n_2(n_1+1)}$ in place of $\sqrt{(n+1)}$ in J–C model and $\sqrt{(n+1)(n+2)}$ in the two-photon single mode J–C model [9,15,16]. The atomic population inversion $W(t)$, defined as the difference in the excited and ground state populations is given by

$$W(t) = |a_{n_1,n_2}(t)|^2 - |b_{n_1+1,n_2-1}(t)|^2 = \cos 2gt\sqrt{(n_1+1)n_2}.$$

The Rabi frequency $\Omega(n_1, n_2) = 2g\sqrt{(n_1+1)n_2}$ couples the two modes so that $W(t) = \cos[\Omega(n_1, n_2)t]$. For the two field modes initially prepared in a coherent state (written as infinite sum over a Poisson distribution of photon numbers) the population inversion is given by

$$W(t) = \sum_{n_1=0}^{\infty} P(n_1) \sum_{n_2=0}^{\infty} P(n_2) \cos\left[2gt\sqrt{(n_1+1)n_2}\right]. \tag{15}$$

For the normalized Poisson distribution

$$P(n) = \mathrm{e}^{-\bar{n}}\frac{\bar{n}^n}{n!},$$

where $\bar{n}$ is the average number of photons.

Cardimona *et al* [18] have shown that the double sum in eq. (15) can be broken up into a series of Buck–Sukumar [9]-like sums. We, on the other hand, study the dynamics of the population inversion in the special case when the Stokes radiation mode is prepared initially in a coherent state whereas the pump radiation mode in a vacuum state that is, for $n_1 = 0$ and

$$P(n_2) = \mathrm{e}^{-\bar{n}_2}\frac{\bar{n}_2^{n_2}}{n_2!}.$$

The expression (15) for population inversion then becomes

$$W(t) = \mathrm{e}^{-\bar{n}_2} \sum_{n_2=0}^{\infty} \frac{\bar{n}_2^{n_2}}{n_2!} \cos[2gt\sqrt{n_2}]. \tag{16}$$

3. Two-mode multiphoton interaction model

In the RWA the Hamiltonian for two-mode electromagnetic field interacting with a two-level atom via k-photon process is given by [10,24]

$$\hat{H} = \frac{1}{2}\hbar\omega_0\hat{\sigma}_3 + \hbar\omega_1\hat{a}_1^\dagger\hat{a}_1 + \hbar\omega_2\hat{a}_2^\dagger\hat{a}_2 + \hbar g[(\hat{a}_2^\dagger)^k\hat{a}_1^k\sigma_+ + (\hat{a}_1^\dagger)^k\hat{a}_2^k\sigma_-]. \quad (17)$$

Defining the operators

$$\hat{E}_+ = (\hat{a}_2^\dagger)^k(\hat{a}_1^k)\hat{\sigma}_+, \quad \hat{E}_- = (\hat{a}_1^\dagger)^k(\hat{a}_2^k)\hat{\sigma}_-, \quad (18)$$

the Hamiltonian given by eq. (17) reads

$$\hat{H} = \frac{1}{2}\hbar\omega_0\hat{\sigma}_3 + \hbar\omega_1\hat{a}_1^\dagger\hat{a}_1 + \hbar\omega_2\hat{a}_2^\dagger\hat{a}_2 + \hbar g(\hat{E}_+ + \hat{E}_-). \quad (19)$$

Since $\hat{H}$ given by eq. (17) is time-independent in the Heisenberg representation under the condition $k(\omega_1 - \omega_2) \approx \omega_0$, a straightforward integration of eq. (5) with $\hat{H}$ given by eq. (17) gives the transformation operator representing the $2k$ (two-mode k photon) interaction as

$$\hat{T}(t) = \exp\left[-it\left\{\frac{\omega_0}{2}\hat{\sigma}_3 + \omega_1\hat{a}_1^\dagger\hat{a}_1 + \omega_2\hat{a}_2^\dagger\hat{a}_2 + g(\hat{E}_+ + \hat{E}_-)\right\}\right]. \quad (20)$$

As in §2, the right-hand side of eq. (8) is obtained by expanding $\hat{T}(t)$ given by eq. (20) and then operating $|a, n_1, n_2\rangle$ by each term of the expansion. When this is done we get

$$\begin{aligned}
|\psi(t)\rangle = \Bigg[&1 - it\left(\omega_1 n_1 + \omega_2 n_2 + \frac{\omega_0}{2}\right) - \frac{t^2}{2!}\Bigg\{\left(\omega_1 n_1 + \omega_2 n_2 + \frac{\omega_0}{2}\right)^2 \\
&+g^2(n_1+1)(n_1+2)\cdots(n_1+k)n_2(n_2-1)\cdots(n_2-k+1)\Bigg\} \\
&+\frac{it^3}{3!}\Bigg\{\left(\omega_1 n_1 + \omega_2 n_2 + \frac{\omega_0}{2}\right)^3 \\
&+g^2(n_1+1)(n_1+2)\cdots(n_1+k)n_2(n_2-1)\cdots(n_2-k+1) \\
&\times\Bigg(\frac{\omega_0}{2} + 2\omega_1 n_1 + \omega_1(n_1+k) \\
&+2\omega_2 n_2 + \omega_2(n_2-k)\Bigg)\Bigg\} + \cdots\Bigg]|a, n_1, n_2\rangle \\
&+\Bigg[-igt\sqrt{(n_1+1)(n_1+2)\cdots(n_1+k)n_2(n_2-1)\cdots(n_2-k+1)} \\
&-\left(\frac{t^2}{2!}\right)\Bigg\{(g\sqrt{(n_1+1)(n_1+2)\cdots(n_1+k)n_2(n_2-1)\cdots(n_2-k+1)}) \\
&\times(\omega_1(n_1+k) + \omega_1 n_1 + \omega_2(n_2-k) + \omega_2 n_2)\Bigg\}
\end{aligned}$$

$$+\frac{it^3}{3!}\Bigg\{\left(g\sqrt{(n_1+1)(n_1+2)\cdots(n_1+k)n_2(n_2-1)\cdots(n_2-k+1)}\right)$$
$$\times\Bigg(\frac{\omega_0^2}{4}+\omega_1^2(n_1+k)^2+\omega_1^2 n_1(n_1+k)+g^2(n_1+1)$$
$$\times(n_1+2)\cdots(n_1+k)n_2(n_2-1)\cdots(n_2-k+1)$$
$$+\omega_1^2 n_1^2+\omega_2^2(n_2-k)^2+\omega_2^2 n_2(n_2-k)+\omega_2^2 n_2^2+k\omega_2\frac{\omega_0}{2}$$
$$-k\omega_1\frac{\omega_0}{2}+\omega_1(n_1+k)\omega_2 n_2+2\omega_1\omega_2(n_1+k)(n_2-k)$$
$$+2\omega_1 n_1\omega_2 n_2+\omega_1 n_1\omega_2(n_2-k)\Bigg)\Bigg\}+\cdots\Bigg]|b,n_1+k,n_2-k\rangle. \tag{21}$$

At $2k$ photon (two-mode k-photon) resonance in which case $k(\omega_1-\omega_2)\approx\omega_0$ the operators $\hat{E}_+$ and $\hat{E}_-$ become time-independent in the Heisenberg picture and the following properties that have been used in obtaining the above follow from the definitions of the operators

$$\begin{aligned}
&\hat{E}_+|b,n_1,n_2\rangle\\
&\quad=\sqrt{[(n_2+1)(n_2+2)\cdots(n_2+k)n_1(n_1-1)\cdots(n_1-k+1)]}\\
&\qquad\times|a,n_1-k,n_2+k\rangle;\\
&\hat{E}_-|a,n_1,n_2\rangle\\
&\quad=\sqrt{[(n_1+1)(n_1+2)\cdots(n_1+k)n_2(n_2-1)\cdots(n_2-k+1)]}\\
&\qquad\times|b,n_1+k,n_2-k\rangle;\\
&\hat{E}_+\hat{E}_-|a,n_1,n_2\rangle\\
&\quad=[(n_1+1)(n_1+2)\cdots(n_1+k)n_2(n_2-1)\cdots(n_2-k+1)]\\
&\qquad\times|a,n_1,n_2\rangle;\\
&\hat{E}_-\hat{E}_+|b,n_1,n_2\rangle\\
&\quad=[n_1(n_1-1)\cdots(n_1-k+1)(n_2+1)(n_2+2)\cdots(n_2+k)]\\
&\qquad\times|b,n_1,n_2\rangle;\\
&\hat{E}_+|a,n_1,n_2\rangle=0;\quad \hat{E}_-|b,n_1,n_2\rangle=0;\\
&\qquad\hat{E}_+\hat{E}_-|b,n_1,n_2\rangle=0;\quad \hat{E}_-\hat{E}_+|a,n_1,n_2\rangle=0;\ldots.
\end{aligned}\tag{22}$$

Collecting the coefficients of $|b,n_1+k,n_2-k\rangle$ on the right-hand side of eq. (21) and making algebraic simplifications the transition probability for $2k$ (two-mode k-photon) process is obtained as

$$|b_{n_1+k,n_2-k}(t)|^2=\frac{4g^2Y}{[4g^2Y+\{k(\omega_1-\omega_2)-\omega_0\}^2]}\times\left[\sin^2\frac{t}{2}\sqrt{4g^2Y+\{k(\omega_1-\omega_2)-\omega_0\}^2}\right],$$

where $Y=(n_1+1)(n_1+2)\cdots(n_1+k)n_2(n_2-1)\cdots(n_2-k+1)$.

The transition probability for $2k$-photon process under the condition $k(\omega_1-\omega_2) \approx \omega_0$ is obtained as

$$|b_{n_1+k,n_2-k}(t)^2 = \left[\sin^2 tg\sqrt{(n_1+1)(n_1+2)\cdots(n_1+k)n_2(n_2-1)\cdots(n_2-k+1)}\right]. \tag{23}$$

The atomic population inversion and Rabi frequency are then given by

$$W(t) = \cos 2gt\sqrt{(n_1+1)(n_1+2)\cdots(n_1+k)n_2(n_2-1)\cdots(n_2-k+1)} \tag{24}$$

and

$$\Omega(n_1,n_2) = 2g\sqrt{(n_1+1)(n_1+2)\cdots(n_1+k)n_2(n_2-1)\cdots(n_2-k+1)}. \tag{25}$$

In the situation when Stokes radiation mode is initially prepared in a coherent state and the pump mode in a vacuum state the expression (24) for population inversion becomes

$$W(t) = \mathrm{e}^{-\bar{n}_2}\sum_{n_2=0}^{\infty}\frac{\bar{n}_2^{n_2}}{n_2!}\cos\left[2gt\sqrt{k!n_2(n_2-1)\cdots(n_2-k+1)}\right]. \tag{26}$$

For $k=2$, i.e. for two-mode two-photon case the above expression yields

$$W(t) = \mathrm{e}^{-\bar{n}_2}\sum_{n_2=0}^{\infty}\frac{\bar{n}_2^{n_2}}{n_2!}\cos\left[2gt\sqrt{2n_2(n_2-1)}\right]. \tag{27}$$

4. Two-mode Raman model with an intensity-dependent coupling

For two-mode Raman model with intensity-dependent coupling the Hamiltonian is obtained from eq. (2) as [9,10,24]

$$\hat{H} = \frac{\hbar\omega_0\hat{\sigma}_3}{2} + \hbar\omega_1\hat{a}_1^{\dagger}\hat{a}_1 + \hbar\omega_2\hat{a}_2^{\dagger}\hat{a}_2 + \hbar g(\hat{L}_+ + \hat{L}_-), \tag{28}$$

where

$$\hat{L}_+ = \sqrt{\hat{a}_2^{\dagger}\hat{a}_2}\hat{a}_2^{\dagger}\hat{a}_1\sqrt{\hat{a}_1^{\dagger}\hat{a}_1}\sigma_+ \quad \text{and} \quad \hat{L}_- = \sqrt{\hat{a}_1^{\dagger}\hat{a}_1}\hat{a}_1^{\dagger}\hat{a}_2\sqrt{\hat{a}_2^{\dagger}\hat{a}_2}\sigma_-. \tag{29}$$

In the Heisenberg representation since $\hat{H}$ given by eq. (28) is time-independent under the condition $\omega_1-\omega_2 \approx \omega_0$, an integration of eq. (5) with $\hat{H}$ given by eq. (28) gives the transformation operator representing the two-mode Raman model with an intensity-dependent coupling as

$$\hat{T} = \exp\left[-it\left\{\frac{\omega_0}{2}\hat{\sigma}_3 + \omega_1\hat{a}_1^\dagger\hat{a}_1 + \omega_2\hat{a}_2^\dagger\hat{a}_2 + g(\hat{L}_+ + \hat{L}_-)\right\}\right]. \quad (30)$$

Considering the initial state of the system as $|a, n_1, n_2\rangle$ and proceeding as in §2, the transition probability is obtained as

$$|b_{n_1+1,n_2-1}|^2 = \frac{4g^2(n_1+1)^2 n_2^2}{[\Delta^2 + 4g^2(n_1+1)^2 n_2^2]}\left[\sin^2\frac{t}{2}\sqrt{\Delta^2 + 4g^2(n_1+1)^2 n_2^2}\right]. \quad (31)$$

At the resonance, $\omega_1 - \omega_2 \approx \omega_0$ and eq. (31) becomes

$$|b_{n_1+1,n_2-1}|^2 = \sin^2[gt(n_1+1)n_2]. \quad (32)$$

The expression for atomic population inversion is then given by $W(t) = \cos 2gt(n_1+1)n_2$. From eq. (32) it is seen that in two-mode Raman model with intensity-dependent coupling the energy emitted depends upon $n_2(n_1+1)$.

If initially the Stokes radiation mode is prepared in a coherent state and the pump radiation mode in a vacuum state, the expression for population inversion is obtained as

$$W(t) = \mathrm{e}^{-\bar{n}_2}\sum_{n_2=0}^{\infty}\frac{\bar{n}_2^{n_2}}{n_2!}\cos[2gtn_2]. \quad (33)$$

5. Two-mode multiphoton interaction model with intensity-dependent coupling

For an intensity-dependent coupling two-mode multiphoton interaction model the Hamiltonian is obtained from eq. (17) as [9,10,24]

$$\hat{H} = \frac{1}{2}\hbar\omega_0\hat{\sigma}_3 + \hbar\omega_1\hat{a}_1^\dagger\hat{a}_1 + \hbar\omega_2\hat{a}_2^\dagger\hat{a}_2 + \hbar g(\hat{M}_+ + \hat{M}_-), \quad (34)$$

where

$$\hat{M}_+ = \sqrt{\hat{a}_2^\dagger\hat{a}_2}(\hat{a}_2^\dagger)^k\hat{a}_1^k\sqrt{\hat{a}_1^\dagger\hat{a}_1}\sigma_+ \quad \text{and} \quad \hat{M}_- = \sqrt{\hat{a}_1^\dagger\hat{a}_1}(\hat{a}_1^\dagger)^k\hat{a}_2^k\sqrt{\hat{a}_2^\dagger\hat{a}_2}\sigma_-. \quad (35)$$

Since $\hat{H}$ given by eq. (34) is time-independent under the condition $k(\omega_1-\omega_2) \approx \omega_0$, a straightforward integration of eq. (5) with $\hat{H}$ given by eq. (34) gives the transformation operator representing the $2k$ (two-mode k photon) intensity-dependent interaction as

$$\hat{T}(t) = \exp\left[-it\left\{\frac{\omega_0}{2}\hat{\sigma}_3 + \omega_1\hat{a}_1^\dagger\hat{a}_1 + \omega_2\hat{a}_2^\dagger\hat{a}_2 + g(\hat{M}_+ + \hat{M}_-)\right\}\right]. \quad (36)$$

Considering the initial state of the system as $|a, n_1, n_2\rangle$ and proceeding as in §3, the transition probability for $2k$ (two-mode k-photon) intensity-dependent interaction is obtained as

$$|b_{n_1+k,n_2-k}(t)|^2 = \frac{4g^2A}{[4g^2A + \{k(\omega_1 - \omega_2) - \omega_0\}^2]} \times \left[\sin^2 \frac{t}{2}\sqrt{4g^2A + \{k(\omega_1 - \omega_2) - \omega_0\}^2}\right],$$

where $A = (n_1 + k)(n_1 + 1)(n_1 + 2)\cdots(n_1 + k)n_2 n_2(n_2 - 1)\cdots(n_2 - k + 1)$.

The transition probability for $2k$ intensity-dependent process under the condition $k(\omega_1 - \omega_2) \approx \omega_0$ is obtained as

$$|b_{n_1+k,n_2-k}|^2 = [\sin^2 tg(n_1 + k)n_2 \times \sqrt{(n_1 + 1)(n_1 + 2)\cdots(n_1 + k - 1)(n_2 - 1)(n_2 - 2)\cdots(n_2 - k + 1)}]. \quad (37)$$

The expressions for the atomic population inversion and Rabi frequency then become

$$W(t) = \cos 2gt(n_1 + k)n_2 \times \sqrt{(n_1 + 1)(n_1 + 2)\cdots(n_1 + k - 1)(n_2 - 1)(n_2 - 2)\cdots(n_2 - k + 1)}, \quad (38)$$

$$\Omega = 2g(n_1 + k)n_2 \times \sqrt{(n_1 + 1)(n_1 + 2)\cdots(n_1 + k - 1)(n_2 - 1)(n_2 - 2)\cdots(n_2 - k + 1)}. \quad (39)$$

For the case when the Stokes radiation mode is initially prepared in a coherent state and the pump radiation mode in a vacuum state, the expression (38) for population inversion becomes

$$W(t) = \mathrm{e}^{-\bar{n}_2} \sum_{n_2=0}^{\infty} \frac{\bar{n}_2^{n_2}}{n_2!} \times \cos\left[2gtkn_2\sqrt{(k-1)!(n_2 - 1)(n_2 - 2)\cdots(n_2 - k + 1)}\right]. \quad (40)$$

For $k = 2$, i.e. for two-mode two-photon case under intensity-dependent coupling the above expression yields

$$W(t) = \mathrm{e}^{-\bar{n}_2} \sum_{n_2=0}^{\infty} \frac{\bar{n}_2^{n_2}}{n_2!} \cos\left[4gtn_2\sqrt{(n_2 - 1)}\right]. \quad (41)$$

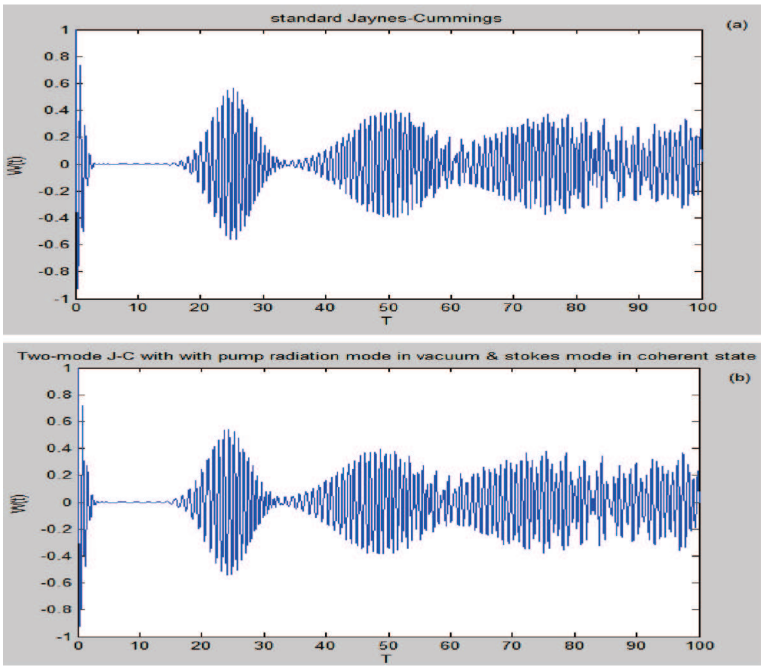

Figure 1. The population inversion dynamics are compared for (**a**) standard J–C model with the field initially in a coherent state $\bar{n} = 15$, (**b**) two-mode Raman model (given by eq. (16)) with the Stokes mode prepared initially in a coherent state $\bar{n}_2 = 15$ while the pump mode in a vacuum state.

6. Discussion and conclusion

In order to study the dynamics of population inversion we plot the atomic population inversion $W(t)$ (for the Stokes mode initially prepared in a coherent state while the pump mode in a vacuum state) versus the normalized time $T = gt$ for the two-mode Raman model (eq. (16)), two-mode two-photon case (eq. (27)) and the two-mode Raman model with intensity–dependent coupling (eq. (33)). Comparisons can be made pertaining to the various models on the basis of figures for the standard Jaynes–Cummings model (figure 1a), two-photon single mode model (figure 2a), to some two-mode models considered (figures 1b, 2b and 2c).

In case of the standard J–C model (figure 1a) for the initial coherent photon distribution the collapse–revival phenomenon in the time-evolution process of the atomic population inversion has been found [25]. As is well understood now, the collapses occur because the many different components in the summation get out of phase initially. After that they acquire a common phase and this process is repeated to obtain a series of collapses and revivals. The revivals are a manifestation of the quantum nature of the electromagnetic field which is mathematically reflected in the discrete summation. The collapse time t_c can be obtained from the time–frequency uncertainty relation $t_c[\Omega(\bar{n}+\Delta n) - \Omega(\bar{n}-\Delta n)] \approx 1$, where $\bar{n}$ is the average photon number. In case of the standard J–C model, $\Omega(\bar{n}) = 2g\sqrt{\bar{n}+1} \approx 2g\sqrt{\bar{n}}$. For the

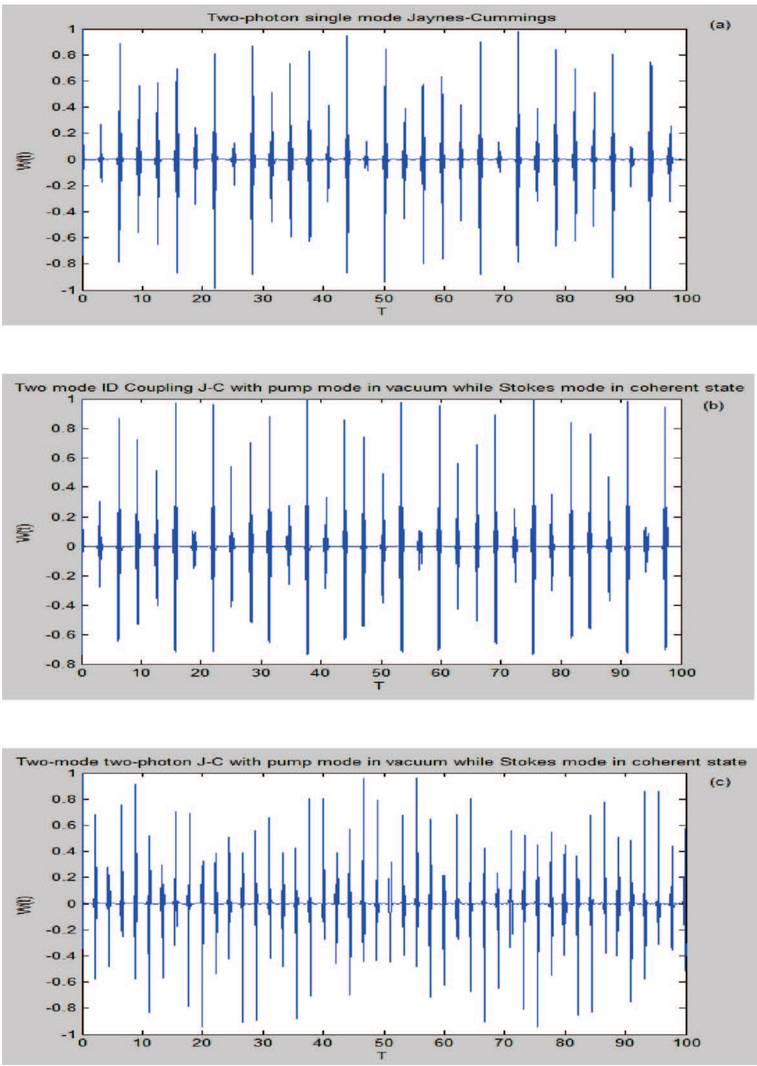

Figure 2. The plot of population inversion $W(t)$ vs. the normalized time $T = gt$ for (**a**) two-photon single mode J–C model with the field initially in a coherent state $\bar{n} = 15$, (**b**) two-mode I–D coupling J–C model given by eq. (33) and (**c**) two-mode two-photon J–C model given by eq. (28). In the case of both (b) and (c), pump radiation mode has been initially prepared in a vacuum state while the Stokes mode in a coherent state $\bar{n}_2 = 15$. These two models behave almost like the exactly periodic degenerate mode case (figure 2a) studied by Buck and Sukumar.

Poisson distribution for the initial coherent field, the root-mean-square deviation in the photon number $\Delta n = \sqrt{\bar{n}}$ so that $\Omega(\bar{n} \pm \bar{n}^{1/2}) \approx 2g[\bar{n} \pm \bar{n}^{1/2}]^{1/2} \approx 2g\bar{n}^{1/2} \pm g$, giving the collapse time $t_{\rm c} = (2g)^{-1}$ which is independent of mean number of photons $\bar{n}$. The intervals between revivals $t_{\rm R}$ are found from the condition $[\Omega(\bar{n}) - \Omega(\bar{n}-1)]t_{\rm R} = 2\pi m$, $m = 1, 2, 3, \ldots$. In the limit $\bar{n} \gg 1$, $t_{\rm R} = (2\pi/g)\sqrt{\bar{n}}m$ indicating that revivals take place at regular intervals. The ratio of collapse to revival time, i.e. $(t_{\rm c}/t_{\rm R}) = 1/4\pi\sqrt{\bar{n}}$.

The standard Jaynes–Cummings model (figure 1a) with the initial coherent photon distribution and two-mode Raman model (figure 1b) de-phase after a certain amount of time due to their incommensurate energy spectrum going as $\sqrt{n}$ (the square root of the quantum number). For the same average photon number the above two models become erratic much sooner than the other models.

The only similarity that all the five models possess is the quantum collapses and revivals.

The two-photon single mode J–C model [9] (figure 2a) sustains these due to the commensurability in its energy $[\propto (n^{1/2})^2]$.

In the two-mode Raman model with an intensity-dependent coupling (figure 2b) and two-mode two-photon J–C model (figure 2c), the spectrum is effectively linear in quantum number. This explains why the dynamics of this model retain their collapse and revival structure for a much longer time than the standard J–C model. The collapse and revival time for the two-mode Raman model with intensity-dependent coupling is obtained as $t'_{\rm c} = 1/4g\sqrt{\bar{n}}$ and $t'_{\rm R} = \pi m/g$. For the two-mode two-photon J–C model (figure 2c), it becomes $t''_{\rm c} = 1/4g\sqrt{2\bar{n}}$ and $t''_{\rm R} = \pi m/\sqrt{2}g$, i.e. decreases by a factor of $\sqrt{2}$. It is interesting to note that the ratio of the collapse to revival time for the two models is the same as in the case of the standard J–C model.

From figures 2b and 2c it is observed that these two models behave almost like the exactly periodic degenerate mode case (figure 2a) studied by Buck and Sukumar [9].

The unitary transformation method presented here has the unique advantage that it not only solves the time-dependent problem but also permits a determination of the eigensolutions of the interacting Hamiltonian at the same time.

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