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K-causal structure of space-time in general relativity

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Abstract. Using K-causal relation introduced by Sorkin and Woolgar [1], we generalize results of Garcia-Parrado and Senovilla [2,3] on causal maps. We also introduce causality conditions with respect to K-causality which are analogous to those in classical causality theory and prove their inter-relationships. We introduce a new causality condition following the work of Bombelli and Noldus [4] and show that this condition lies in between global hyperbolicity and causal simplicity. This approach is simpler and more general as compared to traditional causal approach [5,6] and it has been used by Penrose *et al* [7] in giving a new proof of positivity of mass theorem. C^0-space-time structures arise in many mathematical and physical situations like conical singularities, discontinuous matter distributions, phenomena of topology-change in quantum field theory etc.

Keywords. C^0-space-times; causal maps; causality conditions; K-causality.

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1. Introduction

The condition that no material particle signals can travel faster than the velocity of light fixes the causal structure for Minkowski space-time. In general, causality is one of the most important concepts in physical theories, and in particular, in all relativistic theories based on a Lorentz manifold. Thus in particular, in general relativity, locally the causal relations are the same as in the Minkowski space-time. However globally, there could be important differences in the causal structure due to a change in space-time topology, strong gravitational fields etc. From the physical point of view, concept of causalities embodies the concept of time evolution, finite speed of signal propagation and accessible communications. Causality concepts were crucially used to formulate and prove the singularity theorems [5,8]. Several types of causality conditions are usually required on space-times in order that these space-times are physically reasonable.

It is also well-known that classical general relativity employs a Lorentzian space-time metric whereas all reasonable approaches to quantum gravity are free of such a metric background. Thus we can ask the question whether there exists a structure which captures essential features of light cones and hence notion of causality in a purely topological and order theoretic manner without *a priori* assumption of a Lorentzian metric. The results in §§2 and 3 in this paper show that this is possible. Thus, by using only topology and order defined on the cones, one can recover different causality conditions, and also establish their inter-relationships.

The order theoretic structures, namely causal sets, have been extensively used by Sorkin and his co-workers in developing a new approach to quantum gravity [9–11]. The use of domain theory, a branch of computer science, by Martin [12] shows that only order is necessary to define causal structures and the topology is implicitly described by the order at an abstract level. We shall comment more on this at the end of the paper. Furthermore, the conformally invariant causal cone structures have been used in resolving factor-ordering problems in the quantization in mini-superspace models of higher dimensional Einstein gravity by Rainer [13,14]. Thus, we feel that the approach and results in this paper will be helpful in studying such phenomena and also phenomena like topology change in quantum gravity in a more general manner.

As far as C^0-Lorentzian manifolds are concerned the use of C^0 structure does not need any differentiability condition and thus may be useful to analyze a wide variety of situations of physical and mathematical interest, like conical singularities (occurring in quantum field theory) and metric describing discontinuous matter distributions.

Thus in §2, using C^0-Lorentzian manifold structure, we define K-causal maps and prove their properties. Motivation for this study is given in the beginning of §2.

In §3, we define a number of causality conditions with respect to K^+ analogous to [5,6,8], and prove some interesting properties of space-times satisfying these conditions and their inter-relationships.

In §4, we discuss the hierarchy of causality conditions on the basis of results proved in §3 and make some concluding remarks.

2. K-causal maps

In 1996, Sorkin and Woolgar [1] introduced a new causal relation K^+ which is a generalization of chronological relation I^+ and causal relation J^+ (Penrose [6] and Hawking and Ellis [5]). Theory of causal structures developed by Penrose and Hawking played an important role in establishing singularity theorems in general relativity and quite exhaustive work has been done in this area since 1964. Books by Hawking and Ellis [5], Wald [15], Joshi [8] and Beem *et al* [16] are a proof of it. The causal relation K^+ introduced in [1] is order-theoretic in nature and no smoothness assumption is needed to develop the ideas in general relativity. In Sorkin and Woolgar [1], the results are proved for C^0-Lorentzian manifolds (a C^1-manifold endowed with a C^0-metric) and the framework developed is of wider applicability as compared to previous work and also conceptually simple. The authors have made use of Vietoris topology to establish some basic results. An important result proved is the compactness of the space of K-causal curves in a globally hyperbolic space-time. More, recently Garcia-Parrado and Senovilla [2,3] introduced the concept of causal mappings and proved a series of results in causal structure theory of space-times in general relativity. These mappings are more general than conformal mappings and the authors have also discussed a number of examples to illustrate their ideas. Causal isomorphisms used in [2,3] are a generalization of 'chronal isomorphisms' first used by Zeeman in the context of Minkowski space and later generalized by Joshi, Akolia and Vyas (see Joshi [8], §4.8) for space-times in general relativity.

Dowker *et al* [17] have proved that in a stably causal space-time, K-causal future is equal to Seifert future. Recently, Minguzzi [18] also proved some interesting results which may lead to the proof of equivalence of K-causality and stable causality. We shall comment more on this in the last section. Since we are interested in the results not depending on metric, but only on topology and order, we shall restrict to K-causal considerations.

In this and the following section, we combine the concept of K-causality with causal mappings and prove a series of results which are generalizations of results of Garcia-Parrado and Senovilla. In fact, our aim is to recast 'global causal analysis' along order-theoretic and general-topological lines, following the definitions and basic results of Sorkin and Woolgar [1] and following other works by Hawking and Sachs [19], Beem [20], Geroch [21], Joshi [8] and Diekmann [22].

Thus we first recall basic definitions from Sorkin and Woolgar [1] and then define K-causal maps and derive their properties. We then discuss briefly the algebraic structure of the set of all K-causal maps on a C^0-Lorentzian manifold. We also define K-conformal maps and as expected, these maps preserve all K-causal properties.

To begin with, we give basic definitions and some results from Sorkin and Woolgar [1] that will be used in this paper. Throughout this paper, by a C^0-space-time V, we mean a C^0-Lorentzian manifold with metric g_{ab} (a C^1-manifold endowed with a C^0-metric).

DEFINITION I

Let V be a C^0-space-time and u^a be any vector field defining its time-orientation. A time-like or light-like vector v^a is future-pointing if $g_{ab}v^a u^b < 0$ and past-pointing if $g_{ab}v^a u^b > 0$. Now let $I = [0, 1]$. A future time-like path in V is a piecewise C^1 continuous function γ: $[0, 1] \to M$ whose tangent vector $\gamma^a(t) = (\mathrm{d}\gamma(t)/\mathrm{d}t)^a$ is future pointing time-like whenever it is defined. A past time-like path is defined dually. The image of a future- or past-time-like path is a time-like curve. Let O be an open subset of V. If there is a future-time-like curve in O from p to q, we write $q \in I^+(p, O)$, and we call $I^+(p, O)$, the chronological future of p relative to O. Past-time-like paths and curves are defined analogously.

DEFINITION II

K^+ is the smallest relation containing I^+ that is topologically closed and transitive. If q is in $K^+(p)$ then we write $p \prec q$.

That is, we define the relation K^+, regarded as a subset of $V \times V$, to be the intersection of all closed subsets $R \supseteq I^+$ with the property $(p, q) \in R$ and $(q, r) \in R$ implies $(p, r) \in R$. (Such sets R exist because $V \times V$ is one of them.) One can also describe K^+ as the closed-transitive relation generated by I^+.

Remark. We note here that to define K^+, we need I^+ and the topology of space-time manifold. I^+ can be defined if, *a priori*, a cone structure is given. Thus a cone structure and topology are sufficient to define K^+.

DEFINITION III

An open set O is K-causal iff the relation $\prec$ induces a reflexive partial ordering on O, i.e., $p \prec q$ and $q \prec p$ together imply $p = q$.

DEFINITION IV

A subset of V is K-convex iff it contains along with p and q any $r \in V$ for which $p \prec r \prec q$.

DEFINITION V

A K-causal curve Γ from p to q is the image of a C^0 map γ: $[0, 1] \to V$ with $p = \gamma(0), q = \gamma(1)$ such that for each $t \in (0, 1)$ and each open set $O \ni \gamma(t)$ there is a positive number ϵ such that $t' \in (t, t + \epsilon) \Rightarrow \gamma(t) \prec_O \gamma(t'), t' \in (t - \epsilon, t) \Rightarrow \gamma(t') \prec_O \gamma(t)$.

Here $\prec_O$ denotes K^+ relation relative to O.

Remark. In the standard formalism, a causal curve has to satisfy a differentiability condition because one has to define its tangent vector in order to know if the tangent vector lies within the light cone or not. The differentiability condition differs from one author to another. For example, Penrose [6] requires time-like curves to be everywhere differentiable and smooth. However, he works mostly with trips, which are only piecewise-smooth. We know that, not all continuous curves are differentiable, smooth, or even piecewise smooth. Indeed, there are continuous curves which fail to be differentiable at infinitely many points. If such a curve can be linearly ordered by K, then it is a K-causal curve but not a causal curve.

DEFINITION VI

A K-causal open set $O \subseteq V$ is globally hyperbolic iff for every pair of points $p, q \in O$, the interval $K(p, q) = K^+(p) \cap K^-(q)$ is compact and contained in O.

We shall need the following theorems which are proved in Sorkin and Woolgar [1]:

Theorem i. *If V is K-causal then every element of V possesses an arbitrarily small K-convex open neighbourhood.*

Theorem ii. *A subset Γ of a K-causal space-time is a K-causal curve iff it is compact, connected and linearly ordered by $\prec = K^+$.*

Theorem iii. *In a K-causal space-time, let the K-causal curve Γ be the Vietoris limit of a sequence of K-causal curves Γ_n with initial endpoints p_n and final endpoints q_n. Then p_n converge to the initial endpoint of Γ and q_n to its final endpoint.*

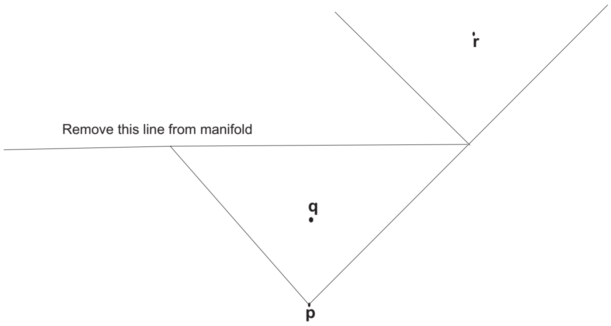

Figure 1. Two-dimensional Minkowski space, where a closed half-line is removed. Here, r is in $K^+(p)$ but not in $I^+(p)$. Also, $\mathrm{cl}(I^+(p)) \neq K^+(p)$.

Theorem iv. *In a K-causal space-time, the Vietoris limit of a sequence (or net) of causal curves is also a causal curve.*

Theorem v. *Let O be a globally hyperbolic open subset of a space-time V and let P and Q be compact subsets of O. Then the space of K-causal curves from P to Q is bicompact.*

With this background we now define a K-causal map. We work with K^+ throughout. Analogous definitions and results for K^- can be derived similarly.

A K-causal map is a causal relation which is a homeomorphism between the two topological spaces and at the same time preserves the order with respect to K^+. To define it, we first define an order preserving map with respect to K^+:

DEFINITION 1

Let V and W be C^0-space-times. A mapping f: $V \to W$ is said to be order preserving with respect to K^+ or simply order preserving if the following condition is satisfied: for $p, q \in V$, if $q \in K^+(p)$, then $f(q) \in K^+(f(p))$. That is, if p precedes q K-causally, then $f(p)$ should precede $f(q)$ K-causally. Symbolically, $p \prec q$ should imply $f(p) \prec f(q)$.

DEFINITION 2

Let V and W be C^0-space-times. A homeomorphism f: $V \to W$ is said to be K-causal if f is order preserving.

Remark. In general, K-causal maps and causal maps defined by Garcia-Parrado and Senovilla [2,3] are not comparable as $r \in K^+(p)$ need not imply that $r \in I^+(p)$ (see figure 1).

Using the definition of K-causal map, we now prove a series of properties which follow directly from the definition. We give their proofs for the sake of completeness: 'A homeomorphism f: $V \to W$ is order preserving iff $f((K^+(x)) \subseteq K^+(f(x)), \forall\, x \in V$'. This is proved as below:

Let f: $V \to W$ be an order preserving homeomorphism and let $x \in V$. Let $y \in f(K^+(x))$. Then $y = f(p), x \prec p$ which implies $f(x) \prec f(p)$ as f is order preserving, i.e., $f(x) \prec y$ or $y \in K^+(f(x))$. Hence $f(K^+(x)) \subseteq K^+(f(x)), \forall x \in V$. Conversely let f: $V \to W$ be a homeomorphism such that $f((K^+(x)) \subseteq K^+(f(x)), x \in V$. Let $p \prec q$. Then $f(q) \in f(K^+(p))$. By hypothesis, this gives $f(q) \in K^+(f(p))$. Hence $f(p) \prec f(q)$. Thus, if f is a K-causal map then $f(K^+(x)) \subseteq K^+(f(x)), \forall x \in V$.

Similarly we have the property that, 'if f: $V \to W$ be a homeomorphism then f^{-1} is order preserving iff $K^+(f(x)) \subseteq f((K^+(x)),\ x \in V$'.

We now define, for $S \subseteq V$, $K^+(S) = \bigcup_{x \in S} K^+(x)$. In general, $K^+(S)$ is neither open nor closed. In §3, we shall show that in a globally hyperbolic C^0-space-time, if S is compact, then $K^+(S)$ is closed. However at present, we can prove the following property that 'if f: $V \to W$ is an order preserving homeomorphism then $f((K^+(S)) \subseteq K^+(f(S)), S \subseteq V$'.

For, if f: $V \to W$ be an order preserving homeomorphism and $S \subseteq V$ then by definition, $K^+(S) = \bigcup_{x \in S} K^+(x)$. Let $y \in f(K^+(S))$. Then there exists x in S such that $y \in f(K^+(x))$. This gives $y \in K^+(f(x))$, i.e., $y \in K^+(f(S))$. Hence $f(K^+(S)) \subseteq K^+(f(S))$. Analogously we have, If f: $V \to W$ be a homeomorphism, and f^{-1} is order preserving then $K^+(f(S)) \subseteq f(K^+(S)), S \subseteq V$.

We know that causal structure of space-times is given by its conformal structure. Thus, two space-times have identical causality properties if they are related by a conformal diffeomorphism. Analogously, we expect that a K-conformal map should preserve K-causal properties. Thus we define a K-conformal map as follows.

DEFINITION 3

A homeomorphism f: $V \longrightarrow W$ is said be K-conformal if both f and f^{-1} are K-causal maps.

Remark. A K-conformal map is a causal automorphism in the sense of Zeeman [23].

This definition is similar to chronal/causal isomorphism of Zeeman [23], Joshi [8] and Garcia-Parrado and Senovilla [2,3].

Combining the above properties, we have the following, namely, if f: $V \to W$ is K-conformal then $f(K^+(x)) = K^+(f(x)), \forall x \in V$.

By definition, K-conformal map will preserve different K-causality conditions defined in §3, below. If a map is only K-causal and not K-conformal, then we have the following properties: To begin with, 'if f: $V \to W$ is a K-causal mapping and W is K-causal, so is V'.

For, let f: $V \to W$ be a K-causal map and W be K-causal. Let $p \prec q$ and $q \prec p, p, q \in V$. Then $f(p), f(q) \in W$ such that $f(p) \prec f(q)$ and $f(q) \prec f(p)$ as f is order preserving. Therefore, $f(p) = f(q)$ since W is K-causal. Hence $p = q$.

Analogous result would follow for f^{-1}. In addition, a K-causal mapping takes K-causal curves to K-causal curves. This is given by the property that, 'if V is a K-causal space-time and f: $V \to W$ is a K-causal mapping, then f maps every K-causal curve in V to a K-causal curve in W', which is proved as below:

Let f: $V \to W$ be a K-causal map. Therefore, f is an order preserving homeomorphism. Let Γ be a K-causal curve in V. Then by Theorem ii, Γ is connected, compact and linearly ordered. Since f is continuous, it maps a connected set to a connected set and a compact set to a compact set. Since f is order preserving and Γ is linearly ordered, $f(\Gamma)$ is a K-causal curve in W. Analogous result would follow for f^{-1}.

From the above result we can deduce that, 'if f be a K-causal map from V to W, then for every future directed K-causal curve Γ in V, any two points $x, y \in f(\Gamma)$ satisfy $x \prec y$ or $y \prec x$'.

DEFINITION 4

Let V and W be two C^0-space-times. Then W is said to be K-causally related to V if there exists a K-causal mapping f from V to W, i.e., $V \prec_f W$.

The following property follows easily from this definition, which shows that the relation '$\prec_f$' is transitive also: If $V \prec_f W$ and $W \prec_g U$ then $V \prec_{g\circ f} U$.

Next, we have the property concerning K-convex sets, that 'if $f\colon V \longrightarrow W$ is a K-causal map then $C \subseteq V$ is K-convex if $f(C)$ is a K-convex subset of W'. The proof is as below:

Let $f\colon V \longrightarrow W$ be K-causal and $f(C)$ be a K-convex subset of W. Let $p, q \in C$ and $r \in V$ such that $p \prec r \prec q$. Since f is order preserving we get $f(p) \prec f(r) \prec f(q)$ where $f(p), f(q) \in f(C)$ and $f(r) \in W$. Since $f(C)$ is K-convex, $f(r) \in f(C)$, i.e., $r \in C$. Hence C is a K-convex subset of V.

Remark. Concept of a convex set is needed to define 'strong causality', as we shall see below.

We now discuss briefly the algebraic structure of the set of all K-causal maps from V to V. We define the following:

DEFINITION 5

If V is a C^0-space-time then Hom(V) is defined as the group consisting of all homeomorphisms acting on V.

DEFINITION 6

If V is a C^0-space-time then $K(V)$ is defined as the set of all K-causal maps from V to V.

Then we have the property that '$K(V)$ is a submonoid of Hom(V)', which is more or less obvious, for, if $f_1, f_2, f_3 \in K(V)$ then $f_1 \circ f_2 \in K(V)$. Also, $f_1 \circ (f_2 \circ f_3) = (f_1 \circ f_2) \circ f_3$ and identity homeomorphism exists. Hence $K(V)$ is a submonoid of Hom(V). It is obvious that $K(V)$ is a bigger class than the class of K-conformal maps.

3. K-causal structure

Different causality conditions are imposed on a space-time to make it a physically reasonable one. In other words, this is to avoid different pathological situations which would arise otherwise.

In this section, we define different causality conditions with respect to K-causality analogous to those in causal structure theory and prove their properties, and inter-relationship. We also illustrate with the help of diagrams that these conditions are not equivalent to those in standard causal structure theory. We introduce a condition coined by Noldus and interestingly enough, we show that it is intermediate between global hyperbolicity and causal simplicity. Finally, we get a simpler hierarchy among K-causality conditions. Needless to say that K-conformal maps will preserve all these K-causality conditions.

To begin with, analogous to usual causal structure, we define strongly causal and future distinguishing space-times with respect to K^+ relation.

DEFINITION 7

A C^0-space-time V is said to be strongly causal at p with respect to K^+, if p has arbitrarily small K-convex open neighbourhoods.

V is said to be strongly causal with respect to K^+, if it is strongly causal at each and every point of it with respect to K^+. Thus, Lemma 16 of [1] implies that K-causality implies strong causality with respect to K^+. Analogous definition would follow for K^-.

DEFINITION 8

A C^0-space-time V is said to be K-future distinguishing if for every $p \neq q, K^+(p) \neq K^+(q)$. K-past distinguishing spaces can be defined analogously.

DEFINITION 9

A C^0-space-time V is said to be K-distinguishing if it is both K-future and K-past distinguishing.

Since, for every $p \neq q, I^+(p) = I^+(q)$ implies $K^+(p) = K^+(q)$, we have the following property: If a C^0-space-time V is K-future distinguishing then it is future distinguishing.

Remark. In general, future distinguishing need not imply K-future distinguishing (refer figure 3). Figure 2 shows a space-time which is neither future distinguishing nor K-future distinguishing. However, we have the following property: If a C^0-space-time V is K-causal, then it is K-distinguishing. The proof is as follows: Let $p, q \in V$ be such that, $K^+(p) = K^+(q)$. Now $p \in K^+(p)$ implies $p \in K^+(q)$. Similarly, $q \in K^+(p)$. Thus, by K-causality, $p = q$. Hence, V is K-future distinguishing. Similarly, V is K-past distinguishing also. Thus, every K-causal space-time is K-distinguishing.

We have a stronger result in the property, namely, 'if a C^0-space-time V is strongly causal with respect to K^+, then it is K-future distinguishing', the proof of which is as follows: Let V be strongly causal with respect to K^+. Let $p, q \in V$ such that $p \neq q$ and $K^+(p) = K^+(q)$.

Let P and Q be disjoint K-convex neighbourhoods of p and q in V respectively. Choose $x \in K^+(p) \cap P$. Then $x \in K^+(p)$. Therefore, $x \in K^+(q)$. Let y in Q be such that $q \prec y \prec x$. Then y is not in P and $y \in K^+(q)$ which implies $y \in K^+(p)$. So, $p \prec y \prec x$, but y not in P, contradicting the hypothesis that P is K-convex. Hence, $K^+(p) \neq K^+(q)$ whenever $p \neq q$.

Analogous result would follow for K^-. Hence, in a C^0-space-time V, strong causality with respect to K implies K-distinguishing.

Remark. K-conformal maps preserve K-distinguishing, strongly causal with respect to K^+ and globally hyperbolic properties.

A reflecting condition is a useful causality condition and as Clarke and Joshi [24] have proved, any stationary space-time is reflecting. In our case, we may define K-reflecting space-times as follows:

DEFINITION 10

A C^0-space-time V is said to be K-reflecting if $K^+(p) \supseteq K^+(q) \Leftrightarrow K^-(q) \supseteq K^-(p)$.

However, since the condition $K^+(p) \supseteq K^+(q)$ always implies $K^-(q) \supseteq K^-(p)$ because of transitivity and $x \in K^+(x)$, and vice versa, a C^0-space-time with K-causal condition is always K-reflecting. We shall comment more on this in the concluding section.

Moreover, in general, K-reflecting need not imply reflecting (refer figure 4). Since, any K-causal space-time is K-reflecting, any non-reflecting open subset of the space-time will be K-causal but non-reflecting.

Our last definition in this section is that of null relation with respect to K^+. We define it as follows:

DEFINITION 11

In V, y is said to be null related to x with respect to K^+, if $K^+(x) \cap K^-(y)$ does not contain any open set. (In this case, we can say that x and y are null-related to each other.)

Then we have the following property: 'if f^{-1}: $W \to V$ is a K-causal map, then f preserves the null relation with respect to K^+', which can be proved as follows:

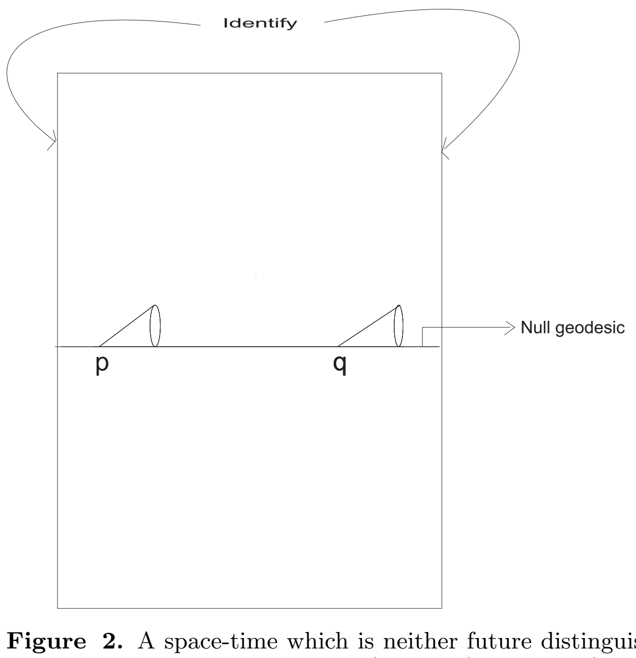

Figure 2. A space-time which is neither future distinguishing nor K-future distinguishing. Here, $p \neq q$, but $I^+(p) = I^+(q)$ and $K^+(p) = K^+(q)$.

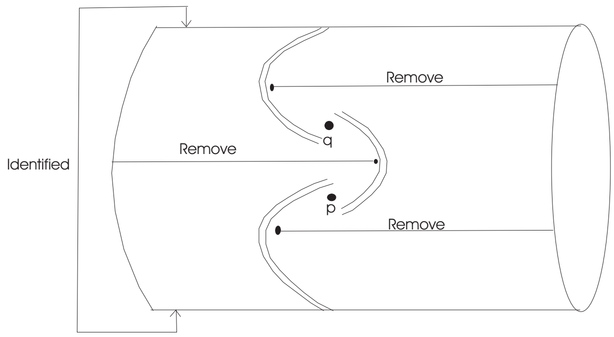

Figure 3. A time-like cylinder derived from M^2 with Cartesian coordinates (t, z) by periodically identifying t. From this space-time, remove three closed half-lines, each parallel to the z-axis: the half-lines starting at (± 1, -1) and moving off to the right, and the half-line starting from (0, 0) and moving off to the left. The resulting manifold has families of time-like curves as depicted, which implies that $p \neq q$, $I^+(p) \neq I^+(q)$, but $K^+(p) = K^+(q)$. This is because $q \in K^+(p)$, and because of identification, $p \in K^+(q)$. Hence, the space-time is future distinguishing but not K-future distinguishing.

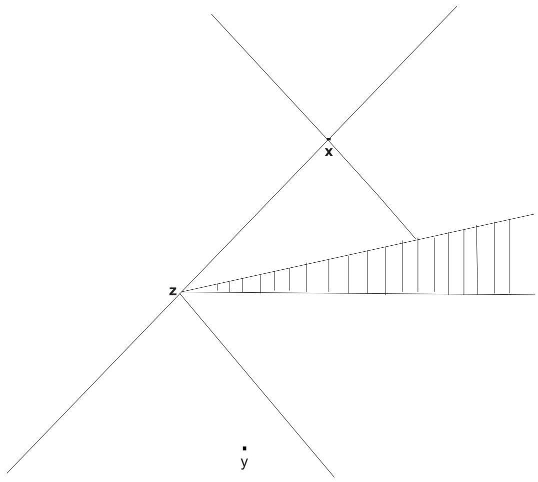

Figure 4. The space-time which is K-reflecting but not reflecting. The shaded region represents closed subset which has been removed. Here z is in $K^-(x)$ but not in $I^-(x)$. Hence, $K^+(x) \subseteq K^+(y) \;\Rightarrow\; K^-(y) \subseteq K^-(x)$ and $I^+(x) \subseteq I^+(y)$ but $I^-(y)$ is not contained in $I^-(x)$. As any K-causal space-time is K-reflecting, any non-reflecting open subset of the space-time will be K-causal but non-reflecting.

Let x and y be null related with respect to K^+. Then, $K^+(x) \cap K^-(y)$ does not contain any open set. Since f is a homeomorphism, $f(K^+(x) \cap K^-(y))$ does not contain any open set, i.e., $f(K^+(x)) \cap f(K^-(y))$ does not contain any open set. Since, f^{-1} is K-causal, $K^+(f(x)) \cap K^-(f(y))$ does not contain any open set. Thus $f(x)$ and $f(y)$ are null related with respect to K^+.

Similarly, if f is a K-causal map, then f^{-1} preserves the null relation with respect to K^+.

Hence, if f is a K-conformal map, then x and y are null-related iff $f(x)$ and $f(y)$ are null-related with respect to K^+. Analogous result can be proved for K^-.

Since global hyperbolicity is the strongest causality condition under study, we expect that a globally hyperbolic space-time with respect to K^+ is a nicely behaving one, that is, there are no pathologies in such a space-time. Thus, we expect that in such a space-time, $K^+(x)$ = closure of $I^+(x)$, $I^+(x)$ = interior of $K^+(x)$, and thus boundaries of $K^+(x)$ and $I^+(x)$ coincide. We prove below that it is indeed the case.

Also, in general $K^+(S)$ is not a closed set. We show that in a globally hyperbolic space-time, if S is compact, then $K^+(S)$ comes out to be a closed set. We prove this as our first result below:

Theorem 1. *Let V be a globally hyperbolic C^0-space-time. If $S \subseteq V$ is compact then $K^+(S)$ is closed.*

Proof. Let $S \subseteq V$ be compact. Let $q \in \mathrm{cl}(K^+(S))$. Then there exists a sequence q_n in $K^+(S)$ such that q_n converges to q. Hence, there exists a sequence p_n in S corresponding to q_n and future directed K-causal curves Γ_n from p_n to q_n. Then p_n has a subsequence p_{n_k} converging to $p \in S$ since S is compact, which gives a subsequence Γ_{n_k} of future directed K-causal curves from p_{n_k} to q_{n_k} where p_{n_k} converges to p and q_{n_k} converges to q.

Define $P = \{p_{n_k}, p\}$ and $Q = \{q_{n_k}, q\}$. Then P and Q are compact subsets of V. Hence, by Theorem iv, the set $\boldsymbol{C}$ of all future directed K-causal curves from P to Q is compact. Now, $\{\Gamma_{n_k}\}$ is a subset of $\boldsymbol{C}$. Thus, $\{\Gamma_{n_k}\}$ is a sequence in a compact set and hence has a convergent subsequence say $\Gamma_{n_{k_l}}$ of future directed K-causal curves from $p_{n_{k_l}}$ to $q_{n_{k_l}}$ where $p_{n_{k_l}}$ converges to p and $q_{n_{k_l}}$ converges to q. Let Γ be the Vietoris limit of $\Gamma_{n_{k_l}}$. Then by Theorem iii, Γ is a future directed K-causal curve from p to q. Since $p \in S$, we have, $q \in K^+(S)$. Hence $\mathrm{cl}(K^+(S)) \subseteq K^+(S)$. Thus $K^+(S)$ is closed.

The next two theorems show that in a globally hyperbolic C^0-space-time V, it is possible to express $K^+(x)$ in terms of $I^+(x)$.

Theorem 2. *If V is a globally hyperbolic C^0-space-time, then* $K^+(p) = \mathrm{cl}(\mathrm{int}$ $(K^+(p)), p \in V$.

Proof. Let V be globally hyperbolic. It is enough to prove that $K^+(p) \subseteq \mathrm{cl}(\mathrm{int}(K^+(p)), p \in V$. For this we show that $\mathrm{cl}(\mathrm{int}(K^+(p))$ is closed with respect to transitivity. So, let $x, y, z \in \mathrm{cl}(\mathrm{int}(K^+(p))$ such that $x \prec y$ and $y \prec z$. We show that $x \prec z$. Since x, y, z are limit points of $\mathrm{int}(K^+(p))$, there are sequences $\{x_n\}, \{y_n\}, \{z_n\}$ in $\mathrm{int}(K^+(p))$ such that $x_n \to x, y_n \to y, z_n \to z$. Using first countability axiom, we may assume, without loss of generality, that these sequences are linearly ordered in the past directed sense (see Martin and Panangaden [25], Lemma 4.3). Thus, for sufficiently large n, we can assume that $x_n \prec y_n$ and $y_n \prec z_n$. Since

$x_n, y_n, z_n \in K^+(p)$, by transitivity, $x_n \prec z_n$ for sufficiently large n. We claim that $x \prec z$. Let x be not in $K^-(z)$. Then as $K^-(z)$ is closed, using local compactness, there exists a compact neighbourhood N of x such that $N \cap K^-(z) = \emptyset$, and so, z is not in $K^+(N)$. Now, by Theorem 1, as V is globally hyperbolic and N is compact, $K^+(N)$ is closed. Hence, there exists a K-convex neighbourhood N' of z such that $N' \cap K^+(N) = \emptyset$, which is a contradiction as $x_n \prec z_n$ for large n. Hence, $x \prec z$. Thus, $\mathrm{cl}(\mathrm{int}(K^+(p))$ is closed with respect to transitivity. Since, by definition, $K^+(p)$ is the smallest closed set which is transitive, we get, $K^+(p) \subseteq \mathrm{cl}(\mathrm{int}(K^+(p))$. Hence $K^+(p) = \mathrm{cl}(\mathrm{int}(K^+(p))$. Similarly, $K^-(p) = \mathrm{cl}(\mathrm{int}(K^-(p))$.

Theorem 3. *If V is a globally hyperbolic C^0-space-time then* $\mathrm{int}(K^{\pm}(x)) = I^{\pm}(x), x \in V$.

Proof. Let V be globally hyperbolic and $x \in V$. That $I^+(x) \subseteq \mathrm{int}(K^+(x))$ is obvious by definition of $K^+(x)$. To prove the reverse inclusion, we prove that, if $x \prec y$ then there exists a K-causal curve from x to y and if $y \in \mathrm{int}(K^+(x))$, then this curve must be a future-directed time-like curve.

Let $x \prec y$ and there is no K-causal curve from x to y. Then image of [0,1] will not be connected, compact or linearly ordered. This is possible, only when a point or a set of points has been removed from the compact set $K^+(x) \cap K^-(y)$, that is, when some of the limit points have been removed from this set, which will imply that this set is not closed.

But, since V is globally hyperbolic, $K^+(x) \cap K^-(y)$ is compact and hence closed. Hence, there must exist a K-causal curve from x to y.

Suppose, $y \in \mathrm{int}(K^+(x))$. Then, there exists a neighbourhood $I^+(p) \cap I^-(q)$ of y such that $y \in I^+(p) \cap I^-(q) \subseteq K^+(x)$. To show that a K-causal curve from x to y is time-like, it is enough to prove that x and y are not null-related, that is, there exists a non-empty open set in $K^+(x) \cap K^-(y)$.

Consider, $I^+(p) \cap I^-(q) \cap I^+(x) \cap I^-(y)$, which is open. Take any point say z, on the future-directed time-like curve from p to y. Then, $z \in I^+(p) \cap I^-(q) \cap I^+(x) \cap I^-(y) \subseteq K^+(x) \cap K^-(y)$. (Here, $z \in I^+(x)$ because, if x and z are null-related then $K^+(x) \cap K^-(z)$ will not contain an open set. But $I^+(p) \cap I^-(z) \subseteq K^+(x) \cap K^-(z)$.) That is, $K^+(x) \cap K^-(y)$ has a non-empty open subset. Hence, x and y are not null-related, and so, the K-causal curve from x to y is time-like. That is, $y \in I^+(x)$. Thus, $\mathrm{int}(K^+(x)) \subseteq I^+(x)$ which proves that $\mathrm{int}(K^+(x)) = I^+(x)$. Similarly, we can prove that $\mathrm{int}(K^-(x)) = I^-(x)$.

From Theorems 2 and 3, we have the following result: 'for a globally hyperbolic space-time V, $K^{\pm}(x) = \mathrm{cl}(I^{\pm}(x))$'.

We now introduce the following condition given by Luca Bombelli and Johan Noldus [4]: '$K^+(p) \subseteq K^+(q)$ and $K^-(q) \subseteq K^-(p) \Rightarrow I^+(p) \subseteq I^+(q)$ and $I^-(q) \subseteq I^-(p)$'. We call this as 'Noldus condition'. Bombelli and Noldus have given an interesting example of a space-time where this condition is not satisfied. We then have the following:

Theorem 4. *The Noldus condition is equivalent to* $\mathrm{int}(K^{\pm}(x)) = I^{\pm}(x)$ *in a K-causal space-time.*

Proof. Let V be a K-causal space-time and $p, q \in V$. Let $\mathrm{int}(K^{\pm}(x)) = I^{\pm}(x)$. Then, $K^+(p) \subseteq K^+(q)$ and $K^-(q) \subseteq K^-(p) \Rightarrow \mathrm{int}(K^+(p)) \subseteq \mathrm{int}(K^+(q))$ and $\mathrm{int}(K^-(q)) \subseteq \mathrm{int}(K^-(p))$. That is, $I^+(p) \subseteq I^+(q)$ and $I^-(q) \subseteq I^-(p)$.

Conversely, let us assume the Noldus condition. Let $y \in \text{int}(K^+(x))$. Then, there exists an open neighbourhood say, $I^+(p) \cap I^-(q)$ of y such that, $y \in I^+(p) \cap I^-(q) \subseteq K^+(x)$. Therefore, $y \in I^+(p)$ and $q \in I^+(y)$.

Now, $y \in I^+(p)$ implies there exists a future directed time-like curve from p to y. Choose a point z on this trip. Then, $z \in I^+(p)$ and $y \in I^+(z)$ and hence $z \in I^+(p) \cap I^-(q) \subseteq K^+(x)$. Thus $x \prec z$. This then implies $K^+(z) \subseteq K^+(x)$ and also $K^-(x) \subseteq K^-(z)$ which implies $I^+(z) \subseteq I^+(x)$ and $I^-(x) \subseteq I^-(z)$. Thus, we have, $y \in I^+(z)$ and $I^+(z) \subseteq I^+(x)$ which gives $y \in I^+(x)$. Hence, $\text{int}(K^+(x)) \subseteq I^+(x)$ and so, $\text{int}(K^+(x)) = I^+(x)$. Similarly, we can prove $\text{int}(K^-(x)) = I^-(x)$.

Hence from Theorems 3 and 4, it follows that global hyperbolicity implies Noldus condition. We thus have,

Theorem 5. *In a C^2-space-time which is* K*-causal, if Noldus condition is assumed then J^+ and K^+ are equal.*

Proof. Let $x \in V$ and $y \in K^+(x)$. Then, $K^+(y) \subseteq K^+(x)$ and $K^-(x) \subseteq K^-(y)$ and hence by assumption, $I^+(y) \subseteq I^+(x)$ and $I^-(x) \subseteq I^-(y)$. Thus, $y \in J^+(x)$. Therefore, $K^+(x) \subseteq J^+(x)$, and so, $K^+(x) = J^+(x)$.

Hence, we have the following:

COROLLARY

In a C^2-space-time, Noldus condition implies $J^+(x)$ and $J^-(x)$ are closed and hence such a space-time is causally simple.

From Theorems 3–5 and the above corollary, we have the following:

COROLLARY

Noldus condition lies in between global hyperbolicity and causal simplicity.

4. K-causality hierarchy and concluding remarks

(i) In this paper we have generalized concepts from causal structure theory in terms of K-causal relations and causal maps in the light of the work of Sorkin and Woolgar [1] and Garcia-Parrado and Senovilla [2,3]. We have proved a number of results in this context as it is seen from the text of the paper.

In §3, we have proved that strong causality with respect to K^+ implies K-future distinguishing. Thus K-causality implies strongly causality with respect to K which implies K-distinguishing. Since a K-causal space-time is always K-reflecting as remarked in §3, it follows that the K-causal space-time is K-reflecting as well as K-distinguishing. In the classical theory, such a space-time is called causally continuous [19]. Such space-times have been useful particularly in the study of topology change in quantum gravity [26]. Thus if we define a K-causally continuous space-time analogously, then we get the result that a K-causal C^0-space-time is K-causally continuous. Moreover, since $K^{\pm}(x)$ are topologically closed by definition, analogue of causal simplicity is redundant. Since causal continuity implies stable causality in the classical sense [19], we expect that K-causal space-time is stably causal. As remarked in [1], stable causality implies K-causality. Thus we shall then have equivalence of these two conditions. Recently, certain steps towards this equivalence have been proved by Minguzzi [18], though complete equivalence is yet to be proved. Assuming this equivalence, K-causal hierarchy would read as follows:

Global hyperbolicity $\Rightarrow$ Stably causal $\Leftrightarrow$ K-causality $\Rightarrow$ Strong causality $\Rightarrow$ Distinguishing

The recent results proved by Minguzzi [18] and earlier results proved by Dowker *et al* [17] support and complement the results proved in §3 and also this hierarchy. In [17], the definition of causal continuity has been kept intact as in the classical sense, and hence the results differ from those proved here. However, Dowker *et al* [17] discuss in addition, the role of K^+ in topology changing Morse space-times both with and without degeneracies and find further characterizations of causal continuity. Thus they prove that a Morse space-time is causally continuous if and only if the functions $\mathrm{Int}(K^+(\cdot))$ and $\mathrm{Int}(K^-(\cdot))$ are inner continuous. On the other hand, our paper discusses K-causal maps and their properties as an additional feature, as compared to the papers [17] and [18].

(ii) Martin and Panangaden [25,27] have defined a bicontinuous poset $(X, \leq)$ as globally hyperbolic, if the intervals $[a, b]$ are compact in the interval topology. Similar definition has been given, using causal intervals in the theory of convex cones in the book *Causal symmetric spaces* [28]. In Martin and Panangaden [25] (see Theorem 4.1), it is proved that in a globally hyperbolic poset, its partial order $\leq$ is a closed subset of $X \times X$.

In a K-causal space-time, since the relation we have studied in this paper is a closed partial order, and global hyperbolicity is defined in terms of this relation, it is natural to ask whether the results in Martin and Panangaden [25,27] can be proved for K-causal space-time. Work is in progress in this direction.

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