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An approach to one-dimensional elliptic quasi-exactly solvable models

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Abstract. One-dimensional Jacobian elliptic quasi-exactly solvable second-order differential equations are obtained by introducing the generalized third master functions. It is shown that the solutions of these differential equations are generating functions for a new set of polynomials in terms of energy with factorization property. The roots of these polynomials are the same as the eigenvalues of the differential equations. Some one-dimensional elliptic quasi-exactly quantum solvable models are obtained from these differential equations.

Keywords. Quasi-exactly solvable potential; master function.

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1. Introduction

The spectral problems of non-relativistic quantum mechanics fall within two general categories. In the former category, we have a small number of the so-called exactly solvable problems, that is Schrödinger operators, the entire spectrum of which can be determined with algebraic methods. The simplest example of such a problem is given by harmonic oscillator. In the latter one, we have the Schrödinger operators, the complete spectrum of which cannot be computed exactly, but can only be approximated numerically at the very best. Over the past few decades, there has been a fair amount of interest in trying to construct physically significant systems which may not be exactly solvable, but in which only a part of the spectrum can be computed exactly by algebraic methods. In the early 1980s, Alhassid *et al*, Balazs and Voros, Levine and collaborators and Gonzalez-Lopez *et al* [1–4] introduced the concept of a 'spectrum generating algebra' to construct models for complicated molecules whose point spectrum could be analysed algebraically. Independently, Ushveridze [5] and Turbiner [6] were led to define a class of spectral problems which they called 'quasi-exactly solvable'. During the last decade plenty of quasi-exactly solvable spectral problems have been introduced [6,10–13]. Several important physical applications of elliptic quasi-exactly solvable potentials have been recently proposed in [14–19].

These potentials are especially interesting in that the quantum mechanical problem of a particle moving in a one-dimensional periodic potential is widespread in different areas of solid state physics, periodic instanton theory (in double-well potential to find static configuration with non-zero energy we are led to Lame potential) [20,21], molecular vibrations, etc. On the other hand, these potentials are employed to study the spin systems. In recent years, important affairs on spin systems have been conducted, for example the ones of Ulyanov and Zaslavskii [14]. They studied the one-dimensional spin systems, using the effective potential method on the one- and two-axis paramagnetic materials. To describe the spin systems, one may introduce a potential so that the energy spectrum of the spin system coincides with lower $2S+1$ energy levels of a particle moving in this potential [14]. The higher energy levels have nothing to do with the under-consideration spin systems and this is the same as factorization in quasi-exactly solvable models [11,13].

In ref. [13], some one-dimensional quasi-exactly solvable models have been obtained using the generalized master function approach. As far as elliptic quasi-exactly solvable potentials are concerned, these models have not been investigated, using this method. Therefore, the importance of elliptic quasi-exactly solvable potentials in different areas of physics (see above) motivated us to study these potentials and find some new elliptic potentials using generalized master function approach.

The paper is organized as follows: In §2 we generalize the usual second-order master function to a master function of at most three-order polynomials, then the most general elliptic quasi-exactly solvable differential operators which are related to the generalized master functions of degree $k=3$ are given. Also by expanding their solutions in powers of variable X, and considering eigenvalue equation, we get 3-term recursion relations among their coefficients.

In §3 we derive possible one-dimensional elliptic quasi-exactly solvable differential equations for $k=3$ and relative quantum Hamiltonians using refs [22,23]. Finally, §4 is our conclusion.

2. Quasi-exactly solvable differential equations associated with generalized master function

By generalizing master function $A(x)$ of the order up to two [17] to polynomial of the order up to k along with the non-negative weight function $W(x)$, defined in the interval (a,b) such that $\frac{1}{W(x)}\frac{\mathrm{d}}{\mathrm{d}x}(A(x)W(x))$ be a polynomial of degree at most $(k-1)$, we can define the operator

$$L=-\frac{1}{W(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left(A(x)W(x)\frac{\mathrm{d}}{\mathrm{d}x}\right)+B(x), \tag{2.1}$$

where $B(x)$ is a polynomial of order up to $(k-2)$. It seems that by a simple change of the independent variable,

$$X = \int W(x)\mathrm{d}x \quad \text{or} \quad \mathrm{d}x = \frac{\mathrm{d}X}{W(x)}. \tag{2.2}$$

One can significantly simplify the present approach, reducing eq. (2.1) to the form

$$L = -\tilde{A}(X)\frac{\mathrm{d}^2}{\mathrm{d}X^2} - (\tilde{A}(X))'\frac{\mathrm{d}}{\mathrm{d}X} + B(X), \quad (\tilde{A}(X))' = \frac{\mathrm{d}\tilde{A}(X)}{\mathrm{d}X}, \tag{2.3}$$

where $\tilde{A} = A(X)W^2(X)$. The interval (a, b) is chosen so that we have

$$\tilde{A}(a) = \tilde{A}(b) = 0. \tag{2.4}$$

It is straightforward to show that the operator L is a self-adjoint linear operator which at most, maps a given polynomial of the order m to another polynomial of the order $(m + k - 2)$. Now, by appropriate choices for $B(X)$ the operator L have an invariant subspace of the polynomials of the order up to n.

Then by choosing the set of orthogonal polynomials $\{\phi_0, \phi_1, ..., \phi_n\}$ defined in the interval (a, b):

$$\int_a^b \phi_m(X)\phi_n(X)\mathrm{d}X = 0, \quad \text{for } m \neq n, \tag{2.5}$$

as the bases, the matrix elements of the operator L on these bases will have the following block diagonal form:

$$L_{ij} = 0, \quad \text{if } \{i \leq n \text{ and } j \geq n+1\} \text{ or } \{i \geq n+1 \text{ and } j \leq n\}. \tag{2.6}$$

Since, according to the well-known theorem of orthogonal polynomials, $\phi_n(X)$ is orthogonal to any polynomial of the order up to $n - 1$. Therefore, for matrix L we get

$$L = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix}, \tag{2.7}$$

where M is an $(n+1) \times (n+1)$ matrix with matrix elements

$$M_{ij} = \int_a^b \mathrm{d}X\phi_i(X)L(X)\phi_j(X), \quad i, j = 0, 1, 2, ..., n, \tag{2.8}$$

and N is a matrix with the infinite elements defined as above with $i, j \geq n+1$. The block diagonal form of the operator L indicates that by diagonalizing the $(n+1) \times (n+1)$ matrix M, we can find $(n+1)$ eigenvalues of the operator L together with the related eigenfunctions as linear functions of orthogonal polynomials $\{\phi_0, \phi_1, ..., \phi_n\}$.

In order to determine the appropriate $B(X)$ for a given generalized master function $\tilde{A}(X)$, we Taylor expand those functions:

$$\tilde{A}(X) = \sum_{i=0}^{k} \frac{\tilde{A}^{(i)}(0)}{i!} X^i, \quad \text{where} \quad \tilde{A}^{(i)}(0) = \left. \frac{\mathrm{d}^i \tilde{A}(X)}{\mathrm{d}X^i} \right|_{X=0}, \tag{2.9}$$

$$B(X) = \sum_{i=0}^{k-2} \frac{B^{(i)}(0)}{i!} X^i, \quad \text{where} \quad B^{(i)}(0) = \left. \frac{\mathrm{d}^i B(X)}{\mathrm{d}X^i} \right|_{X=0}. \tag{2.10}$$

Then, the existence of invariant subspace of the polynomials of order n of the operator L leads to the following linear equations between the coefficients of the above Taylor expansion:

$$-\frac{\tilde{A}^{(i+2)}}{(i+2)!} l(l+i+1) + \frac{B^{(i)}}{i!} = 0, \tag{2.11}$$

where

$$\begin{cases} l = n, & \text{and } i = 1, \quad 2, \quad \ldots, \quad k-2 \\ l = n-1, & \text{and } i = 2, \quad 3, \quad \ldots, \quad k-2 \\ \ldots & \ldots \ \ldots \quad \ldots \quad \ldots \quad \ldots \\ l = n-k+4, & \text{and } i = k-3, \quad k-2 \\ l = n-k+3, & \text{and } i = k-2 \end{cases}. \tag{2.12}$$

The number of the above equations, for a given value of k, is $\frac{(k-1)(k-2)}{2}$. If we determine only the unknown function $B(X)$, then the above $\frac{(k-1)(k-2)}{2}$ equations should satisfy with $(k-2)$ coefficients of Taylor expansion of B as the only unknowns, since $B^{(0)}$ can be absorbed in the eigenspectrum operator L. Therefore, we are left with $(k-2)$ unknowns to be determined, where the compatibility of eq. (2.11) requires $k = 3$, at most. On the other hand, if we add the coefficients of Taylor expansions of $\tilde{A}(X)$ to our list of unknowns (to be determined by solving eq. (2.11)), then their compatibility conditions require that

$$2(k-1) \geq \frac{(k-1)(k-2)}{2}, \tag{2.13}$$

or $k \leq 6$, where further investigations show that we have at most $k = 3$, since for $k \geq 4$ the coefficients $A^{(k)}(0)$ will vanish. Here we summarize the above-mentioned discussion for $k = 3$.

In this case, $B(X)$ is a first-order polynomial where $B^{(1)}$ can be determined by solving eq. (2.11):

$$B^{(1)} = \frac{n(n+2)}{3!} \tilde{A}^{(3)}(0), \tag{2.14}$$

which is the only unknown in this case.

Definitely, we can determine $n+1$ eigenspectrum of the operator L, simply by diagonalizing the $(n+1) \times (n+1)$ matrix M, since it is a self-adjoint operator in Hilbert space of the polynomials and it has a block diagonal form given in eq. (2.7). At the end of this section, we can determine its eigenspectrum analytically, using some recursion relations.

2.1 *Recursion relation*

Now we show that the eigenfunctions of the operator L are generating functions for a new set of polynomials $P_m(E)$ where the eigenvalue equation of the operator L leads to the recursion relation between these polynomials. Quasi-exactly solvable constraints (2.11) will lead to their factorization, that is, $P_{n+N+1}(E) = P_{n+1}(E)Q_N$ for $N \geq 0$, where Q_N is a polynomial of the order N and roots of the polynomials $P_{n+1}(E)$ turn out to be the eigenvalues of the operator L. To achieve these results, first we expand $\psi(X)$, the eigenfunction of L as

$$\psi(X) = \sum_{m=0}^{\infty} P_m(E)X^m, \tag{2.15}$$

where eigenfunction equation

$$L\psi(X) = E\psi(X), \tag{2.16}$$

can be expressed as

$$\begin{aligned} -\tilde{A}(X)\sum_{m=2}^{\infty} m(m-1)P_m(E)X^{m-2} - (\tilde{A}(X))'\sum_{m=1}^{\infty} mP_m(E)X^{m-1} \\ +B(X)\sum_{m=0}^{\infty} P_m(E)X^m = E\sum_{m=0}^{\infty} P_m(E)X^m, \end{aligned} \tag{2.17}$$

which leads to the following recursion relations for the coefficients $P_m(E)$:

$$\begin{aligned} &\left((m+1)^2\tilde{A}^{(1)}\right)P_{m+2}(E) + \left(\frac{(m+1)(m+2)}{2!}\tilde{A}^{(2)} + E\right)P_{m+1}(E) \\ &+\left(\frac{m(m+2)}{3!}\tilde{A}^{(3)} - B^{(1)}\right)P_m(E) = 0. \end{aligned} \tag{2.18}$$

In order to have finite eigenspectrum, that is, quasi-integrable differential equation, the above recursion relation should be truncated for some value of $m = n$, which is obviously possible by an appropriate choice of $B^{(1)}$ as in eq. (2.14), which is in agreement with the result of the previous subsection. Using the recursion relation eq. (2.18), with $B^{(1)}$ given in eq. (2.14), we get a factorization of the polynomial $P_{n+N+1}(E)$ for $N \geq 0$ in terms of $P_{n+1}(E)$ as follows:

$$P_{n+N+1}(E) = P_{n+1}(E)Q_N(E), \quad N \geq 0, \tag{2.19}$$

where, by choosing the eigenvalue E as the roots of the polynomials $P_{n+1}(E)$, all of the polynomials of the order higher than n will vanish.

By using eq. (2.15) we obtain eigenfunction $\psi_i(X)$ as

$$\psi_i(X) = \sum_{m=0}^{n} P_m(E_i)X^m, \quad i = 0, 1, \ldots, n, \tag{2.20}$$

where E_i are the roots of the polynomial $P_{n+1}(E)$.

The above eigenfunctions are the polynomials of order n, and hence they have at most n roots in the interval (a, b), where, according to the well-known oscillation and comparison theorem of the second-order linear differential equation, these numbers order the eigenvalues according to the number of roots of the corresponding eigenfunctions. Therefore, we can say that the eigenvalues thus obtained are the first $n+1$ eigenvalues of the operator L. Using the recursion relation eq. (2.18), we can evaluate the polynomials $P_m(E)$ in terms of $P_0(E)$, where we have chosen $P_0(E) = 1$. We have evaluated the first three polynomials appeared in Appendix B.

3. Quasi-exactly potential associated with generalized master function

By considering the eigenvalue equation

$$L\psi(X) = E\psi(X), \tag{3.1}$$

and using the similarity transformation [22,23]

$$H(t) = \tilde{A}^{1/4}(X)L(X)\tilde{A}^{-1/4}(X), \tag{3.2}$$

the eigenvalue equation of the operator L reduces to the Schrödinger equation as

$$H(t)\phi(t) = \left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} + V(t)\right)\phi(t) = E\phi(t), \tag{3.3}$$

with the same eigenvalue E, where the eigenfunction $\psi(t)$ is defined as

$$\phi(t) = \tilde{A}^{1/4}(X)\psi(X). \tag{3.4}$$

The new variable t has been defined in terms of X by $\frac{\mathrm{d}X}{\mathrm{d}t} = \sqrt{\tilde{A}(X)}$, and potential V in terms of t is

$$V(t) = \frac{1}{4}\frac{\ddot{\tilde{A}}(t)}{\tilde{A}} - \frac{3}{16}\frac{(\dot{\tilde{A}})^2(t)}{(\tilde{A})^2(t)} + B(t), \quad \dot{\tilde{A}}(t) = \frac{\mathrm{d}\tilde{A}(t)}{\mathrm{d}t}. \tag{3.5}$$

Also we can introduce the potential V in terms of X:

$$V(X) = \frac{(\tilde{A}(X))''}{4} - \frac{1}{16}\frac{(\tilde{A}(X))'^2}{\tilde{A}(X)} + B(X). \tag{3.6}$$

One can easily show that

$$\int_{\alpha}^{\beta} \mathrm{d}t\phi(t)H(t)\phi(t) = \int_{a}^{b} \mathrm{d}X\psi(X)L(X)\psi(X). \tag{3.7}$$

Hence block diagonalization of L leads to block diagonalization of H.

Table 1. Cubic master functions and corresponding elliptic potentials.

$$A_1 = 4\,X(k^2X-1)(X-1), \quad X = \tfrac{cn(t,k)^2}{dn(t,k)^2}$$

$$V_1 = -\tfrac{1}{16}\,\tfrac{(-12\,kX^2-8\,X+4)^2}{-4\,kX^3-4\,X^2+4\,X} - 6\,kX - 2 - 4\,kn(n+2)X$$

$$A_2 = 4\,X(X-k^2)(X-1), \quad X = \tfrac{dn^2(t,k)}{cn^2(t,k)}$$

$$V_2 = -\tfrac{1}{16}\,\tfrac{(12\,X^2+2\,(-4\,k^2-4)X+4\,k^2)^2}{4\,X^3+(-4\,k^2-4)X^2+4\,k^2X} + 6\,X - 2\,k^2 - 2 + 4\,n(n+2)X$$

$$A_3 = 4\,X(1+k^2X)(1+(k^2-1)X), \quad X = \tfrac{sn^2(t,k)}{dn^2(t,k)}$$

$$V_3 = -\tfrac{1}{16}\,\tfrac{(12\,k^2(k^2-1)X^2+2\,(8\,k^2-4)X+4)^2}{4\,k^2(k^2-1)X^3+(8\,k^2-4)X^2+4\,X} + 6\,k^2(k^2-1)X + 4\,k^2 - 2$$
$$+4\,k^2(k^2-1)n(n+2)X$$

$$A_4 = 4\,X(X-1)(X-k^2), \quad X = \tfrac{1}{sn^2(t,k)}$$

$$V_4 = -\tfrac{1}{16}\,\tfrac{(12\,X^2+2\,(-4-4\,k^2)X+4\,k^2)^2}{4\,X^3+(-4-4\,k^2)X^2+4\,k^2X} + 6\,X - 2 - 2\,k^2 + 4\,n(n+2)X$$

$$A_5 = 4\,X(X-1)((1-k^2)X+k^2), \quad X = \tfrac{1}{cn^2(t,k)}$$

$$V_5 = -\tfrac{1}{16}\,\tfrac{(3\,(4-4\,k^2)X^2+2\,(-4+8\,k^2)X-4\,k^2)^2}{(4-4\,k^2)X^3+(-4+8\,k^2)X^2-4\,k^2X} + \tfrac{3}{2}\,(4-4\,k^2)X - 2 + 4\,k^2$$
$$+\tfrac{1}{6}\,(24-24\,k^2)n(n+2)X$$

$$A_6 = 4\,X(1-X)(1-k^2+k^2X), \quad X = cn^2(t,k)$$

$$V_6 = -\tfrac{1}{16}\,\tfrac{(-12\,k^2X^2+2\,(8\,k^2-4)X+4-4\,k^2)^2}{-4\,k^2X^3+(8\,k^2-4)X^2+(4-4\,k^2)X} - 6\,k^2X + 4\,k^2 - 2 - 4\,k^2n(n+2)X$$

$$A_7 = 4\,X(1-X)(k^2-1+X), \quad X = dn^2(t,k)$$

$$V_7 = -\tfrac{1}{16}\,\tfrac{(-12\,X^2+2\,(8-4\,k^2)X-4+4\,k^2)^2}{-4\,X^3+(8-4\,k^2)X^2+(-4+4\,k^2)X} - 6\,X + 4 - 2\,k^2 - 4\,n(n+2)X$$

$$A_8 = 4\,X(X-1)((k^2-1)X+1), \quad X = \tfrac{1}{dn^2(t,k)}$$

$$V_8 = \tfrac{1}{16}\,\tfrac{(3\,(4\,k^2-4)X^2+2\,(-4\,k^2+8)X-4)^2}{(4\,k^2-4)X^3+(-4\,k^2+8)X^2-4\,X}$$

$$A_9 = 4\,X(1+X)(1-k^2+X), \quad X = \tfrac{cn^2(t,k)}{sn^2(t,k)}$$

$$V_9 = -\tfrac{1}{16}\,\tfrac{(12\,X^2+2\,(8-4\,k^2)X+4-4\,k^2)^2}{4\,X^3+(8-4\,k^2)X^2+(4-4\,k^2)X} + 6\,X + 4 - 2\,k^2 + 4\,n(n+2)X$$

$$A_{10} = 4\,X(1-X)(1-k^2X), \quad X = sn^2(t,k),$$

$$V_{10} = -\tfrac{1}{16}\,\tfrac{(12\,k^2X^2+2\,(-4\,k^2-4)X+4)^2}{4\,k^2X^3+(-4\,k^2-4)X^2+4\,X} + 6\,k^2X - 2\,k^2 - 2 + 4\,k^2n(n+2)X$$

$$A_{11} = 4\,X(1+X)(1+(1-k^2)X), \quad X = \tfrac{sn^2(t,k)}{cn^2(t,k)}$$

$$V_{11} = -\tfrac{1}{16}\,\tfrac{(3\,(4-4\,k^2)X^2+2\,(8-4\,k^2)X+4)^2}{(4-4\,k^2)X^3+(8-4\,k^2)X^2+4\,X} + \tfrac{3}{2}\,(4-4\,k^2)X + 4 - 2\,k^2$$
$$+\tfrac{1}{6}\,(24-24\,k^2)n(n+2)X$$

$$A_{12} = 4\,X(k^2+X)(X+k^2-1), \quad X = \tfrac{dn^2(t,k)}{sn^2(t,k)}$$

$$V_{12} = -\tfrac{1}{16}\,\tfrac{(12\,X^2+2\,(8\,k^2-4)X+4\,k^2(k^2-1))^2}{4\,X^3+(8\,k^2-4)X^2+4\,k^2(k^2-1)X} + 6\,X + 4\,k^2 - 2 + 4\,n(n+2)X$$

3.1 *Elliptic quasi-exactly solvable potential*

Generalized master function $\tilde{A}(X)$ has a key role to find elliptic quasi-exactly solvable potential. Then, the selection of $\tilde{A}(X)$ which leads to elliptic potentials, is of crucial importance. Considering the relation $\frac{\mathrm{d}X}{\mathrm{d}t} = \sqrt{\tilde{A}(X)}$, we select the master functions so that X comes into the form of elliptic Jacobi functions (the interval of X is between the roots of $\tilde{A}(X)$). Therefore, from eq. (3.5) the potential V will have elliptic form. After determining B^1 from eq. (2.14), we obtain $B(X)$ as

$$B(X) = B^{(1)}X.$$

Using $\tilde{A}(X)$, $(\tilde{A}(X))'$ and $B(X)$, we can determine the operator L and the potential V respectively from eqs (2.3) and (3.5) or (3.6).

In table 1 we introduce the possible 12 elliptic quasi-exactly solvable potentials V and the cubic generalized master function $\tilde{A}(X)$ giving rise to these potentials.

Notice that all the Jacobi elliptic sn, cn and dn appearing in the potentials are the functions of t and modulus k.

4. Conclusion

By introducing the generalized third-order master functions, one-dimensional Jacobian elliptic quasi-exactly solvable second-order differential equations are obtained. The solutions of these differential equations are generating functions for a new set of polynomials in terms of energy with factorization property. The roots of these polynomials are the same as eigenvalues of the differential equations. We have obtained some one-dimensional elliptic quasi-exactly quantum solvable models from these differential equations. PT symmetric and pseudohermitian quasi-exactly solvable models are under investigation.

Appendix A: Jacobian elliptic functions

Jacobian elliptic functions are similar to trigonometric functions and they can be defined as the inversion of Legendre's elliptic integral of the first kind [24]. Therefore, $sn(u,k)$ is defined as

$$u = \int_0^{sn(u)} \frac{\mathrm{d}X}{\sqrt{(1-X^2)(1-k^2X^2)}}. \tag{A.1}$$

Then the functions $cn(u,k)$ and $dn(u,k)$ are defined by

$$cn(u,k) = \sqrt{1-(sn(u,k))^2}, \quad dn(u,k) = \sqrt{1-k^2(sn(u,k))^2}. \tag{A.2}$$

The above relations can also be represented by the equations

$$sn^2(u,k) + cn^2(u,k) = 1$$

and

$$dn^2(u,k) + k^2 sn^2(u,k) = 1.$$

By differentiating $(A-1)$ and using $(A-2)$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}u} sn(u,k) = cn(u,k)dn(u,k).$$

Similarly, by differentiating $(A-2)$ we get

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}u}(cn(u,k)) = -sn(u,k)dn(u,k) \\ \dfrac{\mathrm{d}}{\mathrm{d}u}(dn(u,k)) = -k^2 sn(u,k)cn(u,k) \end{cases}.$$

Jacobian elliptic functions have the following addition formulas:

$$sn(u+v,k) = \frac{sn(u,k)cn(v,k)dn(v,k) + sn(v,k)cn(u,k)dn(u,k)}{1-k^2 sn^2(v,k)},$$
$$cn(u+v,k) = \frac{cn(u,k)cn(v,k) - sn(u,k)sn(v,k)dn(u,k)dn(v,k)}{1-k^2 sn^2(v,k)},$$
$$dn(u+v,k) = \frac{dn(u,k)dn(v,k) - k^2 sn(u,k)sn(v,k)cn(u,k)cn(v,k)}{1-k^2 sn^2(v,k)}.$$

From the above relations, we arrive at the following duplicate formulas:

$$sn(2u,k) = \frac{2sn(u,k)\sqrt{1-(sn(u,k))^2}\sqrt{1-(sn(v,k))^2}}{1-k^2(sn(u,k))^4},$$
$$cn(2u,k) = \frac{cn(u,k) - \sqrt{1-(cn(u,k))^2}\sqrt{1-k^2(1-(cn(v,k))^2)}}{1-k^2(1-dn^2(u,k))^2},$$
$$dn(2v,k) = \frac{dn^2(v,k) - (1-dn^2(v,k))(1-\frac{1}{k^2}+\frac{dn^2(v,k)}{k^2}}{1-\frac{1}{k^2}(1-dn^2(v,k))^2}.$$

We have also exploited the following Glaiser's symbols:

$$sc(u,k) = \frac{sn(u,k)}{cn(u,k)}, \quad sd(u,k) = \frac{sn(u,k)}{dn(u,k)}, \quad cd(u,k) = \frac{cn(u,k)}{dn(u,k)},$$
$$cs(u,k) = \frac{cn(u,k)}{sn(u,k)}, \quad ds(u,k) = \frac{dn(u,k)}{sn(u,k)}, \quad dc(u,k) = \frac{dn(u,k)}{cn(u,k)}.$$

Appendix B: The first four polynomials $p_n(E)$ for k =3

To summarize, we set $F^{(i-1)} = i\tilde{A}^{(i)}$.

$$P_0 = 1, \quad P_1 = -\frac{E}{F^0}$$
$$P_2 = 1/2\,\frac{B^1F^0 + EF^1 + E^2}{F^0(A^1 + F^0)}$$

$$\begin{aligned}P_3 = &-(2EB^{(1)}\tilde{A}^{(1)} + \tilde{A}^{(2)}E^2 + 2F^{(1)}B^{(0)}F^{(0)}\\ &+\tilde{A}^{(2)}B^{(1)}F^{(0)} + \tilde{A}^{(2)}EF^{(1)} + 3EB^{(1)}F^{(0)} + E^3\\ &+2EF^{(1)^2} + 3F^{(1)}E^2 - EF^{(2)}\tilde{A}^{(1)} - EF^{(2)}F^{(0)})\\ &/(6F^{(0)}2\tilde{A}^{(1)^2} + 3\tilde{A}^{(1)}F^{(0)} + F^{(0)^2})\end{aligned}$$

$$\begin{aligned}P_4 = &(-\tilde{A}^{(3)}EF^{(1)}F^{(0)} + 4\tilde{A}^{(2)}E^3 + 6F^{(1)}E^3 + 6EF^{(1)^3}\\ &+11F^{(1)^2}E^2 + 3B^{(1)^2}F^{(0)^2} - 2\tilde{A}^{(3)}E^2\tilde{A}^{(1)} + 3\tilde{A}^{(2)^2}B^{(1)}F^{(0)}\\ &+3\tilde{A}^{(2)^2}EF^{(1)} + 6F^{(1)^2}B^{(1)}F^{(0)} + 8E^2B^{(1)}\tilde{A}^{(1)}\\ &+6E^2B^{(1)}F^{(0)} - 7E^2F^{(2)}\tilde{A}^{(1)} - 4E^2F^{(2)}F^{(0)} + 9\tilde{A}^{(2)}EF^{(1)^2}\\ &+13\tilde{A}^{(2)}F^{(1)}E^2 - 3F^{(2)}B^{(1)}F^{(0)^2} + 6\tilde{A}^{(1)}B^{(1)^2}F^{(0)}\\ &-2\tilde{A}^{(3)}EF^{(1)}\tilde{A}^{(1)} + 6\tilde{A}^{(2)}EB^{(1)}\tilde{A}^{(1)} + 9\tilde{A}^{(2)}F^{(1)}B^{(1)}F^{(0)}\\ &+10\tilde{A}^{(2)}EB^{(1)}F^{(0)} - 3\tilde{A}^{(2)}EF^{(2)}A^{(1)} - 3\tilde{A}^{(2)}EF^{(2)}F^{(0)}\\ &+12F^{(1)}EB^{(1)}\tilde{A}^{(1)} + 14F^{(1)}EB^{(1)}F^{(0)} - 9F^{(1)}EF^{(2)}\tilde{A}^{(1)}\\ &-6F^{(1)}EF^{(2)}F^{(0)} - 6\tilde{A}^{(1)}F^{(2)}B^{(1)}F^{(0)} - 2\tilde{A}^{(1)}\tilde{A}^{(3)}B^{(1)}F^{(0)}\\ &-\tilde{A}^3B^1F^{(0)^2} - \tilde{A}^{(3)}E^2F^{(0)} + 3\tilde{A}^{(2)^2}E^2 + E^4)\\ &/(24F^{(0)}(6\tilde{A}^{(1)^3} + 11\tilde{A}^{(1)^2}F^{(0)} + 6\tilde{A}^{(1)}F^{(0)^2} + F^{(0)^3})).\end{aligned}$$

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