PRAMANA — journal of physics © Indian Academy of Sciences Vol. 70, No. 3 March 2008 pp. 417–438

Generalized N-coupled maps with invariant measure in Bose–Mesner algebra perspective

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MS received 21 December 2006; revised 9 August 2007; accepted 28 August 2007

Abstract. By choosing a dynamical system with d different couplings, one can rearrange a system based on the graph with a given vertex dependent on the dynamical system elements. The relation between the dynamical elements (coupling) is replaced by a relation between the vertexes. Based on the E_0 transverse projection operator, we addressed synchronization problem of an array of the linearly coupled map lattices of identical discrete time systems. The synchronization rate is determined by the second largest eigenvalue of the transition probability matrix. Algebraic properties of the Bose–Mesner algebra with an associated scheme with definite spectrum has been used in order to study the stability of the coupled map lattice. Associated schemes play a key role and may lead to analytical methods in studying the stability of the dynamical systems. The relation between the coupling parameters and the chaotic region is presented. It is shown that the feasible region is analytically determined by the number of couplings (i.e. by increasing the number of coupled maps, the feasible region is restricted). It is very easy to apply our criteria to the system being studied and they encompass a wide range of coupling schemes including most of the popularly used ones in the literature.

Keywords. Synchronization; coupled maps; dynamical systems; Bose–Mesner algebra.

PACS Nos 05.45.Xt; 84.35.Xt; 89.75.Fb; 05.45.Ra

1. Introduction

Simulation of the natural phenomena is one of the most important research fields and coupled map lattices are a paradigm for studying fundamental questions in spatially extended dynamical systems. The early definition of the coupled map lattice goes back to the Kaneko's paper [1]. Now we can divide them in two categories: internal and external coupled map lattices [2]. Globally coupled map is one of the most well-known examples of the external coupling, with different number of elements [3]. A variety of couplings such as the week coupling [4], noisy coupling [5] and functional coupling [6] are used to make the new coupled map.

On the other hand, many areas of research lie on the interface of algebra and physics. Bose–Mesner algebra arose independently in three areas: statistical designs, centralizer algebras of permutation groups, and distance-transitive graphs. Associated schemes are combinatorial objects that allow us to solve problems in several branches of mathematics. They have been used in the study of permutation groups and graphs and also in the design of experiments, coding theory, partition designs etc. In this paper the authors are interested to use the ability of algebraic properties of the Bose–Mesner algebra associated with an association scheme with known spectrum in order to study the stability of the coupled map lattice, since elements of Bose–Mesner algebra have capability that can be diagonalized simultaneously [7,8].

The aim of the present paper is three-fold: introducing a generalized model of the coupled map lattice which covers internal and external couplings in a form of associated schemes. By taking into account that idempotents are another basis of this algebra (E_0 is the longitude projection operator, E_β is the transverse projection operator), in our proposed model all the elements have a common prolongation. Idempotents are independent of coupling and coupling method influences the stability of the coupled map lattices.

We addressed synchronization problem of an array of the linearly coupled map lattices of identical discrete-time systems. We present the proof that the synchronization rate is mainly determined by the second largest eigenvalue of the Floquet multipliers. We also prove that topological behavior can be studied by Sinai–Bowen rule (SBR) measure for the coupled map lattice of measurable dynamical system at the complete synchronized state.

In order to detect the feasible region, the stability analysis of the coupled map lattice with regard to the Lyapunov exponent is possible. We define the Lyapunov function by referring to the projection operators E_α's in order to confirm the stability analysis based on the Lyapunov exponents. Lyapunov function takes 0 value at synchronized state. We have investigated some conditions on transition from regularity to chaos with Li–Yorke theorem in a coupled map network, whose individual nodes are non-chaotic before being connected to the network. It has been discovered that, such a transition to chaos depends not only on the network topology but also on the original node dynamics.

In order to determine whether our results in this paper are generic or they are only limited to the specific model, we study several other coupled maps. It was proved that the global coupled map lattice corresponds to the complete graph; and the nearest-neighbor coupled map can be studied based on the model of associated schemes. It is very easy to apply our criteria to the system being studied, and they encompass a wide range of coupling schemes including most of the popularly used ones in the literature.

The paper is organized as follows: In §2 of the paper we present a brief outline of some of the main features of the associated schemes, such as adjacency matrix, distance-regular graphs, stratification and orthonormal basis of the strata. In §3, the generic model of the discrete-time coupled map lattice (internal and coupled map lattice) is introduced in adjacency matric perspective. In §3, the synchronization conditions are also studied. Section 4 discusses the existence of SBR measure at the synchronized state in the light of a measurable dynamical system. In §5, the stability analysis of the coupled map lattice based on the discussion of Floquet multipliers of coupled map lattice is done through three subsections; Lyapunov exponent calculation, Lyapunov function analysis and Li–Yorke theorem. In §6, we present some examples of the coupled map lattice model and their correspondent graphs such as the complete graph, strongly regular graphs and cycle graph. This section is followed by an outlook section. Respective appendix can be found at the end of paper.

2. Association schemes

In this section we give a brief outline of some of the main features of association scheme, and the reader is referred to [9] for further information on the associated schemes. Recall that a finite graph is a finite set Γ, whose elements are called vertices described by vertex set V, together with a set of $2-$ *subset* of Γ called edges. An association scheme with d associate classes on a finite set Γ is the coloring of all the edges of the graph by s colors such that the set of edges with the same color form a nonempty relation subsets on Γ (i.e. subset of $\Gamma \times \Gamma$), denoted by R_i, where $i = 1, ..., d$ corresponds to different colors. Then the pair $Y = (\Gamma, \{R_i\}_{0 \le i \le d})$ consisting of a set Γ and a set of relations $\{R_i\}_{0 \le i \le d}$ together with the following four conditions is called an association scheme.

(1) $\{R_i\}_{0 \le i \le d}$ is a partition of $\Gamma \times \Gamma$

(2) $R_0 = \{(\alpha, \alpha) : \alpha \in \Gamma\}$

(3) $R_i = R_i^t$ for $0 \le i \le d$, where $R_i^t = \{(\beta, \alpha) : (\alpha, \beta) \in R_i\}$

(4) Given $(\alpha, \beta) \in R_k, p_{ij}^k = |\{\gamma \in \Gamma : (\alpha, \gamma) \in R_i \;\text{ and }\; (\gamma, \beta) \in R_j\}|$,

where the constants p_{ij}^k are called the intersection numbers, depending only on i, j and k and not on the choice of $(\alpha, \beta) \in R_k$. Then the number N of the vertices V is called the order of the association scheme and R_i is called a relation or associate class. Let $\Gamma = (V, R)$ denote a finite, undirected, connected graph, with vertex set V, edge set R, path-length distance function ∂, and diameter $d := \max\{\partial(\alpha, \beta) : \alpha, \beta \in V\}$. For all α, $\beta \in V$ and all integer i, we set $\Gamma_i = (V, R_i) = \{(\alpha, \beta) : \alpha, \beta \in V : \partial(\alpha, \beta) = i\}$ so that $\Gamma_i(\alpha) = \{\beta \in V : \partial(\alpha, \beta) = i\}$.

3. The Bose–Mesner algebra

Let R denote the field of complex numbers. By $\mathrm{Mat}_V(R)$ we mean the R-algebra consisting of all matrices whose entries are in R and whose rows and columns are indexed by V. For each integer i $(0 \le \alpha \le d)$, let A_α denote the matrix in $\mathrm{Mat}_V(R)$ with (i, j)-entry

$$(A_\alpha)_{i,j} = \begin{cases} 1 & \text{if } (i,j) \in R_\alpha, \\ 0 & \text{otherwise} \end{cases} \qquad (i,j \in \Gamma). \tag{3.1}$$

The matrix A_α is called an adjacency matrix of the association scheme [10]. Then we have $A_0 = I$ (by (2) above) and

$$A_\alpha A_\beta = \sum_{\gamma=0}^{d} p_{\alpha\beta}^{\gamma} A_\gamma, \tag{3.2}$$

(by (4) above). So $A_0, A_1, ..., A_d$ form a basis for a commutative algebra $\mathbf{A}$ of $\mathrm{Mat}_V(R)$, where $\mathbf{A}$ is known as the Bose–Mesner algebra of Y. Since the matrices A_i commute, they can be diagonalized simultaneously, that is, there exists a matrix S such that for each $A \in \mathbf{A}$, $S^{-1}AS$ is a diagonal matrix. Therefore, $\mathbf{A}$ is semi-simple and has a second basis $E_0, ..., E_d$ (see [11]). These are matrices satisfying

$$E_0 = \frac{1}{N} J_N, \quad E_\alpha E_\beta = \delta_{\alpha\beta} E_\alpha, \quad \sum_{\alpha=0}^{d} E_\alpha = I_N. \tag{3.3}$$

The matrix $\frac{1}{N} J_N$ (where J_N is the all-one matrix in $\mathbf{A}$ and $N = |\Gamma|$) is a minimal idempotent (idempotent is clear, and minimal follows since rank matrix J_N equals 1). E_α, for $(0 \le \alpha, \beta \le d)$ are known as the primitive idempotent of Y. Let P and Q be the matrices relating our two bases for $\mathbf{A}$:

$$A_\beta = \sum_{\alpha=0}^{d} P_{\alpha\beta} E_\alpha, \quad 0 \le \beta \le d, \tag{3.4}$$

$$E_\beta = \frac{1}{N} \sum_{\alpha=0}^{d} Q_{\alpha\beta} A_\alpha, \quad 0 \le \beta \le d. \tag{3.5}$$

Then clearly

$$PQ = QP = NI_N. \tag{3.6}$$

It also follows that

$$A_\beta E_\alpha = P_{\alpha\beta} E_\alpha, \tag{3.7}$$

which shows that $P_{\alpha\beta}$ (resp. $Q_{\alpha\beta}$) is the αth eigenvalues (resp. the αth dual eigenvalues) of A_β (resp. E_β) and that the columns of E_α are the corresponding eigenvectors. Thus $m_\alpha = \mathrm{rank}(E_\alpha)$ is the multiplicity of the eigenvalue $P_{\alpha\beta}$ of A_β (provided $P_{\alpha\beta} \ne P_{\gamma\beta}$ for $\gamma \ne \alpha$). We see that $m_0 = 1, \sum_\alpha m_\alpha = N$, and $m_\alpha = \mathrm{trace} E_\alpha = N(E_\alpha)_{\beta\beta}$ (indeed, E_α has only eigenvalues 0 and 1, so $\mathrm{rank}(E_\gamma)$ equals the sum of the eigenvalues). Also, the eigenvalues and dual eigenvalues satisfy

$$m_\beta P_{\beta\alpha} = k_\alpha Q_{\alpha\beta}, \qquad 0 \le \alpha, \beta \le d, \tag{3.8}$$

where for all integer α $(0 \le \alpha \le d)$, set $k_\alpha = p_{\alpha\alpha}^{0} = (N-1)^\alpha C_N^\alpha$, and note that $k_\alpha \ne 0$, since R_α is nonempty.

3.1 *Definition of N-coupled map*

Coupled map lattices are arrays of states whose values are continuous, usually within the unit interval, or discrete space and time. The N-coupled map dynamical system can be considered as an N-dimensional dynamical map defined as

$$\psi(x_1(n), ..., x_N(n)) = \begin{cases} x_1(n+1) = \Phi_1(x_1(n), ..., x_N(n)) \\ x_2(n+1) = \Phi_2(x_1(n), ..., x_N(n)) \\ \vdots \\ x_N(n+1) = \Phi_N(x_1(n), ..., x_N(n)). \end{cases} \tag{3.9}$$

Adjacency matrix element such as A_0 can cover the elements of the dynamical systems in the coupled map lattice. A_1, A_2, etc., have the capability of generating different coupled map matrices such as nearest neighbor coupled map, second nearest neighbor coupled map and globally coupled maps. Definition of the coupled map model based on the adjacency matrix allows us to use the spectrum of graph based on the associated scheme. Therefore, N-coupled map is defined as follows:

$$\begin{aligned} x_i(n+1) = \Bigg[& \epsilon \sum_{\alpha=0}^{d} \frac{\epsilon'_\alpha}{k_\alpha} \left(\sum_{x \in \Gamma_\alpha(x_i(n))} \Phi(x(n)) \right) \\ & + (1-\epsilon)\Phi\left(\sum_{\alpha=0}^{d} \frac{\epsilon''_\alpha}{k_\alpha} \left(\sum_{x \in \Gamma_\alpha(x_i(n))} x(n) \right) \right) \Bigg], \end{aligned} \tag{3.10}$$

where n represents the time, N is the number of coupled maps, ϵ corresponds to internal and external coupling, $\epsilon'_\alpha(\epsilon''_\alpha)$ stand for coupling parameters and Φ is a 1D map in this model. For $\epsilon'_\alpha(\epsilon''_\alpha) \to 0$, there is no coupling at all, and hence, local neighborhoods have no influence on the behavior of the coupled map lattices. Using Bose–Mesner algebra language, we can write

$$x_i(n+1) = \left[\epsilon \sum_j A'_{ij} \Phi\left(x_j(n)\right) + (1-\epsilon)\Phi\left(\sum_j A''_{ij} x_j(n) \right) \right], \quad i, j = 1, ..., N, \tag{3.11}$$

where

$$A' = \sum_\alpha \frac{\epsilon'_\alpha}{k_\alpha} A_\alpha, \quad A'' = \sum_\alpha \frac{\epsilon''_\alpha}{k_\alpha} A_\alpha. \tag{3.12}$$

A' and A'' are the elements of Bose–Mesner algebra. ϵ'_α and ϵ''_α with the same relation in the associated schemes are the coupling constants in the coupled map lattice

$$\sum_{\alpha=0}^{d} \epsilon'_\alpha = 1, \quad \sum_{\alpha=0}^{d} \epsilon''_\alpha = 1. \tag{3.13}$$

Dynamical system symmetry consists of permutations which do not affect the adjacency matrices and elements of Bose–Mesner algebra.

(a) By considering $\epsilon = 0$, we can generate internal-coupled maps as follows:

$$x_i(n+1) = \Phi\left(\sum_{j=0} A'_{ij} x_j(n)\right) = \Phi\left(\sum_{\alpha=0}^{d} \frac{\epsilon''_\alpha}{k_\alpha}\left(\sum_{x\in\Gamma_\alpha(x_i(n))} x(n)\right)\right). \tag{3.14}$$

(b) By considering $\epsilon = 1$, we can generate external-coupled maps as follows:

$$x_i(n+1) = \sum_{j=0} A''_{ij}\Phi\left(x_j(n)\right) = \sum_{\alpha=0}^{d} \frac{\epsilon'_\alpha}{k_\alpha}\left(\sum_{x\in\Gamma_\alpha(x_i(n))} \Phi(x(n))\right). \tag{3.15}$$

3.2 *Local and globally coupled maps*

We generalize the models (3.11) to the coupled map lattices with local-global couplings on a one-dimensional lattice of length N with periodic boundary conditions [12]. These maps follow eq. (3.11) if we modify the coupling constants in the following way:

$$\frac{\epsilon'_\alpha}{k_\alpha} = \frac{\tilde{\epsilon}'_\alpha}{k_\alpha} + \frac{\gamma}{N}, \quad \alpha = 0, 1, ..., d, \tag{3.16}$$

where $\tilde{\epsilon}_\alpha$ and γ are coupling parameters. By substituting ϵ_α in (3.11):

$$\begin{aligned} x_i(n+1) = {} & \epsilon\left(1 - \sum_{\alpha=1}^{d} \tilde{\epsilon}'_\alpha - \gamma\right)\Phi(x(n)) \\ & + \epsilon \sum_{\alpha=1}^{d} \frac{\tilde{\epsilon}'_\alpha}{k_\alpha}\left(\sum_{x\in\Gamma_\alpha(x_i(n))} \Phi(x(n))\right) \\ & + (1-\epsilon)\Phi\left[\left(1 - \sum_{\alpha=1}^{d} \tilde{\epsilon}''_\alpha - \gamma\right) x(n)\right. \\ & \left. + (1-\epsilon)\sum_{\alpha=1}^{d} \frac{\epsilon''_\alpha}{k_\alpha}\left(\sum_{x\in\Gamma_\alpha(x_i(n))} x(n)\right)\right] \\ & + \frac{\gamma}{N}\left[\epsilon \sum_{\alpha=0}^{d}\left(\sum_{x\in\Gamma_\alpha(x_i(n))} \Phi(x(n))\right)\right. \\ & \left. + (1-\epsilon)\Phi\left(\sum_{\alpha=0}^{d}\left(\sum_{x\in\Gamma_\alpha(x_i(n))} x(n)\right)\right)\right]. \end{aligned} \tag{3.17}$$

3.3 *Synchronization of coupled maps*

Synchronization of two (or more) chaotic dynamical systems (starting with different initial conditions) means that their chaotic trajectories remain in step with each other during the temporal evolution. Complete synchronization was popularized after the seminal papers of Pecora and Carroll [13].

Complete synchronization in the coupled map means the existence of an invariant one-dimensional sub-manifold $x_1 = \cdots = x_N$ or $(x_1(n) = \cdots = x_N(n) \Leftrightarrow x_1(n+1) = \cdots = x_N(n+1))$.

As was discussed in §3, the matrices A' and A'' appearing in coupled map (3.11) are semi-simple, and hence possess the minimal idempotent $E_0, \ldots, E_d$, where

$$E_0 = \frac{1}{N} J_N = \frac{1}{N} \begin{pmatrix} 1 & .. & 1 \end{pmatrix} \begin{pmatrix} 1 \\ . \\ 1 \end{pmatrix} \tag{3.18}$$

projects coupled dynamical system on synchronized state. The remaining ones, on the other hand, will project it on the transverse modes. By choosing $i = j$ and considering (3.12) the generic model of coupled map lattice eq. (3.11) is reduced to $x_i(n+1) = \Phi(x_i(n))$. Thus, at synchronized state, the behavior of coupled map lattice is studied by the behavior of single dynamical element. In this article we study the synchronization of the coupled map lattices with different coupling topologies such as the global coupling, the nearest neighbor coupling, the coupling varying with distance etc.

In this paper we are interested in explaining the generic model based on the hierarchy of one and many-parameter chaotic maps which are introduced in our previous papers [14–16]. One-parameter families of chaotic maps of the interval [0, 1] with an invariant measure can be defined as the ratio of polynomials of degree N (see [17] for more details):

$$\Phi_M^{(1,2)}(x, \alpha) = \frac{\alpha^2 F}{1 + (\alpha^2 - 1)F}. \tag{3.19}$$

If F is substituted with Chebyshev polynomial of type one $T_M(x)$, we will get $\Phi_M^{(1)}(x, \alpha)$ and if F is substituted with Chebyshev polynomial of type two $U_M(x)$, we will get $\Phi_M^{(2)}(x, \alpha)$. In the hierarchy of elliptic chaotic maps F is substituted by Jacobian elliptic functions of **cn** and **sn** types. Also, we present some examples of discrete-time one-dimensional dynamical systems in Appendix B.

4. Invariant measure at synchronized state

The measure which describes the ergodic properties with respect to the typical initial conditions is usually called SRB measure [18]. The difficulty in proving rigorously that a given coupled map lattice exhibits spatio-temporal chaos lies in finding such a SRB measure, which has the following important properties:

(a) The measure is invariant with respect to symmetry transformation of N-dimensional dynamical system, namely it corresponds to the scalar representation of its symmetry group.

(b) The measure is smooth along unstable eigendirections in the phase space.

(c) The measure has strong ergodic properties, including mixing and positive KS-entropy [19].

Symmetric transformations are the kinds of transformations which leave the adjacency matrices invariant. As we will see later, by choosing the one-dimensional maps with an invariant measure, such as logistics map or the ones introduced by authors in [14,15], the coupled maps can display the invariant measure at synchronized state. The corresponding Forbenious–Perron (FP) equation for N-coupled map (3.9) is shown here:

$$\begin{aligned}\mu(x_1(n+1),...,x_N(n+1)) = &\int \mathrm{d}x_1 ... \int \mathrm{d}x_N \delta(x_1(n+1)\\ &-\Phi_1(x_1(n),...,x_N(n)))...\delta(x_N(n+1)\\ &-\Phi_N(x_1(n),...,x_N(n)))\mu(x_1(n),...,x_N(n)).\end{aligned} \tag{4.1}$$

Considering the above-mentioned properties of the invariant measure, it is obvious that, at synchronized state it should have the following form:

$$\mu(x_1,...,x_N) = \delta(x_2 - x_1)\cdots\delta(x_N - x_1)\mu(x_1), \tag{4.2}$$

where $\mu(x_1)$ corresponds to the invariant measure of one-dimensional map defined in one-dimensional invariant manifold $x_1 = x_2 = \cdots = x_N$. By exercising the synchronization condition in eq. (4.1), we have

$$\begin{aligned}\mu(x_1(n+1),...,x_N(n+1)) &= \int \mathrm{d}x_1 ... \int \mathrm{d}x_N \delta(x_1(n+1)\\ &\quad -\Phi_1(x_1(n),...,x_N(n)))...\delta(x_N(n+1)\\ &\quad -\Phi_N(x_1(n),...,x_N(n)))\delta(x_2(n)\\ &\quad -x_1(n))...\delta(x_N(n) - x_1(n))\mu(x_1)\\ &= \int \mathrm{d}x_1 \delta(x_1(n+1)\\ &\quad -\Phi_1(x_1(n),...,x_1(n)))...\delta(x_N(n+1)\\ &\quad -\Phi_N(x_1(n),...,x_1(n)))\mu(x_1)\\ &= \delta(x_2(n+1) - x_1(n+1))...\delta(x_N(n+1) - x_1(n+1))\\ &\quad \times \int \mathrm{d}x_1 \delta(x_1(n+1) - \Phi_1(x_1(n),...,x_1(n)))\mu(x_1(n)).\end{aligned}$$

Now, if the one-dimensional map $x(n+1) = \Phi(x_1(n),...,x_1(n))$ possesses the invariant measure $\mu(x_1(n))$, we can write

$$\mu(x(n+1)) = \int \delta(x(n+1) - \Phi(x_1(n),...,x_1(n))\mathrm{d}\mu(x_1). \tag{4.3}$$

Then, we have

$$\mu(x_1(n+1),...,x_N(n+1))$$
$$= \delta(x_1(n+1) - x_2(n+1))$$
$$....\delta(x_N(n+1) - x_1(n+1))\mu(x_1(n+1)). \qquad (4.4)$$

We have already derived analytically the invariant measure for one-parameter families of chaotic maps (3.19) by using arbitrary values of the control parameter α and for each integer values of N.

$$\mu_{\Phi_M^{(1,2)}(x,\alpha)}(x,\beta) = \frac{1}{\pi}\frac{\sqrt{\beta}}{\sqrt{x(1-x)}(\beta + (1-\beta)x)}, \quad \beta > 0. \qquad (4.5)$$

It should be mentioned that reader may refer to our previous papers for derivation and the relation between the control parameter and β [15,16,18].

5. Stability analysis at synchronized state

In this section we present the stability analysis of the coupled map lattice at synchronized state of fixed point type (i.e., one-dimensional map (3.19) at synchronized state possesses fixed point) and synchronized chaotic type (one-dimensional map at synchronized state is chaotic and possesses an invariant measure). This section also studies Li–Yorke-type of chaos as one-synchronized state becomes unstable.

5.1 *Stability analysis by Lyapunov exponent spectra*

The stability of the coupled map can be assessed by computing its Lyapunov exponent spectrum. The spectrum of Lyapunov exponents of coupled map lattice with respect to the synchronization state can be evaluated in a way similar to that of one-dimensional local maps [20]. At synchronized state, the Lyapunov exponents Λ_β of N-dimensional dynamical system described by the map (3.11) can be obtained by perturbing the state $x_i(n+1)$ infinitesimally around the synchronized state $(x_1(0) = x, ..., x = x_N(0))$; one can show that $\delta x(m+1)$, the corresponding perturbed state at the time m, becomes [18]

$$\delta x_i(m+1) = \sum_j \left(\frac{\partial x_i(m+1)}{\partial x_j(m)}\right)_{x_1=...=x_N} \delta x_i(m). \qquad (5.1)$$

Then, by taking derivative of $x_i(m+1)$ with respect to $x_j(m)$, in eq. (3.11), and exercising the result in (5.1), we obtain

$$\delta x(m+1) = \Phi'(x(m))(\epsilon A' + (1-\epsilon)A'')\delta x(m). \qquad (5.2)$$

By iterating (5.2) we obtain

$$\delta x(n) = \left(\prod_{m=0}^{n-1} \Phi'(x(m))\right) (\epsilon A' + (1-\epsilon)A'')^m \delta x(0), \tag{5.3}$$

with

$$\lambda_\beta = \sum_\alpha \frac{\eta_\alpha}{k_\alpha} P_{\beta\alpha}, \tag{5.4}$$

where $\eta_\alpha = \epsilon\epsilon'_\alpha + (1-\epsilon)\epsilon''_\alpha$. By using eq. (3.5) and idempotency property of E_α, we obtain

$$(\epsilon A' + (1-\epsilon)A'')^m = \sum_\beta \lambda_\beta^m E_\beta.$$

Substituting in (5.3), we get

$$\delta x(n) = \prod_{m=0}^{n-1} \acute{\Phi}(x(m)) \sum_\beta \lambda_\beta^m E_\beta \delta x(0),$$

where

$$\delta x(n) = \prod_{m=0}^{n-1} \acute{\Phi}(x(m)) E_0 \delta x(0) + \sum_{\beta=1}^{d} \prod_{m=0}^{n-1} \acute{\Phi}(x(m)) \lambda_\beta^m E_\beta \delta x(0). \tag{5.5}$$

It should be reminded that in Bose–Mesner algebra perspective, by resorting to projection operator, one could diagonalize the adjacency matrix, where diagonalization is the common problem in the coupled map lattice.

$E_0\delta x(0)$ is a prolongation which depends on synchronization state whereas $E_\beta \delta x(0)$ $(\beta = 1, ..., d)$ prolongations depend on transverse modes. The eigenvalues $(\prod_{m=0}^{n-1} \acute{\Phi}(x(m))\lambda_\beta^m)$ are called the Floquet (stability) multipliers of the orbit. The dynamical system phase space is formed by sum-direct of sub-space due to its idempotent project. Thus, for stabilities of subspace, the absolute value of Floquet multipliers should be less than 1.

$$\rho_0 = \prod_{m=0}^{n-1} \acute{\Phi}(x(m)), \quad \rho_\beta = \prod_{m=0}^{n-1} \acute{\Phi}(x(m))\lambda_\beta^m \quad (\beta = 1, ..., d). \tag{5.6}$$

Note that ρ_0 is the eigenvalue which corresponds to synchronous periodic orbit while ρ_β corresponds to transverse states. The eigenvalues of the Floquet (stability) multipliers are real having only one value in magnitude. We will show them in non-increasing order:

$$|\rho_0| \geq |\rho_2| \geq \cdots \geq |\rho_d|.$$

As eqs (5.4,6) show, ρ_β directly corresponds to E_β (projection operator). So, by considering eq. (3.6), one can diagonalize adjacency matrix spontaneously. This simplifies the calculation of ρ_β. Stability analysis of the coupled map lattice based on the stability of synchronous periodic orbit and transverse state, should be done in one of these three categories:

(a) Both synchronous periodic orbit and transverse states are stable. So, feasible region is studied by Lyapunov exponent and Lyapunov function.

(b) Synchronous periodic orbit is unstable, but transverse states are stable. So, feasible region is studied by Lyapunov exponent and Lyapunov function.

(c) Both synchronous periodic orbit and transverse states are unstable. So, stability analysis is studied by Li–Yorke theorem.

By taking the logarithm of the absolute value of Floquet multipliers and dividing it over n, one can study analytically the stability of dynamical systems with Lyapunov exponent, whereas Floquet multipliers should be studied numerically. Notice that, $\delta x(0)$ belongs to the corresponding subspace of E_α. So, the Lyapunov exponents of N-dimensional dynamical system Λ_β are defined as

$$\Lambda_\beta = \lim_{n\to\infty} \frac{1}{n} \ln\left(\frac{\|\delta x(n)\|}{\|\delta x(0)\|}\right) = \ln|\lambda_\beta| + \lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n} \ln|\Phi'(x(m))|. \quad (5.7)$$

By considering one-dimensional ergodic map as a dynamical element, the last part of the above-mentioned eq. (5.7) is reduced to

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=0}^{n} \ln|\Phi'(x(m))| = \int_0^\infty \mathrm{d}(\mu(x)) \ln|\Phi'(x(m))| = \lambda_{\mathrm{L}}(\Phi), \quad (5.8)$$

where λ_{L} shows the Lyapunov exponent of one-dimensional map. Therefore, the Lyapunov exponent of N-dimensional dynamical system is defined as

$$\Lambda_\beta = \ln|\lambda_\beta| + \lambda_{\mathrm{L}}(\Phi). \quad (5.9)$$

If in eq. (5.9) β is substituted by 0, the equation would be reduced to

$$\lambda_0 = \sum_\alpha \frac{\eta_\alpha}{k_\alpha} P_{0\alpha} = 1. \quad (5.10)$$

Hence, $\Lambda_0 = \lambda_{\mathrm{L}}(\Phi)$. According to item (b), for stability of transverse modes it is necessary to have $\Lambda_\beta < 0$ $(\beta = 1, ..., d)$ and therefore

$$\left|\sum_\alpha \frac{\eta_\alpha}{k_\alpha} P_{\beta\alpha}\right| \le \mathrm{e}^{-\lambda_{\mathrm{L}}(\Phi)} \quad (5.11)$$

as $\sum_{\alpha=0}^{d} \eta_\alpha = 1$. Therefore, synchronized state makes the coupling constants meet the following inequality condition:

$$1 - \mathrm{e}^{-\lambda_{\mathrm{L}}(\Phi)} \le \sum_{\alpha=1}^{d} (k_\alpha - P_{\beta\alpha}) \frac{\eta_\alpha}{k_\alpha} \le 1 + \mathrm{e}^{-\lambda_{\mathrm{L}}(\Phi)}, \quad \beta = 1, ..., d. \quad (5.12)$$

By considering that the multiples of η_α are positive, the gradient of this planes is in the first region. So, eq. (5.12) defines a hyper-parallelogram in the hyperspace formed by η_α's. If $d = 2$, this region would be a parallelogram to the first region. Considering eq. (5.12) as a semi-linear one, we should use linear function and its minimal condition to distinguish the feasible region. The region is defined as

$$k_\alpha(1 - \mathrm{e}^{-\lambda_{\mathrm{L}}(\Phi)}) \leq \epsilon_\alpha \leq k_\alpha(1 + \mathrm{e}^{-\lambda_{\mathrm{L}}(\Phi)}),$$

in ref. [15] is a subset of feasible region in eq. (5.12). According to Appendix A, we have $-k_\alpha \leq P_{\beta\alpha} \leq k_\alpha$ and by noting that $J = NE_0$ and considering eq. (3.4) we can write

$$\left(N - \sum_{\beta=0}^{d} P_{0\beta}\right) E_0 = \left(\sum_{\alpha=1}^{d}\sum_{\beta=0}^{d} P_{\alpha\beta}\right) E_\alpha \tag{5.13}$$

because $E_\alpha^{\prime s}$ $(\alpha = 0, ..., d)$ are independent, we may conclude

$$\sum_{\beta=0}^{d} P_{0\beta} = N, \quad \sum_{\alpha=1}^{d} P_{\alpha\beta} = 0$$

according (3.7), $P_{\alpha 0} = 1$ and by using relation (5.12)

$$k_\alpha(1 - \mathrm{e}^{-\lambda_{\mathrm{L}}(\Phi)}) < \sum_{\alpha}(k_\alpha - P_{\beta\alpha})\eta_\alpha < k_\alpha(1 + \mathrm{e}^{-\lambda_{\mathrm{L}}(\Phi)}) \tag{5.14}$$

which confirms the result of ref. [15]. To investigate the abolition of asynchronization and also in order to calculate the scaling exponent of its suppression, we linearize the recursion map (3.11) near the fixed direction (synchronization state) of this map:

$$\lim_{n \longrightarrow \infty} x_i(n) = x. \tag{5.15}$$

This leads us to write $x_i(n) = x + \delta x_i(n)$. As expected, the maximum eigenvalue corresponds to synchronization state, giving the power scaling exponent of dynamical system [21]. Then,

$$\left|\lim_{n \longrightarrow \infty} \frac{x_k}{x_l} - 1\right| = \mathrm{e}^{-n\overline{\rho}} \tag{5.16}$$

by regarding eq. (5.16) and by noting that ρ_{SLF} is the second largest Floquet multiplier, the scaling exponent of suppression of the asynchronization, $\overline{\rho}$ is defined as

$$\overline{\rho} = \ln\left|\frac{\rho_{\max}}{\rho_{\mathrm{SLF}}}\right| = -\ln(\text{second largest eigenvalue of } A) = -\ln|\lambda_{\mathrm{SLE}}(A)|, \tag{5.17}$$

where $\lambda_{\mathrm{SLE}}(A)$ means second largest eigenvalue of A. We can describe the transition probability with respect to synchronized state by

$$||x_N(n) - x_{\text{syn}}|| \simeq \mathrm{e}^{-n\lambda_{\text{syn}}}.$$

Transition probability changes in each time step ($n \to n+1$), and takes positive values if $\epsilon > 0$. The smaller the second largest eigenvalue modulus, the faster the Lyapunov exponent converges to its equilibrium distribution (synchronization state). This problem can be formulated as a convex optimization problem, which can in turn be expressed as a semi-definite program [22]. Restoration of synchronization time t_{RS} is a time in which the system transits from synchronization state into asynchronization state. It is in fact the inverse of $\overline{\rho}$

$$t_{\text{RS}} = \frac{1}{\overline{\rho}} = -\frac{1}{\ln|\lambda_{\max}(A)|}. \tag{5.18}$$

From (5.18) it is clear that, synchronization time t is independent of Lyapunov exponent of the map [23].

5.2 *Stability analysis with Lyapunov function*

Lyapunov function [24] is the generalization of an energy-like function that is decreased along trajectories. If the Lyapunov function exists, then closed orbit will be forbidden. The formation of synchronization can be studied through the Lyapunov function. It states that for all asymptotically stable systems, there exists a Lyapunov function whose derivative along the trajectories of the system takes a negative value. To study the stability region of the coupled map lattices, one can define Lyapunov function in terms of E_α as follows:

$$\sum_{\alpha \neq 0} E_\alpha = I_N - \frac{J_N}{N} = E_{AS}. \tag{5.19}$$

So, we can define the Lyapunov function as

$$V(n) = \sum_{i,j}^{N} x_i(n)(E_{AS})_{ij} x_j(n) = \vec{x}^t(n)\left(I_n - \frac{J_n}{n}\right)\vec{x}(n). \tag{5.20}$$

Lyapunov function takes 0 value at synchronized state and in step n it becomes

$$V(n) = \sum_{i=1}^{N} x_i(n)^2 - \frac{(\sum_{i=1}^{N} x_i(n))^2}{N}. \tag{5.21}$$

Clearly $V(n) \geq 0$ and the equality represents the exact synchronized state. For the asymptotic global stability of the synchronized state, Lyapunov function must satisfy the following condition in the region of stability $V(n+1) < V(n)$ [25]. We can write $\vec{x}$ state in the following form, $\vec{x}(n) = \vec{x}_s(n) + \delta\vec{x}(n)$, where

$$\vec{x}_s(n) = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} x(n) \tag{5.22}$$

$\vec{x}_s(n)$ is the synchronized state. By considering (4.9), $V(n) = \delta\vec{x}^t(n)\delta\vec{x}(n)$, substituting $\vec{x}(n)$ (corresponding to $\vec{x}(n{+}1)$) in (3.11) and by noting that $\delta\vec{x}(n)\cdot\vec{x}_s(n) = 0$,

$$\delta\vec{x}(n+1) = \acute{\Phi}(x(n))[\epsilon A' + (1-\epsilon)A'']\delta\vec{x}(n) \tag{5.23}$$

according to (5.21), we can write the ratio of the Lyapunov function in step $n+1$ to Lyapunov function in step n

$$\begin{aligned}\frac{V(n+1)}{V(n)} &= \frac{\delta\vec{x}^t(n+1)\delta\vec{x}(n+1)}{\delta\vec{x}^t(n)\delta\vec{x}(n)} \\ &= \frac{\delta\vec{x}^t(n)[\epsilon A' + (1-\epsilon)A'']^2\delta\vec{x}(n)\acute{\Phi}^2(x(n))}{\delta\vec{x}^t(n)\delta\vec{x}(n)},\end{aligned} \tag{5.24}$$

where according to ref. [15], $\acute{\Phi}(x)$ with $N =$ even equals N^2/a^2. By noting $|\lambda_1| \geq \cdots \geq |\lambda_d|$, $A = \epsilon A' + (1-\epsilon)A''$ and according to Appendix A, one can write

$$|\delta x^t(n)A^2\delta x(n)| \leq ||\delta x(n)||^2 \max(|\lambda_A|^2). \tag{5.25}$$

Therefore,

$$\frac{V(n+1)}{V(n)} \leq \max\{|\acute{\Phi}(x(n))|^2 : x(n) \in [0,1]\}(|\lambda|^2)_{\max} \leq 1. \tag{5.26}$$

Equation (5.26) is confirmed by the results of studies that have already been obtained in ref. [25]. The above-mentioned result is converted to Lyapunov exponent method in §5.1. For this purpose, we write the log of geometric mean of square root of Lyapunov function as

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\frac{1}{n}\ln\left(\sqrt{\prod_{m=0}^{n-1}\left|\frac{V(m+1)}{V(m)}\right|}\right) \\ &= \lim_{n\longrightarrow\infty}\frac{1}{n}\prod_{m=0}^{n-1}\acute{\Phi}(x(n))^2\sqrt{\frac{\delta\vec{x}^t(0)A^{2n}\delta\vec{x}(0)}{\delta\vec{x}^t(0)\delta\vec{x}(0)}}\end{aligned} \tag{5.27}$$

by choosing $\delta x(0)$ as the idempotent of $E_\alpha (\alpha = 1, ..., d)$. Obviously, the right-hand side of (5.33) equals $\lambda_{\rm L}(\Phi) + \ln|\lambda_\alpha|$. Otherwise, if $\delta\vec{x}^t(0)E_{\max}\delta\vec{x}(0) \neq 0$, it becomes $\lambda_{\rm L}(\Phi) + \ln|\lambda_{\max}|$, where $AE_{\max} = \lambda_{\max}E_{\max}$. It means that our results confirm Lyapunov exponent approach in §5.1.

5.2.1 *Stability of periodic orbit.* In order to prove the stability of synchronized fixed point we can rewrite the Lyapunov function and Lyapunov spectra at a synchronized state. By referring to the definition of the coupled map lattice Lyapunov spectra (5.7) at fix point $x^\star$, $\Lambda_\beta(x^\star) = \ln|\lambda_\beta\Phi'(x^\star)|$. Both λ_β and $|\Phi'(x^\star)|$ are smaller than 1. Therefore, coupled map is stable. Also by referring to the definition of the Lyapunov function at the synchronized state, we can prove the stability of fixed point at the synchronized state. We can rewrite relation (5.22) at fixed point $x_\star$ as follows:

$$\vec{x}_s = x^\star \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \tag{5.28}$$

where $x^\star$ is the coordinate of the synchronized state. According to Appendix A,

$$\frac{V(n+1)}{V(n)} \le |\acute{\Phi}^2(x^\star(n))| \le 1 \tag{5.29}$$

by noting that $x^\star$ is a fixed point, then $\acute{\Phi}(x^\star)$ is smaller than 1. So, for the stability of the coupled map the following condition should be met:

$$\frac{V(n+1)}{V(n)} \le 1. \tag{5.30}$$

5.3 *Study of unstable synchronized and transverse state with Li–Yorke method*

Here we present the study of the topological effects on the dynamic behaviors of a coupled complex network in transition to chaos based on Li–Yorke idea. We considered a network consisting of nodes that are in non-chaotic states with parameters in non-chaotic regions before they are coupled together. It was shown that if these non-chaotic nodes are linked together through a suitable structural topology, positive Lyapunov exponents of the coupled network Λ_β can be generated by choosing a certain uniform coupling strength $\epsilon'(\epsilon'')$, and the threshold for this coupling strength is determined by the complexity of the network topology.

Li–Yorke theorem [26] clarifies the existence of a snapback repeller, which implies the existence of chaos. This theory was generalized by Marotto to higher-dimensional discrete dynamical systems [27]. By employing Marottos theorem the transition to chaotic procedure is as follows: Consider the N-dimensional difference equation

$$x(n+1) = \Phi(x(n)); \qquad x(n) \in R^n. \tag{5.31}$$

Suppose that eq. (5.31) has a fixed point $x^\star$. This fixed point $x^\star$ is called a snap-back repeller if

(a) Φ is differentiable in a neighborhood $B(x^\star, r)$ of $x^\star$ with radius $r > 0$, such that all eigenvalues of the Jacobian $[D\Phi]$ are strictly larger than 1 in absolute values.

(b) There exists a point $x(0) \in B(x^\star, r)$, with $x(0) \ne x^\star$, for some integer $m > 0$, $\Phi^m(x(0)) = x^\star$ and Φ^m is differentiable at $x(0)$ with $\det[D\Phi^m(x(0))] \ne 0$.

If eq. (5.28) has a snap-back repeller, then (5.31) is chaotic in the sense of Li–Yorke. At the synchronized state, the Lyapunov exponents Λ_β of the N-dimensional dynamical system are described by (5.9). Now, by considering (5.9) at fixed point $(x^\star)$ we have

$$\Lambda_\beta(x^\star) = \ln|\lambda_\beta| + \ln|\Phi'(x^\star)|. \quad (5.32)$$

The transition to chaotic state takes place when

$$|\lambda_\beta|_{\max} \geq \frac{1}{|\Phi'(x^\star)|}. \quad (5.33)$$

The second condition is

$$\det[D\Phi^m(x(0))] = \left(\prod_{i=0}^{m-1} \Phi'(x(i)))\det(\epsilon A' + (1-\epsilon)A''\right)^m = \prod_{i=0}^{m-1} \Phi'(x(i)) \prod_{\beta=1}^{d} \lambda_\beta^m \neq 0. \quad (5.34)$$

Therefore, we need to choose coupling constants ϵ'_α and ϵ''_α so that all $\lambda_\beta(\beta = 0, ..., d)$ become nonzero. Also, $x(0)$ is chosen in such a way that it satisfies the condition $\Phi(x(0)) = x^\star$ and $\Phi'(x(0)) = \frac{N^2}{a^2} \neq 0$.

As an example for the trigonometric chaotic maps [15], we introduce the above-mentioned conditions:

- For $\Phi_N^{(1)}(x,\alpha)$ (N = even), as it was discussed [15], that maps have only a fixed point attractor ($x_\star = 0$), then $x_0 = 1$.
- For $\Phi_N^{(2)}(x,\alpha)$ (N = even), it was shown [15], that maps have only a fixed point attractor ($x_\star = 1$), then $x_0 = 0$.
- By choosing odd value of N in $\Phi_N^{(1,2)}(x,\alpha)$, another fixed point is presented for which the Li–Yorke condition is met too.

The ratio of each polynomial volume to the total defined volume is equal to 0 [28]. Then $\prod_{\beta=1}^{d} \lambda_\beta^m \neq 0$ and Li–Yorke chaos happened.

6. Some examples of well-known coupled map with corresponding graph

Globally coupled maps are one of the favorite models in the study of spatially extended dynamical systems [29]. These models of the coupled map lattice correspond to the complete graph. A complete graph is a graph in which each pair of graph vertices is connected by an edge. The complete graph with **N** graph vertices is denoted by K_N and has $\frac{N(N-1)}{2}$ (the triangular numbers) undirected edges [30]. Globally coupled maps correspond to the trivial scheme with agency matrices:

$$A_0 = I_N, \quad A_1 = I_N - J_N. \quad (6.1)$$

By considering $E_1 = I_N - \frac{J_N}{N}$ and $E_0 = \frac{J_N}{N}$ and by using an arbitrary element of Bose–Mesner algebra, the equation $A = \epsilon A' + (1-\epsilon)A''$ is reduced to

$$A = E_0 + \left(1 - N\frac{\eta_1}{k_1}E_1\right). \quad (6.2)$$

Ten eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 1 - N\frac{\eta_1}{k_1}$. According to (5.18) and (A.2), it is clear that restoration of synchronization time $t_{\rm RS}$ is reduced to

$$t_{\rm RS} = -\frac{1}{\ln|1 - N\frac{\eta_1}{k_1}|}.$$

Based on (5.12)

$$\begin{aligned}&\frac{k_1}{N}(1 - \exp(-\lambda_{\rm L}(x_{n+1} = \Phi(x_n)))) \le \eta_1 \\ &\quad \le \frac{k_1}{N}(1 + \exp(-\lambda_{\rm L}(x_{n+1} = \Phi(x_n)))).\end{aligned} \tag{6.3}$$

Similarly, through Lyapunov function approach the following condition is met too:

$$\frac{k_1}{N}\left(1 - \frac{1}{|\max\Phi'(x_n)|}\right) \le \eta_1 \le \frac{k_1}{N}\left(1 + \frac{1}{|\max\Phi'(x_n)|}\right). \tag{6.4}$$

Coupling coefficients of the global coupled map (see as an example [31]) generate the agency matrix following (6.1,2). So, their stability analysis, stable region, can be calculated by relations (6.3, 4). Also, in Appendix C some of the well-known graphs have been presented to illustrate the model.

7. Conclusion and outlook

In conclusion, it was found that a variety of coupled maps (internal and external) display an identical pattern. Moreover, the synchronization capabilities of various kinds of coupling schemes were analyzed. We presented the proofs that the synchronization rate is mainly determined by the second largest eigenvalue of the Floquet multipliers.

In order to determine whether our results in this paper are generic or they only depend on the specific model, we also studied several other coupled maps. In future, the study may cover the following issues:

- By applying the renormalization group theory in coupled dynamical systems, the scheme picture of the model would be presented. As was shown, near the synchronized state, different dynamical systems show the same patterns. This may lead us to this question 'do different dynamical systems follow the same pattern at synchronization state?'.
- As it was shown in this study and in the author's previous works [23,32], in nonlinear maps, dynamical element represents the scheme picture, which may lead us to another interesting question, 'can we attribute the scheme to every nonlinear dynamical systems and coupled dynamical system too?'

Further, we do hope that our obtained results through this paper will pave the way for further studies on nonlinear dynamical systems.

Appendix A

It was proved [33], that if A is a Hermitian matrix, then we have $|x^t Ax| \leq \max(|\lambda_A|)||x||^2$ and if x is the largest absolute eigenvalue then we have, $\max(|\lambda_A|)$, $|x^t A_\alpha x| \leq k_\alpha ||x||^2$. Thus for adjacency matrix A_α, we have $\max(|\lambda_{A_\alpha}|) \leq k_\alpha$ and so $P_{\beta\alpha} \leq |k_\alpha|$ for $A = \sum_\alpha \eta_\alpha A_\alpha$:

$$|x^t Ax| = \sum_\alpha |\eta_\alpha||x^t A_\alpha x| \leq \sum_\alpha k_\alpha |\eta_\alpha|||x||^2. \tag{A.1}$$

Taking $\sum_\alpha \eta_\alpha k_\alpha = 1$ and by assuming that η_α is positive, we have

$$\frac{|x^t Ax|}{||x||^2} \leq 1. \tag{A.2}$$

Appendix B

We present some examples of the discrete one-dimensional dynamical system with their invariant measure and Lyapunov exponent in order to simplify the generation of coupled map model based on Bose–Mesner algebra:

1. Bernuli shift map:

$$\phi(x) = \frac{1}{p_k}\left(x - \sum_{i=1}^{k-1} p_i\right) \quad \text{for} \quad \sum_{i=1}^{k-1} p_i \leq x \leq \sum_{i=1}^{k} p_i \quad (k = 2, 3, ...).$$

$$\mu = \frac{1}{p_k}, \quad \lambda = -\sum_{i=1}^{k} p_i \ln p_i.$$

2. Generalized tent map:

$$\phi(x) = \frac{(-1)^{k+1}}{p_k}\left(x - \sum_{i=1}^{k-1} p_i\right) + \frac{1}{2}(1 + (-1)^k)$$
$$\text{for} \sum_{i=1}^{k-1} p_i \leq x \leq \sum_{i=1}^{k} p_i \ (k = 2, 3, ...)$$

$$\mu = \frac{1}{p_k}, \quad \lambda = -\sum_{i=1}^{k} p_i \ln p_i.$$

3. Gauss map:

$$\phi(x) = \frac{1}{x} - \left[\frac{1}{x}\right], \quad \mu = \frac{1}{(1+x)\ln 2}, \quad \lambda = \frac{\pi^2}{6 \log 2}.$$

4. Hut map:

$$\phi(x) = \frac{1}{2}\left(-1 + \sqrt{9 - 16\left|x - \frac{1}{2}\right|}\right),$$

$$\mu = x + \frac{1}{2}, \quad \lambda = \frac{1}{2} + 2\ln 2 - \frac{9}{8}\ln 3.$$

5. Piecewise parabolic map:

$$\phi(x) = \frac{1 + r - \sqrt{(1-r)^2 + 4r|1-2x|}}{2r} \quad \text{for} \quad r \in [-1, 1],$$

$$\mu = 1 + r(1-2x),$$

$$\lambda = \ln 2 + \frac{(1-r)^2}{4r}\ln(1-r) - \frac{5r^2 - 2r + 1}{4r}\ln(1+r) + \frac{r}{2} + 1.$$

Appendix C

In this appendix we present some examples of the graph in order to simplify the generation of the coupled map model based on Bose–Mesner algebra.

(a) *Strongly regular graph.* A regular graph G of degree r that is neither empty nor complete is called strongly regular if every pair of adjacent vertices has exactly u common neighbors and every pair of non-adjacent vertices has exactly ν common neighbors. The numbers r, u and ν are the parameters of the graph. It is proved that a regular connected graph G of degree r is strongly regular if and only if it has exactly three distinct eigenvalues $\lambda_1 = r > \lambda_2 > \lambda_3$. If G is a strongly regular graph with parameters u and ν then $u = r + \lambda_2\lambda_3 + \lambda_2 + \lambda_3$, and $\nu = r + \lambda_2\lambda_3$ or

$$\lambda_{2,3} = \frac{1}{2}[u - \nu \pm \sqrt{(u-\nu)^2 - 4(\nu - r)}].$$

Then according to (5.18) and (A.2) it is clear that restoration of synchronization time t_{RS} is

$$t_{\mathrm{RS}} = -\frac{1}{\ln|\frac{1}{2}[u - \nu + \sqrt{(u-\nu)^2 - 4(\nu - r)}]|}.$$

The parameters N, r, u and ν determine not only each eigenvalue, but also the multiplicities of each eigenvalue. Since G is connected, the multiplicity of λ_1 is 1. The multiplicities m_2 and m_3 of λ_2 and λ_3 must sum to $N-1$ and by noting that the sum of the eigenvalues must be zero, the sum of the squares of the eigenvalues must be equal to $N\lambda_1 = Nr$. Thus m_2 and m_3 are

$$\frac{1}{2}\left(\nu - 1 \mp \frac{(\lambda_2 + \lambda_3)(\nu - 1) + 2r}{\lambda_2 - \lambda_3}\right).$$

An example for strongly regular graph is Hamming graphs with diameter 2. $H(2,q)$ is an SRG $(q^2, 2(q-1), (q-2), 2)$ with

$$\mathrm{Spec}(H(2,q)) = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ 1 & m_1 & m_2 \end{pmatrix} = \begin{pmatrix} 2(q-1) & q-2 & -2 \\ 1 & 2(q-1) & (q-1)^2 \end{pmatrix}.$$

If $q = 2$, then

$$\frac{1-\mathrm{e}^{-\lambda_{\mathrm{L}}(\Phi)}}{4} \leq \eta_1 \leq \frac{1+\mathrm{e}^{-\lambda_{\mathrm{L}}(\Phi)}}{4},$$

$$\frac{1-\mathrm{e}^{-\lambda_{\mathrm{L}}(\Phi)}}{2} \leq \eta_1 + \eta_2 \leq \frac{1+\mathrm{e}^{-\lambda_{\mathrm{L}}(\Phi)}}{2}.$$

If $q > 2$, using eq. (5.14) we find a feasible region in the following form:

$$\frac{1-\mathrm{e}^{-\lambda_{\mathrm{L}}(\Phi)}}{q} \leq \eta_1 + (q-1)\eta_2 \leq \frac{1+\mathrm{e}^{-\lambda_{\mathrm{L}}(\Phi)}}{q},$$

$$\frac{1+(3-2q)\mathrm{e}^{-\lambda_{\mathrm{L}}(\Phi)}}{q^2} \leq \eta_1 \leq \frac{1+(-3+2q)\mathrm{e}^{-\lambda_{\mathrm{L}}(\Phi)}}{q^2}.$$

Then according to (5.18) and (A.2) it is clear that restoration of synchronization time:

$$t_{\mathrm{RS}} = -\frac{1}{\ln(q-2)}.$$

(b) *Cycle graph*. Another family of the N-coupled maps generated with cycle graph [30], is introduced as follows:

$$S = \begin{pmatrix} 0 & 1 & 0 & \ldots & \\ \vdots & 0 & 1 & \ldots & \\ & \ldots & 0 & 0 & \end{pmatrix},$$

where S is the shift matrix. In cycle graph, adjancy matrices are introduced in the following form:

$$A_0 = I_N, \quad A_i = S^i + S^{-i}, \quad i = 1, ..., \left[\frac{N}{2}\right], \quad N = \text{odd}$$

$$A_0 = I_N, \quad A_i = S^i + S^{-i}, \quad i = 1, ..., \frac{N}{2} - 1, \quad A_{\frac{N}{2}} = S^{\frac{N}{2}}, \quad N = \text{even}.$$

Nearest neighbor coupled map lattice corresponds to cycle graph. Nearest neighbor coupled map [29] is used as a model in the simulation of natural phenomena such as clouds, smoke, fire and water. It is one of the most important research areas in computer graphics. By using (3.12) and $A = \epsilon A' + (1-\epsilon)A''$

$$A = (1-\eta_1)I + \frac{\eta_1}{2}(S + S^{-1}),$$

where $\eta_1 = \epsilon\epsilon_1' + (1-\epsilon)\epsilon_1''$. Applying Fourier method we can write idempotents:

$$E_j = \begin{pmatrix} 1 \\ \omega^j \\ \vdots \\ \omega^{(n-1)j} \end{pmatrix} (1, \omega^{-j}, ..., \omega^{-(N-1)j}),$$

where $\omega = \exp(2\pi i/N)$, so eigenvalues are

$$\tau_j = 1 - \eta_1 + \eta_1 \cos\frac{2\pi j}{N}.$$

The stability condition for cycle graph:

$$\eta_1 \leq \frac{1 + \mathrm{e}^{-\lambda_{\mathrm{L}}(X=\Phi(x))}}{2(1+\cos\frac{\pi}{N})}, \quad \text{for odd } N,$$

$$\eta_1 \geq \frac{1 - \mathrm{e}^{-\lambda_{\mathrm{L}}(X=\Phi(x))}}{2(1-\cos\frac{2\pi}{N})}, \text{ for even } N.$$

According to (5.18) and (A.2) it is clear that restoration of synchronization time

$$t_{\mathrm{RS}} = -\frac{1}{\ln|\cos\frac{2\pi}{N}|}.$$

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