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Bound values for Hall conductivity of heterogeneous medium under quantum Hall effect conditions

V E ARKHINCHEEV*
Buryat Science Center of Russian Academy of Sciences, Str. Sakhyanovoi 6, Ulan-Ude, 670047 Russia
*Present address: Federal Urdu University of Arts, Science and Technology, Karachi, Pakistan
E-mail: varkhin@yahoo.com

Abstract. Bound values for Hall conductivity under quantum Hall effect (QHE) conditions in inhomogeneous medium has been studied. It is shown that bound values for Hall conductivity differ from bound values for metallic conductivity. This is due to the unusual character of current percolation under quantum Hall effect conditions.

Keywords. Conductivity; percolation; quantum Hall effect; bound values.

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1. Introduction

When we study the current percolation in inhomogeneous medium, we need to study the problem of the effective conductivity of the medium. By definition, the effective conductivity $\sigma_{\rm e}$ is the coefficient of proportionality between average current $\vec{J} = \langle \vec{j}\,\rangle$ and average field $\vec{E} = \langle \vec{e}\,\rangle$, i.e.,

$$\vec{J} = \sigma_{\rm e}\vec{E}. \tag{1}$$

In the inhomogeneous medium, the bound values for effective conductivity are [1,2]:

$$\left\langle \frac{1}{\sigma} \right\rangle^{-1} \leq \sigma_{\rm e} \leq \langle \sigma \rangle. \tag{2}$$

Here $\langle a \rangle$ means the average value of the quantity a. These estimations have been obtained from Joule's dissipation, which is described by the following formula:

$$Q = \frac{1}{V}\int (\vec{j},\ \vec{e})\mathrm{d}V = \vec{J}\vec{E} = \sigma_{\rm e}\vec{E}^2. \tag{3}$$

The lower bound value has been obtained for a case $\langle\vec{J}\rangle$ = const. and the upper bound value has been obtained for a case $\langle\vec{E}\rangle$ = const. It means that if we insert the value for average current $\langle\vec{J}\rangle$ in eq. (3) we obtain the lower bound value for effective conductivity (2). If we insert the average field $\langle\vec{E}\rangle$, we obtain the upper bound value of formula (2).

In the case of current percolation in heterogeneous two-dimensional medium under quantum Hall effect (QHE) conditions $\sigma_{xx} = 0,\ \sigma_{xy} = \text{const.}$ [2,3], we cannot find the bound values from formula (3) because under QHE regime the electric current always flows perpendicular to the electric field.

$$\vec{j} = \sigma_{xy}[\vec{n}, \vec{e}]. \tag{4}$$

Here $\vec{n}$ is the unit vector, which is perpendicular to the considered plane and directed along magnetic field. So Joule's heating (3) is always zero at QHE conditions:

$$Q = 0. \tag{5}$$

So Hall phases are non-dissipative phases. Also from standard boundary conditions and the expression for Hall current (5), we obtain the new boundary conditions:

$$j_{1n} = j_{2n} = 0. \tag{6}$$

According to these boundary conditions, current (transfer) will not pass through the interface of phases, except through a few singular points. It seems that due to these new boundary conditions (6) the effective Hall conductivity must be equal to zero.

The aim of this paper is to show that non-zero value of Hall conductivity under QHE conditions is formed due to percolation through singular points. In the case of the layered media this singular point is placed at infinity, and in the case of 'check-board' structure these singular points, which are responsible for percolation, are placed at the corners of the 'check-board' cell. Exact expressions for Hall conductivity in inhomogeneous medium under QHE conditions have been obtained for layered and for randomly inhomogeneous medium. The lower and upper bound values have been established.

2. Features of percolation in layered media under QHE regime

To understand the features of current percolation under QHE conditions, let us consider a simple model of layered media, consisting of two alternating layers with different Hall conductivities $\sigma_{xy}^{(1)}$ and $\sigma_{xy}^{(2)}$. When electric current flows perpendicular to the interfaces of layers, then according to (4) electric field directs along layers. From standard boundary conditions for tangential components we obtain that the electric field has constant value and it is equal to the average value. So we obtain the formulae for effective conductivity as the average value of Hall conductivities in this case:

$$\sigma_{xy}^{e} = \frac{\sigma_{xy}^{(1)} + \sigma_{xy}^{(2)}}{2}. \tag{7}$$

To check this result we calculate the distributions of electric fields and currents at phases, using the definitions of averaged field and averaged current. After simple calculations we obtain the formulae for electric fields (currents) at every phase:

$$\langle e\rangle_1 = E\frac{\sigma^e_{xy}-\sigma_{(2)}}{\sigma^{(1)}_{xy}-\sigma^{(2)}_{xy}}, \quad \langle e\rangle_2 = E\frac{\sigma^e_{xy}-\sigma_{(1)}}{\sigma^{(1)}_{xy}-\sigma^{(2)}_{xy}}. \tag{8}$$

By inserting formula (7) into expressions (8) it is easy to check that we find the solution with *constant electric field*. The obtained result for effective conductivity (7) needs to clear its physical sense. Because according to the boundary conditions (6) the Hall current cannot cross interfaces of conducting phases, exept a few singular points. In the case of layered medium this singular point has been placed at infinity. The considered Hall edge currents, which flow along phase interfaces, have crossed at this singular point. As a result, the value for effective Hall conductivity (7) became non-zero.

Let us find the solution with constant electric current. For this, we study another case, when electric current flows along layers. Let us suppose that in this case the normal component of electric current has a constant value:

$$j_{1t} = j_{2t} = J_y/2. \tag{9}$$

Obviously, it is not so, but we want to obtain solution in such a form. After averaging we obtain the formula for Hall conductivity in the form

$$\sigma^e_{yx} = \frac{2\sigma^{(1)}_{xy}\sigma^{(2)}_{xy}}{\sigma^{(1)}_{xy}+\sigma^{(2)}_{xy}}. \tag{10}$$

Inserting this formula (10) into expressions for electric fields at every phases (8) we confirm our previous supposition (9). Consequently, the effective Hall conductivity for layered media has a tensor form:

$$\hat{\sigma}^e_{xy} = \begin{pmatrix} 0 & \frac{1}{2}(\sigma^{(1)}_{xy}+\sigma^{(2)}_{xy}) \\ \frac{2\sigma^{(1)}_{xy}\sigma^{(2)}_{xy}}{\sigma^{(1)}_{xy}+\sigma^{(2)}_{xy}} & 0 \end{pmatrix}.$$

It is interesting to note that components of effective Hall conductivity tensor are not equal:

$$\sigma^e_{xy} \neq \sigma^e_{yx}. \tag{11}$$

It seems that our results are in contradiction to the usual Onsager's duality relations for kinetic coefficients. But these Onsager's relations are correct for symmetrical isotropic and homogeneous medium. In our case of layered two-dimensional media the symmetry has broken, because we have certain direction in the system – direction of layers. So the Onsager's relations are not applicable.

3. Dykhne approach, based on rotational symmetry transformations

To find effective Hall conductivity in heterogeneous medium we use the general approach, based on the rotational symmetry of two-dimensional system [1,4–7]. The two-dimensional equations of DC current

$$\text{div}\ \vec{j} = 0, \quad \text{curl}\ \vec{e} = 0, \quad \vec{j} = \sigma\vec{e} \tag{12}$$

are invariant under local rotational transformations:

$$\vec{j} = b[n\vec{e'}], \quad \vec{e} = d[n\vec{j'}], \tag{13}$$

where $\vec{n}$ is a unit vector, normal to the plane and b, d are constant coefficients of rotational transformations. Due to its linearity, the Ohm's law likewise holds in the new (primed) system:

$$\vec{j'} = \sigma'\vec{e}'. \tag{14}$$

The conductivity of the new primed system equals to

$$\sigma' = \frac{b}{d\sigma}. \tag{15}$$

In a magnetic field $\vec{B}$, directed perpendicular to the plane, the Ohm's law has a tensor form

$$\vec{j} = \hat{\sigma}\vec{e}, \tag{16}$$

where $\hat{\sigma}$ is the conductivity tensor in a magnetic field with components σ_{xx} and σ_{xy}. In the case of magnetic field general linear rotational transformations have been used:

$$\vec{j} = a\vec{j'} + b[n, \vec{e'}], \quad \vec{e} = c\vec{e}' + d[n, \vec{j'}] \tag{17}$$

For the primed system we obtain also the Ohm's law in a tensor form with the following expressions for components:

$$\sigma'_{xx} = \frac{\sigma_{xx}(ac+bd)}{(\sigma_{xx}d)^2 + (\sigma_{xy}d+a)^2}, \quad \sigma'_{xy} = \frac{\sigma_{xx}^2 cd + (\sigma_{xy}c - b)(\sigma_{xy}d+a)}{(\sigma_{xx}d)^2 + (\sigma_{xy}d+a)^2}. \tag{18}$$

4. Percolation at heterogeneous medium under QHE regime

Below we consider the random mixture of the two Hall phases and we show that the effective Hall conductivity of the medium has constant value, which is equal to the value of first or second Hall phases.

Firstly we find all possible symmetry transformations. At least there are symmetrical transformations [4,5], at which the initial system in a magnetic field may be transformed to the new primed system.

(1) Interchange the phases of considered system by places $1 \leftrightarrow 2$.

(2) Interchange the places of phase $1 \leftrightarrow 2$ plus change of the magnetic field direction $B \leftrightarrow -B$.

(3) Change of the magnetic field direction only $B \leftrightarrow -B$.

At the first transformation, the initial two-phase system in a magnetic field may be transformed by the rotational transformations to the dual system:

$$\hat{\sigma}_1' = \hat{\sigma}_2, \quad \hat{\sigma}_2' = \hat{\sigma}_1. \tag{19}$$

The coefficients a, b, c, d have been determined by the condition (19). Consequently, the effective conductivity tensor of the primed system equals the effective conductivity of the dual system:

$$\hat{\sigma}_{\mathrm{e}}'(\epsilon) = \hat{\sigma}_{\mathrm{e}}(-\epsilon). \tag{20}$$

Here ϵ is a deviation from percolation threshold $x_c = 1/2$.

At the second transformation, primed system differs from initial one in the replacement of the phases and change in the direction of magnetic field, i.e., conductivities of initial and primed system are related as follows:

$$\sigma_{xx}'^{(1)} = \sigma_{xx}^2, \quad \sigma_{xx}'^{(2)} = \sigma_{xx}^1, \quad \sigma_{xy}' = -\sigma_{xy}. \tag{21}$$

In this case the components of the two systems are connected by the following relations:

$$\sigma_{xx}^{e\prime}(\epsilon) = \sigma_{xx}^{e}(-\epsilon), \quad \sigma_{xy}^{e\prime}(-\epsilon) = -\sigma_{xy}^{e}(\epsilon). \tag{22}$$

From formulae (18), (21) and (23) the following duality relations for effective characteristics have been obtained:

$$\frac{\sigma_{xx}^e(\epsilon)\sigma_{xy}^e(-\epsilon) \pm \sigma_{xx}^e(-\epsilon)\sigma_{xy}^e(\epsilon)}{\sigma_{xx}(\epsilon) \pm \sigma_{xx}(-\epsilon)} = \frac{\sigma_{xx}^1\sigma_{xy}^2 \pm \sigma_{xx}^2\sigma_{xy}^1}{\sigma_{xx}^1 \pm \sigma_{xx}^2}. \tag{23}$$

At the third transformation, the primed system differs from the initial one only in the inversion of the magnetic field direction:

$$\sigma_{xx}' = \sigma_{xx}, \quad \sigma_{xy}' = -\sigma_{xy}. \tag{24}$$

A general relation, connecting the components of the effective conductivity tensor at arbitrary phase concentrations, is obtained:

$$[(\sigma_{xx}^e(\epsilon))^2 + (\sigma_{xy}^e(\epsilon))^2]cd + \sigma_{xy}^e(\epsilon))(ac - bd) - ab = 0. \tag{25}$$

It is easy to see from relation (25) that under QHE conditions the non-diagonal Hall conductivity has a constant value, and at corresponding choice of coefficients it equals:

$$\sigma_{xy}^e = \sigma_{xy}^{(i)}, \quad i = 1, 2. \tag{26}$$

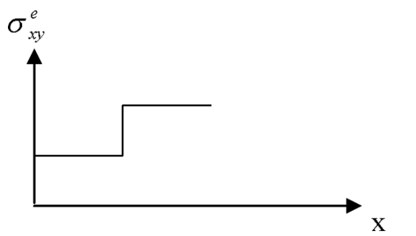

Figure 1. Plateaus of effective Hall conductivity at different concentrations of phases x.

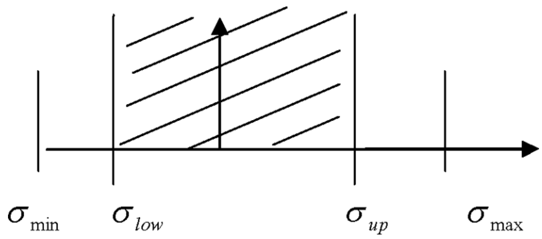

Figure 2. Bound values of Hall conductivity: stroked area is the expected area between usual values σ_{low} and σ_{up} and wider area is with new bound values $\sigma_{\min}$ and $\sigma_{\max}$.

Until there is a percolation at the first phase, the effective Hall conductivity equals the value of first phase conductivity. The electric field in the second phase equals zero according to the formula (8). Similarly, when the concentration of second Hall phase is above the percolation threshold, the effective Hall conductivity equals the value of second phase conductivity (see figure 1).

So, as it follows from the above consideration, we have the new lower and upper bound values for effective Hall conductivity under QHE conditions:

$$\min\left(\sigma_{xy}^{(1)}, \sigma_{xy}^{(2)}\right) \le \sigma_{xy}^{e} \le \max\left(\sigma_{xy}^{(1)}, \sigma_{xy}^{(2)}\right). \tag{27}$$

This result means that bond values under QHE regime depend on the values of conductivities and so on the connectivity of phases. In other words, these bound values have been determined by the topology of percolating ways. So the new bound values have been obtained for effective Hall conductivity under QHE conditions.

5. Conclusion

These results for bound values of effective Hall conductivity have been obtained as a result of unusual percolation of current in the quantum Hall effect regime. In this case the electric Hall current always is perpendicular to an electric field. Then from the equation $\operatorname{div}\vec{j} = 0$ with taking into account the potentiality of electric field $\operatorname{curl}\vec{e} = 0$ we obtain

$$\vec{e} \times \nabla\sigma_{xy} = 0. \tag{28}$$

That is, the current lines do not intersect the lines of constant values of the quantity σ_{xy} [8]. In other words, the Hall current does not percolate from one phase to another and it was 'frozen' in each of the phases. It is explained that the constant value of the plateau at the change of the phase concentration and also the value of the plateau has been determined by the conductivity of the percolating phase only, when there was an infinite cluster of certain first or second phase. It means also that value of plateau depends on the topology of percolating cluster. In this sense bound values of Hall conductivity of heterogeneous medium essentially differ from bound values for diagonal metallic effective conductivity. These results are illustrated by figure 2. The values of Hall conductivity in the strocked (touched) area in figure 2 have been limited by values $\sigma_{\rm low}$ and $\sigma_{\rm up}$, as was expected from estimations of metallic conductivity, and we find a new wider area, which is limited by new bound values $\sigma_{\rm min}$ and $\sigma_{\rm max}$, for Hall conductivity.

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