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Spectral properties of supersymmetric shape invariant potentials

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Abstract. We present the spectral properties of supersymmetric shape invariant potentials (SIPs). Although the folded spectrum is completely random, unfolded spectrum shows that energy levels are highly correlated and absolutely rigid. All the SIPs exhibit harmonic oscillator-type spectral statistics in the unfolded spectrum. We conjecture that this is the reflection of shape invariant symmetry.

Keywords. Supersymmetry; shape invariant potential; spectral statistics.

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1. Introduction

Energy level statistics is one of the most important and well-studied characteristics of quantum systems. This problem has recently attracted new interest in different contexts, because it indicates the type of motion in a quantum system. It is commonly believed that for classically completely integrable systems, the nearest-neighbour level spacing distribution is Poissonian and level repulsion, i.e., Wigner statistics is related to non-integrable chaotic classical motion. There is a lot of numerical evidence [1] in favour of these conjectures. Strong arguments have been given which indicate that for integrable systems with more than one degree of freedom, the $p(s)$ distribution is Poisson-like and the spacings are not correlated [2]. According to Bohigas [3], the Poissonian universality class of spectral fluctuation should obey GOE or GUE ensembles of random matrices depending on the presence (GOE) or absence (GUE) of time reversal symmetry. Later, Casati *et al* [4] and Seligman and Verbaarschot [5], gave simple examples where the distribution for the energy levels for some integrable systems is not uncorrelated Poisson distribution as it is commonly believed. The spectrum was found to be rather rigid. Then Shnirelman [6] showed that the presence of any discrete symmetry in the system results in delta function peak of $p(s)$ at s =0. The spectral distribution of some integrable systems [6] has also been studied. At the opposite extreme end, the spectral distribution is of picket fence type, having maximum correlation between spacings.

Here we are interested in some time-independent integrable systems which are exactly solvable owing to the existence of supersymmetric shape invariance symmetry. In the last decades several ambitious attempts [7–11] have been made to establish that supersymmetry is a necessary ingradient for the unifying approach. In 1981, Witten [12] proposed (0 + 1)-dimensional limit of SUSY quantum field theory, where the supercharges of SUSY quantum mechanics (SUSY QM) generate transformation between two orthogonal eigenstates of a given Hamiltonian with degenerate eigenvalues. Later, all exactly solvable models of quantum mechanics have been revisited [13] in the light of SUSY QM and it was shown by Gendenshtein [14] that exact solvability is simply due to shape invariance condition satisfied by the SUSY partner potentials. Here we study the spectral properties of the full family of shape invariant potentials of SUSY QM. All the potentials exhibit harmonic oscillator-type spectral properties (picket fence) in unfolded spectrum although the folded spectrum is completely random and uncorrelated. We conjecture this as the reflection of shape invariance symmetry in the spectral properties.

The paper is organized as follows. We will introduce *sl* algebra and the method of factorization in §2. Section 3 gives a brief description of fluctuation measures. Section 4 gives the results which are concluded in §5.

2. Factorization of SUSY Hamiltonian and shape invariance condition

The Hamiltonian of SUSY QM is given by

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}, \tag{1}$$

where

$$\begin{aligned} H_{1,2} &= -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_{1,2}(x) \\ V_{1,2} &= W^2 \mp W'(x), \end{aligned} \tag{2}$$

$W(x)$ is called superpotential. Then the supercharges are:

$$Q = \begin{bmatrix} 0 & 0 \\ A & 0 \end{bmatrix} \tag{3}$$

and

$$Q^\dagger = \begin{bmatrix} 0 & A^\dagger \\ 0 & 0 \end{bmatrix}, \tag{4}$$

where

$$A = \frac{\mathrm{d}}{\mathrm{d}x} + W(x), \quad A^\dagger = -\frac{\mathrm{d}}{\mathrm{d}x} + W(x). \tag{5}$$

Then it is easy to present $H_{1,2}$ as the factorization

$$H_1 = A^\dagger A, \quad H_2 = AA^\dagger. \tag{6}$$

Then the following commutation and anti-commutation relation describes the closed superalgebra $sl(1/1)$:

$$[H, Q] = \left[H, Q^\dagger\right] = 0 \tag{7}$$

and

$$\{Q^\dagger, Q^\dagger\} = \{Q, Q\} = 0; \quad \{Q, Q^\dagger\} = H. \tag{8}$$

The fact that the supercharges Q and $Q^\dagger$ commute with H is responsible for the degeneracy. The ground states of H is

$$|0\rangle = \psi_0(x) = \begin{pmatrix} \phi_0^{(1)(x)} \\ \phi_0^{(2)(x)} \end{pmatrix}, \tag{9}$$

where

$$\begin{aligned} A\phi_0^{(1)} &= 0 \Rightarrow \phi_0^{(1)} = N \exp\left(-\int^x W(y)\mathrm{d}y\right) \\ A^\dagger\phi_0^{(2)} &= 0 \Rightarrow \phi_0^{(2)} = N \exp\left(\int^x W(y)\mathrm{d}y\right). \end{aligned} \tag{10}$$

For unbroken supersymmetry

$$Q|0\rangle = Q^\dagger|0\rangle = 0|0\rangle. \tag{11}$$

Thus from eqs (7) and (8), ground state energy must be zero and ground state wave function for the matrix Hamiltonian

$$\psi_0(x) = \begin{pmatrix} \phi_0^{(1)(x)} \\ 0 \end{pmatrix}. \tag{12}$$

Later, with this idea of SUSY QM, the concept of shape invariance was put forward by Gendenshtein [14]. The condition for shape invariance requires same mathematical structure of $V_{1,2}$, i.e

$$V_2(x, a_1) = V_1(x, a_2) + R(a_1), \quad a_2 = f(a_1). \tag{13}$$

This condition is an integrability condition [14] and has been proved sufficient to get exact results. Using the shape invariance condition and repeating SUSY algebra to a hierarchy of Hamiltonians, one can easily obtain the full spectrum as

$$E_n^{(1)} = \sum_{k=1}^{n} R(a_k). \tag{14}$$

3. Fluctuation measures

We briefly introduce the measures frequently used to characterize level fluctuations. For a given energy level sequence, if we plot the cumulative density of eigenvalues $N(E)$ against E, it looks like a staircase function, and we can separate the smooth part which represents the average behaviour and a fluctuating part. So to study the spectral fluctuation, one should get rid of the average behaviour in order to compare the fluctuation properties of different systems whose average behaviours are not the same. This is called unfolding. So, for a given sequence $\{E_i\}$ of discrete spectrum, $N(E)$ is the spectral staircase function which counts the number of levels below E. Now it is possible to separate $N(E)$ in a smooth part $N_{\rm av}(E)$ and a fluctuating part $N_{\rm fl}(E)$ [15], i.e.,

$$N(E) = N_{\rm av}(E) + N_{\rm fl}(E). \tag{15}$$

Before studying fluctuations, one unfolds the original spectrum through some mapping $E \to \epsilon$

$$\{\epsilon_i\} = \{N_{\rm av}(E_i)\}, \quad i = 1, 2, \ldots. \tag{16}$$

The new sequence $\{\epsilon_i\}$ now has unit mean spacing. Thus the varying mean level density is removed in the new sequence. The level spacing $s_i = \epsilon_{i+1} - \epsilon_i$. Now all fluctuation measures will apply to this new sequence $\{\epsilon_i\}$. The nearest neighbour spacing distribution $p(s)$ is defined as the probability for finding the separation s of neighbouring levels in the spectrum. But $p(s)$ contains no information about spacing correlations. A simple measure of spacing correlation is the correlation coefficient c defined as [15]

$$c = \sum_i (s_i - 1)(s_{i+1} - 1) / \sum_i (s_i - 1)^2. \tag{17}$$

For Poisson spectrum $c = 0$. Another very convenient variable which is often used is spectral rigidity Δ_3. Given an interval $[\alpha, \alpha + L]$ of length L, Δ_3 measures the least square deviation of the spectral staircase $N(\epsilon)$ from the best straight line fitting it

$$\Delta_3(\alpha, L) = \frac{1}{L} {\rm Min}_{A,B} \int_\alpha^{\alpha+L} [N(\epsilon) - A\epsilon - B]^2 \, {\rm d}\epsilon. \tag{18}$$

A very convenient way to compute $\Delta_3(L)$ is [15]

$$\begin{aligned} \Delta_3 = {} & \frac{n^2}{16} - \frac{1}{L^2}\left[\sum_{i=1}^n \hat{\epsilon}_i\right]^2 + \frac{3n}{2L^2}\left[\sum_{i=1}^n \hat{\epsilon}_i^2\right] - \frac{3}{L^4}\left[\sum_{i=1}^n \hat{\epsilon}_i^2\right]^2 \\ & + \frac{1}{L}\left[\sum_{i=1}^n (n - 2i + 1)\hat{\epsilon}_i\right], \end{aligned} \tag{19}$$

where $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ is the ordered sequence of points in the interval $[\alpha, \alpha + L]$ and $\hat{\epsilon_i} = \epsilon_i - (\alpha + L/2)$. The value of $\Delta_3(L)$ for $L \geq 1$, for a picket fence spectrum is $\frac{1}{12}$, whereas $\langle \Delta_3(L) \rangle$ is $\frac{L}{15}$ for Poisson spectrum. Here $\langle \Delta_3(L) \rangle$ is the average value of $\Delta_3(\alpha, L)$. We first compute Δ_3 for different non-overlapping but adjacent segments of length L of the spectrum and take the average value.

Two kinds of spectra are important: (a) uncorrelated Poisson spectra, (b) random matrix spectra (GOE/GUE). Their spacing distributions are

$$\begin{aligned} p(s) &= \mathrm{e}^{-s} \text{ (Poisson)} \\ &= \frac{\pi}{2} s \mathrm{e}^{-(\pi/4)s^2} \text{ (GOE)} \\ &= \frac{32}{\pi^2} s^2 \mathrm{e}^{(-4/\pi)s^2} \text{ (GUE)}. \end{aligned} \tag{20}$$

At the opposite extreme, the picket fence having maximum correlation between spacings.

4. Results

In table 1, we present exactly solvable potentials of quantum mechanics having $sl(1/1)$ algebraic structure. For completeness we also present the parameters a_1 and a_2 with energy eigenvalues. The range of potentials are $-\infty \le x \le \infty$, $0 \le r \le \infty$. In the table we exclude shifted oscillator and 3D oscillator whose energy levels are strictly picket fence in folded spectrum. For the remaining potentials in the table, energy levels are completely random and uncorrelated in the folded spectrum. Most supersymmetric potentials in N dimension are related to Calogero or Calogero–Sutherland model. The existence of hidden continuous symmetry results in degenerate supermultiplates. For exact degeneracy $p(s)$ does not exist. So here we will consider some singular potentials in N dimension which are exactly solvable. The radial Schrödinger equation with spherically symmetric potential in N-dimensional space is

$$-\frac{1}{2}\left[\frac{\mathrm{d}^2R}{\mathrm{d}r^2} + \frac{N-1}{r}\frac{\mathrm{d}R}{\mathrm{d}r}\right] + \frac{l(l+N-2)}{2r^2}R = [E - V(r)]\,R. \tag{21}$$

Now taking $\psi(r) = r^{(N-1)/2}R(r)$, eq. (21) can be written as

$$-\frac{1}{2}\frac{\mathrm{d}^2\psi}{\mathrm{d}r^2} + \left[\frac{\Lambda(\Lambda+1)}{2r^2} + V(r)\right]\psi = E\psi, \tag{22}$$

where $\Lambda = l + \frac{N-3}{2}$.

Now its supersymmetric extension is quite straightforward just like one-dimensional Schrödinger equation with $V(r)$ where $a_1 = \Lambda$ and $a_2 = \Lambda + 1$. As an exactly solvable model we take the example of Mie-type potential, which is widely used in molecular physics. It also possesses the general features of true interaction energy and dynamical properties of solids [16]. The general form of this potential

$$V(r) = D_0\left[\frac{p}{q-p}\left(\frac{\sigma}{r}\right)^q - \frac{q}{q-p}\left(\frac{\sigma}{r}\right)^p\right], \tag{23}$$

Table 1. List of exactly solvable shape invariant potentials.

Name of the potential	Potential V_1	a_1	a_2	Restriction on parameters	Eigenvalue
Morse	$A^2 + B^2 e^{-2\alpha x}$ $-2B(A + \frac{\alpha}{2})e^{-\alpha x}$	A	$A - \alpha$	–	$A^2 - (A - n\alpha)^2$
Scarf I	$-A^2 + (A^2 + B^2 - A\alpha)$ $csc^2\alpha x - B(2A - \alpha)$ $\cot \alpha x \, csc\alpha x$	A	$A + \alpha$	$0 \le \alpha x \le \pi$ $A > B$	$(A + n\alpha)^2 - A^2$
Scarf II	$A^2 + (B^2 - A^2 - A\alpha)$ $\mathrm{sech}^2\,\alpha x + B(2A + \alpha)$ $\mathrm{sech}\,\alpha x \tanh \alpha x$	A	$A - \alpha$		$A^2 - (A - n\alpha)^2$
Rosen-Morse I	$A(A - \alpha)csc^2\alpha x$ $+2B \cot\, \alpha x - A^2 + \frac{B^2}{A^2}$	A	$A + \alpha$	$0 \le \alpha x \le \pi$	$(A + n\alpha)^2 - A^2$ $+\frac{B^2}{A^2} - \frac{B^2}{(A+n\alpha)^2}$
Rosen-Morse II	$A^2 + \frac{B^2}{A^2} - A(A + \alpha)$ $\mathrm{sech}^2\,\alpha x + 2B \tanh \alpha x$	A	$A - \alpha$	$B < A^2$	$A^2 - (A - n\alpha)^2$ $+\frac{B^2}{A^2} - \frac{B^2}{(A-n\alpha)^2}$
Pöschl-Teller I	$-(A + B)^2 + A(A - \alpha)$ $\sec^2\,\alpha x + B(B - \alpha)csc^2\alpha x$	(A, B)	$(A + \alpha,$ $B + \alpha)$	$0 \le \alpha x$ $\le \pi/2$	$(A + B + 2n\alpha)^2$ $-(A + B)^2$
Eckart	$A^2 + \frac{B^2}{A^2} + A(A - \alpha)$ $csch^2\alpha r - 2B\coth \alpha r$	A	$A + \alpha$	$B > A^2$	$A^2 - (A + n\alpha)^2$ $+\frac{B^2}{A^2} - \frac{B^2}{(A+n\alpha)^2}$
Gene. Pöschl-Teller	$A^2 + (B^2 + A^2 + A\alpha)$ $csch^2\alpha r - B(2A + \alpha)$ $\coth \alpha r csch\alpha r$	A	$A - \alpha$	$A < B$	$A^2 - (A - n\alpha)^2$
Pöschl-Teller II	$(A - B)^2 - A(A + \alpha)$ $\mathrm{sech}^2\,\alpha r + B(B - \alpha)csch^2\alpha r$	(A, B)	$(A - \alpha,$ $B + \alpha)$	$B < A$	$(A - B)^2$ $-(A - B - 2n\alpha)^2$

where D_0 is the interaction energy between two atoms in a molecular system at $r=\sigma$. For $q = 2p$ and $p = 1$, the potential is exactly solvable, i.e.,

$$V(r) = \frac{A}{r^2} - \frac{B}{r} \tag{24}$$

which is called Kratzer potential with $A = D_0\sigma^2$ and $B = 2D_0\sigma$.

Kasap *et al* [17] used supersymmetric quantum mechanics to find exact results of the Kratzer potential in three dimensions. Recently, these results have been generalized to N dimension [18], in the SSQM framework, which results in

$$E_n = \left(\frac{B/2\beta}{2n + 1 + [(2\Lambda + 1)^2 + A/\beta^2]^{1/2}}\right)^2, \quad n = 0, 1, 2, \ldots, \tag{25}$$

where $\beta = 1/2\sqrt{2}$.

Another very important singular potential is the pseudoharmonic potential [18]

$$\hat{V}(r) = V_0\left(\frac{r}{r_0} - \frac{r_0}{r}\right)^2 = \hat{B}r^2 + \frac{\hat{A}}{r^2} - 2V_0, \tag{26}$$

where r_0 is the equilibrium bond length, $\hat{B} = V_0/r_0^2$, $\hat{A} = V_0 r_0^2$.

This potential has various applications in molecular physics and it is also used in different non-linear systems. The exact full spectrum in N dimension is [18]

$$E_n = 2\beta\sqrt{\hat{B}}\left(4n+2+\left[(2\Lambda+1)^2+\frac{\hat{A}}{\beta^2}\right]^{1/2}\right)-2V_0, \quad n=0,1,2,\ldots. \tag{27}$$

But the energy levels of pseudoharmonic potential are always (even in higher dimension) equispaced both in the folded and unfolded spectra. So its $p(s)$ distribution is not presented.

For each potential we calculate lowest 10,000 levels by using simple analytic expression of E_n. In the folded spectrum the level spacing is *completely random* as expected, as it has varying mean level density. Then we unfold the spectrum using cubic spline smoothing, just to get rid of N_{av}, such that the new energy sequence x_i has a constant mean spacing. Then calculate $p(s)$ distribution from this unfolded spectrum. In figure 1, we plot $p(s)$ for different potentials. We try with different sets of parameters, keeping the restriction as mentioned in table 1. For Morse and Scarf II potentials the width of the delta function in $p(s)$ distribution depend on the parameters slightly. But the average behaviour is the same in all potentials listed in the table. We calculate the correlation coefficient c using eq. (17) and Δ_3 statistics using eq. (18). For Morse potential (in our chosen parameters) $c = 0.52$, but it varies from $c = 0.52$ to $c = 0.99$ with parameters. For the rest of the potentials $c = 0.99$ and $\Delta_3 = 0.083$ in all cases. For Kratzer potential we present our results for $N = 3$, for which $c = 0.92$ and $\Delta_3 = 0.084$. So the average response is the same in all cases. The most remarkable point is that the full family of potentials has the spectral statistics which is very close to that of harmonic oscillator. HO is a picket fence system. But in SUSY potentials, the energy levels are highly uncorrelated in their folded spectrum, having large fluctuating nature. It is of opposite extreme of HO energy levels which has maximum correlation. But to know the origin of this fluctuation we have to study unfolded spectrum. The unfolded spectrum reveals that the energy levels are absolutely rigid, and there is no sign of level clustering.

Next we say that the observation that all the SIPs have similar picket fence type spectral statistics is somehow connected with shape invariance symmetry. Here we try to correlate these two facts qualitatively. From table 1, it is clear that all the SIPs correspond to classically integrable systems and the Schrödinger equation is exactly analytically solvable for all these potentials, and the energy eigenvalues are given in closed analytic form. It implies that once we know the position of one energy level, then the position of any other level can be easily determined independent of its distance from the previous level. This is the insight coming from the shape invariance condition. The same picture is also reflected in the spectral distribution. $p(s)$ distribution is of picket fence type, and it means that it is perfectly uniform and equispaced. So once we know any energy level, the others can be easily determined, no matter how far it is. As all the shape invariant potentials satisfy same shape invariance symmetry, its reflection in the spectral statistics is same.

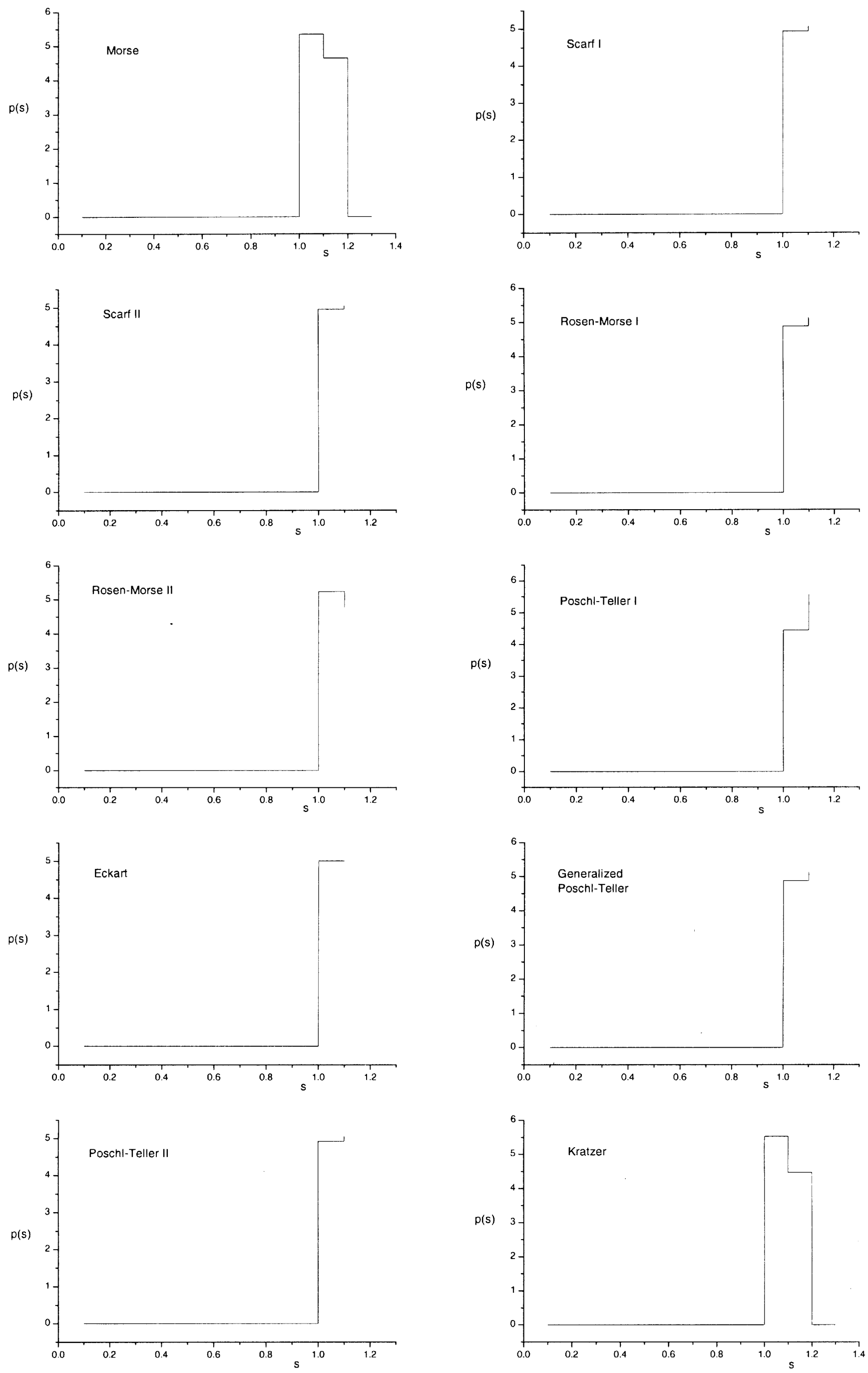

Figure 1. $p(s)$ distribution for the shape invariant potentials.

For the non-SIPs, as very few lower eigenvalues can be known analytically, which are small to calculate spectral fluctuation, we cannot give any deterministic answer for the broad class of non-SIPs. But from the already established conjecture we can say that for the spectrum to be regular, the energy levels will have Poisson distribution and the degree of correlation will depend on the nature of the particular system, whereas in the irregular spectrum levels may repel.

5. Conclusions

It is a well-established fact that systems with more than one degree of freedom have completely random energy level spacings. Lots of theoretical and numerical evidences are put forward in this context. Systems with some discrete symmetry will result in delta function peak at $s = 0$, but after separating this symmetry one will get back Poisson distribution. So spectral statistics is highly related to the symmetry. Here we considered a family of potentials which are exactly solvable due to the existence of supersymmetric shape invariance. We present an exhaustive study for such class of potentials. It results in HO-type statistics in the unfolded spectrum for all cases. So we conjecture that this identical response to HO-type statistics is simply the reflection of shape invariance symmetry. We think that the observation made in this article is completely new and has never been observed. We also try to establish some relevant connection between this observation and the shape invariance condition.

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