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A general approach to bosonization

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Abstract. We summarize recent developments in the field of higher dimensional bosonization made by Setlur and collaborators and propose a general formula for the field operator in terms of currents and densities in one dimension using a new ingredient known as a 'singular complex number'. Using this formalism, we compute the Green function of the homogeneous electron gas in one spatial dimension with short-range interaction leading to the Luttinger liquid and also with long-range interactions that lead to a Wigner crystal whose momentum distribution computed recently exhibits essential singularities. We generalize the formalism to finite temperature by combining with the author's hydrodynamic approach. The one-particle Green function of this system with essential singularities cannot be easily computed using the traditional approach to bosonization which involves the introduction of momentum cutoffs, hence the more general approach of the present formalism is proposed as a suitable alternative.

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1. Introduction

As the term suggests, 'bosonization' is an effort to recast theories involving entities that are not bosons in terms of bosons that are typically expressed in terms of bilinears of the original fields. Indeed it is even possible to recast theories involving bosons in terms of other bosons. This activity is not merely a pedantic exercise, for the end result of these efforts is a nonperturbative technique for studying the original theory. In particular, applying this technique to fermions in one spatial dimension leads to what are known as non(Landau)-Fermi liquids where the momentum distribution is continuous at the Fermi momentum. Thus one can bosonize fermions, spins and even complex scalar fields which are themselves bosons. Bosonization, according to our understanding, is nothing but the polar representation of a complex number. Bosonization of spins is accomplished by polar decomposing the ladder operators which leads to a semiclassical theory of spins. Complex scalar fields may be easily bosonized by polar decomposition as well. One may suspect that a similar decomposition should be feasible for fermions too. This has proved harder than one might hope. In one spatial dimension, the Thirring model which describes relativistic fermions self-interacting via a short-range repulsion was shown to be equivalent to the so-called Sine-Gordon theory which involves scalar fields [1]. This technique has been adapted to study condensed matter problems where the parabolic dispersion of the free fermions in Galilean invariant systems is linearized near the Fermi points so that the fermions now have linear dispersion and are moving with the Fermi velocity. The Fermi velocity takes on the role of the speed of light making the analogy complete. In more than one dimension, this program has not been particularly successful. In hindsight, it appears that it is premature to regard the framework available even in one spatial dimension as a closed subject. The Luttinger model caricature, it seems, is unable to handle some exotic situations involving truly long-range interactions in condensed matter systems. This was pointed out by the present author and this work is a summary and continuation of earlier works [2–6]. The work of Schulz [7] on electrons interacting with potential ($V(x) \sim 1/|x|$) using standard bosonization techniques is able to study the structure factor of the Wigner crystal quite effectively but when it comes to the one-particle Green function, the results are very sketchy and cutoff dependent. It is not clear whether the method of Schulz can be used to study the one-particle Green function of a 1D Fermi system with potential $V(x) \sim -|x|$, since our results show that the momentum distribution does not depend on arbitrary cutoffs, whereas they are mandatory in such standard bosonization methods. Furthermore, there are some technical subtleties involving the so-called Klein factors that ensure fermion commutation rules between Fermi fields in the traditional approach that are far from satisfactory. The purpose of the present article is to summarize recent developments in the subject made by the author and his collaborators and to present a formula for the field operator in terms of currents and densities that is valid in a general sense. We stress that this write up is by no means a review of the entire field of bosonization, it only highlights Setlur and collaborator's the author's contributions, while not completely ignoring the work of others. We then go on to use this formula to calculate the Green functions of interacting systems in one spatial dimension, one with short-range repulsion leading to the Luttinger liquid and also with a specific long-range interaction which leads to the Wigner crystal.

At this stage it is appropriate to survey some relevant literature on the subject of bosonization of fermions in general and higher-dimensional bosonization in particular. This subject, as is well-known, started with the papers of Tomonaga and Luttinger. Later on, Lieb and Mattis, Luther, Luther and Peschel developed it further. All these well-known literature may be found in the reviews and texts [8,9]. This program was taken up by Haldane [10] who coined the term 'Luttinger liquid' to describe systems whose Green functions have power law singularities rather than simple poles. The generalization of these ideas to higher dimensions was attempted by Luther. The effort to generalize these ideas to higher dimensions seemed phenomenologically ill-founded as one does not expect to see non-Fermi liquids in higher dimensions except perhaps for very long-range interactions which are un-physical. However, the phenomenon of high temperature superconductivity suggested the need for such a ground state. In the mid-nineties starting from the work of Haldane [11], Castro-Neto and Fradkin [12], Houghton *et al* [13], Kopietz and collaborators [14] attempted to develop a theory in higher dimensions along the lines of the Tomonaga–Luttinger theory. The works of Kopietz and collaborators deserve special mention since they have persisted with this technique over the years. These efforts closely mimic the theory in one dimension that is valid only in the sense of the random phase approximation (RPA) [15]. Attempts to go beyond this approximation using the original formalism seems futile. Hence Setlur and collaborators developed a new scheme loosely based on the work of Castro-Neto and Fradkin [12] to overcome these difficulties. In what follows, we describe the outcome of these efforts made by Setlur and his collaborators and go on to write down a formula for the field operator in terms of currents and densities.

2. Review of the general formalism

In what follows we describe briefly the main results in Setlur's earlier works. It must be stressed that this write-up is not a substitute for a reading of those works. Let $c_{\mathbf{k}}$ and $c^{\dagger}_{\mathbf{k}}$ be fermion annihilation and creation operators. We may define new operators using these that are called sea-bosons. They are in general, complicated nonlocal combination of number conserving products of Fermi fields [3]. However, in the sense of the random phase approximation (RPA) we may write

$$A_{\mathbf{k}}(\mathbf{q}) \approx n_{\mathrm{F}}(\mathbf{k}-\mathbf{q}/2)(1-n_{\mathrm{F}}(\mathbf{k}+\mathbf{q}/2))c^{\dagger}_{\mathbf{k}-\mathbf{q}/2}c_{\mathbf{k}+\mathbf{q}/2}. \tag{1}$$

Here $n_{\mathrm{F}}(\mathbf{p}) = \theta(k_{\mathrm{F}} - |\mathbf{p}|)$ is the momentum distribution of free fermions. This object $A_{\mathbf{k}}(\mathbf{q})$ has been shown to obey the following commutation rules [3]:

$$\begin{aligned} &[A_{\mathbf{k}}(\mathbf{q}), A^{\dagger}_{\mathbf{k}'}(\mathbf{q}')] = n_{\mathrm{F}}(\mathbf{k}-\mathbf{q}/2)(1-n_{\mathrm{F}}(\mathbf{k}+\mathbf{q}/2))\delta_{\mathbf{k},\mathbf{k}'}\delta_{\mathbf{q},\mathbf{q}'};\\ &[A_{\mathbf{k}}(\mathbf{q}), A_{\mathbf{k}'}(\mathbf{q}')] = 0; \quad [A^{\dagger}_{\mathbf{k}}(\mathbf{q}), A^{\dagger}_{\mathbf{k}'}(\mathbf{q}')] = 0. \end{aligned} \tag{2}$$

The defining equation for $A_{\mathbf{k}}(\mathbf{q})$ may be partially inverted and a formula for the number conserving product of two Fermi fields may be written down in terms of these sea-bosons. Define $c_{\mathbf{k},<} = n_{\mathrm{F}}(\mathbf{k})c_{\mathbf{k}}$ and $c_{\mathbf{k},>} = (1-n_{\mathrm{F}}(\mathbf{k}))c_{\mathbf{k}}$. At the level of RPA we may write, $c^{\dagger}_{\mathbf{k}-\mathbf{q}/2,<}c_{\mathbf{k}+\mathbf{q}/2,>} \approx A_{\mathbf{k}}(\mathbf{q})$ and $c^{\dagger}_{\mathbf{k}+\mathbf{q}/2,<}c_{\mathbf{k}-\mathbf{q}/2,<} \approx 0$ and $c^{\dagger}_{\mathbf{k}+\mathbf{q}/2,>}c_{\mathbf{k}-\mathbf{q}/2,>} \approx 0$. Thus at the RPA level $c^{\dagger}_{\mathbf{k}+\mathbf{q}/2}c_{\mathbf{k}-\mathbf{q}/2} \approx A_{\mathbf{k}}(-\mathbf{q}) + A^{\dagger}_{\mathbf{k}}(\mathbf{q})$. In general (beyond RPA) we may write [3] ($\mathbf{q} \neq 0$),

$$\begin{aligned} c^{\dagger}_{\mathbf{k}+\mathbf{q}/2}c_{\mathbf{k}-\mathbf{q}/2} \approx\; & A_{\mathbf{k}}(-\mathbf{q}) + A^{\dagger}_{\mathbf{k}}(\mathbf{q})\\ & + \sum_{\mathbf{q}_1} A^{\dagger}_{\mathbf{k}+\mathbf{q}/2-\mathbf{q}_1/2}(\mathbf{q}_1)A_{\mathbf{k}-\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1)\\ & - \sum_{\mathbf{q}_1} A^{\dagger}_{\mathbf{k}-\mathbf{q}/2+\mathbf{q}_1/2}(\mathbf{q}_1)A_{\mathbf{k}+\mathbf{q}_1/2}(-\mathbf{q}+\mathbf{q}_1). \end{aligned} \tag{3}$$

The above identification together with eq. (2) can be shown to be sufficient to reproduce the exact commutators between Fermi bilinears. The correspondence in eq. (3) is also sufficient to reproduce all the dynamical correlation functions of the operator $c^{\dagger}_{\mathbf{k}+\mathbf{q}/2}c_{\mathbf{k}-\mathbf{q}/2}$ of the free theory provided we set the kinetic energy operator to be $K = \sum_{\mathbf{kq}} \left(\frac{\mathbf{k}\cdot\mathbf{q}}{m}\right) A^{\dagger}_{\mathbf{k}}(\mathbf{q})A_{\mathbf{k}}(\mathbf{q})$ (for more details, please consult our published works). Thus any theory involving fermions that conserve their total number may be re-expressed in terms of these bosons. The main purpose of the present article is to *fully* invert the defining equation for $A_{\mathbf{k}}(\mathbf{q})$ and express the field operator $c_{\mathbf{p}}$ alone in terms of these bosons. To accomplish this, we first attempt to polar decompose the field operator in real space $\psi(\vec{r}) = \frac{1}{\sqrt{V}}\sum_{\mathbf{p}} \mathrm{e}^{i\mathbf{p}\cdot\vec{r}} c_{\mathbf{p}}$.

$$\psi(\vec{r}) = \mathrm{e}^{i\Lambda([\rho];\vec{r})}\mathrm{e}^{-i\Pi(\vec{r})}\sqrt{\rho(\vec{r})}. \tag{4}$$

Here we have introduced a new variable $\Pi(\vec{r})$ which is a canonical conjugate to $\rho(\vec{r}) = \psi^{\dagger}(\vec{r})\psi(\vec{r})$ [16]. This means that $[\Pi(\vec{r}), \Pi(\vec{r}\,')] = 0$ and $[\Pi(\vec{r}), \rho(\vec{r}\,')] = i\delta(\vec{r} - \vec{r}\,')$ and of course $[\rho(\vec{r}), \rho(\vec{r}\,')] = 0$. As an operator we know that ρ is nonnegative. It is then well-known that Π cannot be self-adjoint. In fact we may write $\Pi(\vec{r}) = X_0 + \tilde{\Pi}(\vec{r})$ where $\tilde{\Pi}(\vec{r})$ is strictly self-adjoint and X_0 is conjugate to the total number $[X_0, \hat{N}] = i$. In what follows we regard the number operator to be equal to infinity (thermodynamic limit) in which case X_0 should be arbitrarily small in order for it to be a conjugate to the number operator. Hence we shall not be too careful and we shall treat X_0 as a real c-number and ignore it altogether.

We now divert our attention and examine the property of the fermion current (density) operator $\mathbf{J}(\vec{r}) = \mathrm{Im}[\psi^{\dagger}(\vec{r})\nabla\psi(\vec{r})]$. In two and three dimensions, we may construct the operator $\mathbf{W} = \rho(\nabla\times\mathbf{J}) - \nabla\rho\times\mathbf{J}$. We first represent currents and densities in real space $\rho(\vec{r}) = \sum_i \delta(\vec{r}-\vec{r}_i)$ and $\mathbf{J}(\vec{r}) = \sum_i \frac{-i\nabla_i}{2}\delta(\vec{r}-\vec{r}_i) + \sum_i \delta(\vec{r}-\vec{r}_i)\frac{-i\nabla_i}{2}$. By acting this on fermionic wave functions in real space we conclude that $\mathbf{W}\equiv 0$. This means that $\mathbf{W} = \rho^2\nabla\times\frac{1}{\rho}\mathbf{J} = 0$. In other words, there exists a scalar Π such that $\mathbf{J} = -\rho\nabla\Pi$. Independently we may argue that a more general ansatz $\mathbf{J} = -\rho\nabla\Pi + \mathbf{C}([\rho];\vec{r})$ fails to reproduce the current–current commutator unless $\mathbf{C}\equiv 0$. It appears then, that the function Λ in eq. (4) should be independent of $\vec{r}$. This Λ is crucial since it determines the statistics of the field ψ. In particular, setting $\Lambda = 0$ describes bosons rather than fermions. In our earlier work [6], we argued that the ansatz in eq. (4) may be used to derive an action in terms of Π and ρ. In the Lagrangian formalism, there are no operators. We denote the operators Π and ρ to the status of real numbers and use eq. (4) in the action for free fermions, $S = \int_0^{-i\beta}\mathrm{d}t\int \mathrm{d}^d x\psi^{\dagger}(i\partial_t + \frac{\nabla^2}{2m})\psi$. This led to the following action for free fermions:

$$S = \int_0^{-i\beta}\mathrm{d}t\int \mathrm{d}^d x\left(\rho\partial_t\Pi - V_{\mathrm{F}}([\rho];\mathbf{x}) - \frac{\rho(\nabla\Pi)^2 + \frac{(\nabla\rho)^2}{4\rho}}{2m}\right). \tag{5}$$

Here V_{F} is a functional of the density that has to be fixed by making contact with the properties of the free theory. We have shown that the RPA limit of the above action may be rigorously derived using sea-bosons [6]. We have also shown [6] that this leads to the following expression for the generating function of density correlations of a homogeneous electron gas in terms of the corresponding quantity for the free theory. If $Z([U])$ is the function that generates density correlations of the homogeneous electron gas where the electrons interact with a potential $v_{\mathbf{q}}$ and $Z_0([U])$ is the corresponding quantity of the free theory (for example $\left(\frac{\delta^2 Z([U])}{\delta U_{\mathbf{q},n}\delta U_{-\mathbf{q},-n}}\right)_{U\equiv 0} \equiv \langle \rho_{\mathbf{q},n}\rho_{-\mathbf{q},-n}\rangle$ and $\langle T\rho(\mathbf{q},t)\rho(-\mathbf{q},t')\rangle = \sum_n \mathrm{e}^{-w_n(t-t')}\langle \rho_{\mathbf{q},n}\rho_{-\mathbf{q},-n}\rangle$ and $w_n = 2\pi n/\beta$ is the Matsubara frequency) then

$$Z([U]) = \int D[U']\mathrm{e}^{\sum_{\mathbf{q}n}\frac{V}{2\beta v_{\mathbf{q}}}(U_{\mathbf{q}n}-U'_{\mathbf{q}n})(U_{-\mathbf{q},-n}-U'_{-\mathbf{q},-n})}Z_0([U']). \tag{6}$$

It appears that we have to judiciously combine the Hamiltonian or the operator version and the Lagrangian version in order to obtain useful results. Thus we wish to now revert to the operator description to try and express the field operator explicitly in terms of currents and densities. We observed that current–current commutator implies that Λ is independent of $\vec{r}$. Unfortunately, this conflicts with the requirement that ψ obey fermion commutation rules. Indeed, imposing these rules on ψ in eq. (4) leads to the following constraint on Λ:

$$\mathrm{e}^{i\Lambda([\rho];\vec{r})}\mathrm{e}^{i\Lambda([\{\rho(\vec{x})+\delta(\vec{x}-\vec{r})\}];\vec{r}\,')} = -\mathrm{e}^{i\Lambda([\rho];\vec{r}\,')}\mathrm{e}^{i\Lambda([\{\rho(\vec{x})+\delta(\vec{x}-\vec{r}\,')\}];\vec{r})}. \tag{7}$$

If Λ is independent of $\vec{r}$ this is impossible. Hence we seem to have reached an impasse. There is a way out of this difficulty using the notion of what may be called 'singular complex numbers'. We describe this concept in the following section.

3. Field operator using singular complex numbers

In our earlier work [6] we argued that the field operator in momentum space may be expressed directly in terms of the sea-bosons provided we invoke the concept of a singular complex number: $w_{\mathbf{p}} = \mathrm{e}^{-iN^0\xi_{\mathbf{p}}}$, where $N^0 \to \infty$ and $\xi_{\mathbf{p}}$ is arbitrary. Therefore, $w_{\mathbf{p}}\bar{w}_{\mathbf{p}'} = \bar{w}_{\mathbf{p}'}w_{\mathbf{p}} = \delta_{\mathbf{p},\mathbf{p}'}$. This rather unusual quantity may be motivated using the following argument. Consider the number operator $n_{\mathbf{k}}$. We may introduce formally a conjugate $P_{\mathbf{k}}$ namely an operator that obeys $[P_{\mathbf{k}}, P_{\mathbf{k}'}] = 0$ and $[P_{\mathbf{k}}, n_{\mathbf{k}'}] = i\delta_{\mathbf{k},\mathbf{k}'}$. If we are going to treat $n_{\mathbf{k}}$ as a c-number namely, $n_{\mathbf{k}} = n_{\mathrm{F}}(\mathbf{k})\mathbf{1}$, then we have to ensure that $P_{\mathbf{k}}$ is a formally infinite c-number in order that it is a conjugate to $n_{\mathbf{k}}$. Thus $w_{\mathbf{k}} = \mathrm{e}^{-iP_{\mathbf{k}}}$ has the properties that we have just described. We argue that the term $\mathrm{e}^{i\Lambda}$ may be rewritten using these complex numbers. Let us invoke the following ansatz that is inspired from our early work [2]:

$$\psi(\vec{r}) = \left(\frac{1}{\sqrt{N^0}}\sum_{\mathbf{p}}\mathrm{e}^{iE_{\mathbf{p}}([\rho];\vec{r})}w_{\mathbf{p}}\mathrm{e}^{ik_{\mathrm{F}}\hat{p}\cdot\vec{r}}n_{\mathrm{F}}(\mathbf{p})\right)\mathrm{e}^{-i\Pi(\vec{r})}\sqrt{\rho(\vec{r})}, \tag{8}$$

$$\psi^{\dagger}(\vec{r}) = \sqrt{\rho(\vec{r})}\mathrm{e}^{i\Pi(\vec{r})}\left(\frac{1}{\sqrt{N^0}}\sum_{\mathbf{p}}\mathrm{e}^{-iE_{\mathbf{p}}([\rho];\vec{r})}\bar{w}_{\mathbf{p}}\mathrm{e}^{-ik_{\mathrm{F}}\hat{p}\cdot\vec{r}}n_{\mathrm{F}}(\mathbf{p})\right). \tag{9}$$

Here $E_{\mathbf{p}}([\rho];\vec{r})$ is real. The idea is that the rapidly varying part is written separately as a multiplying exponent $\mathrm{e}^{\pm ik_{\mathrm{F}}\hat{p}\cdot\vec{r}}$. The slowly varying portion is in the density and phase variables. Notice that the theory presented here is very general and there are no momentum cutoffs at the outset. Therefore, the terms 'slow' and 'fast' are merely suggestive of approximations that will have to be used in the practical computations where such a distinction acquires concrete meaning. If we postulate that $\nabla E_{\mathbf{p}}([\rho];\vec{r}) = -\nabla E_{-\mathbf{p}}([\rho];\vec{r})$ then we find that, $\rho(\vec{r}) = \psi^{\dagger}(\vec{r})\psi(\vec{r})$ and $\mathbf{J}(\vec{r}) = \mathrm{Im}[\psi^{\dagger}(\vec{r})(\nabla\psi(\vec{r}))] = -\rho(\vec{r})\nabla\Pi(\vec{r})$. Therefore, current algebra – the mutual commutation rules between currents and densities is then trivially obeyed. In other words, $\mathrm{e}^{i\Lambda([\rho];\vec{r})} = \frac{1}{\sqrt{N^0}}\sum_{\mathbf{p}} \mathrm{e}^{iE_{\mathbf{p}}([\rho];\vec{r})} w_{\mathbf{p}} \mathrm{e}^{ik_{\mathrm{F}}\hat{p}\cdot\vec{r}} n_{\mathrm{F}}(\mathbf{p})$ is unitary. Now we have to apply fermion commutation rules. We expect to derive a recursion relation similar to eq. (7). Indeed we find that both the requirement $\{\psi(\vec{r}),\psi(\vec{r}\,')\} = 0$ and $\{\psi(\vec{r}),\psi^{\dagger}(\vec{r}\,')\} = \delta(\vec{r}-\vec{r}\,')$ are obeyed if we ensure that the following recursion holds:

$$\mathrm{e}^{-iE_{\mathbf{p}}([\{\rho(\vec{x})+\delta(\vec{x}-\vec{r}\,')];\vec{r})} \mathrm{e}^{iE_{\mathbf{p}}([\rho];\vec{r})} = -\mathrm{e}^{-iE_{\mathbf{p}'}([\{\rho(\vec{x})+\delta(\vec{r}-\vec{x})\}];\vec{r}\,')} \mathrm{e}^{iE_{\mathbf{p}'}([\rho];\vec{r}\,')}. \tag{10}$$

In one dimension, we try the following ansatz. Note that $\hat{p} = \pm 1$ in one dimension. $E_{p'}([\rho];r') = \int_{-\infty}^{\infty} \mathrm{d}x \rho(x) D_{\hat{p}'}(x,r')$. The recursion relation implies, $\mathrm{e}^{-iD_{\hat{p}}(r',r)} = -\mathrm{e}^{-iD_{\hat{p}'}(r,r')}$ whereas the unitarity condition forces us to choose, $D_{\hat{p}'}(x,r') = -\hat{p}' D_{\mathrm{red}}(x,r')$. Both these are satisfied by the choice $D_{\hat{p}'}(x,r') = -\pi\hat{p}'\theta(x-r')$, where $\theta(x)$ is the Heaviside step function ($\theta(x \leq 0) = 0$ and $\theta(x > 0) = 1$). We have to verify that this choice reproduces the Green function of the noninteracting theory. This is shown later on. In higher dimensions we similarly try, $E_{\mathbf{p}}([\rho];\vec{r}) = \int \mathrm{d}^d x \rho(\vec{x}) D_{\hat{p}}(\vec{x},\vec{r})$. The recursion condition in higher dimensions: $\mathrm{e}^{-iD_{\hat{p}}(\vec{r}\,',\vec{r})} = -\mathrm{e}^{-iD_{\hat{p}'}(\vec{r},\vec{r}\,')}$ seems rather hard to satisfy since $\hat{p}$ and $\hat{p}'$ can be parallel to each other, antiparallel, or anything in between. If we are willing to require that only $\{\psi(\vec{r}),\psi^{\dagger}(\vec{r}\,')\} = \delta(\vec{r}-\vec{r}\,')$ be obeyed then a simple choice might be sufficient. In any event it is the propagator that we are interested in and for this it is this rule that is important. In this case $\hat{p} = \hat{p}'$ and we may choose $D_{\hat{p}}(\vec{r},\vec{r}\,') = \pi\theta(\hat{p}\cdot(\vec{r}-\vec{r}\,'))$. This choice obeys both the unitarity condition $\nabla D_{\hat{p}}(\vec{r},\vec{r}\,') = -\nabla D_{-\hat{p}}(\vec{r},\vec{r}\,')$ and the recursion. The recursion breaks down when $\hat{p}\cdot(\vec{r}-\vec{r}\,') = 0$ but these are a set of points of measure zero. Unfortunately, it can be shown that this choice fails to reproduce the free propagator. This suggests that the function $E_{\mathbf{p}}$ may be a nonlinear function of the density in higher dimensions. Therefore, in higher dimensions it is better to try and express the field variable directly in terms of sea-bosons. A partially correct formula has been provided in an earlier work [6]. It appears that more work is needed to make that agenda practically useful. For now we use the expression for the field in terms of the hydrodynamic variables to calculate the Green function in one dimension, first of the Luttinger liquid. In the case of the Wigner crystal we find that a reinterpretation is needed that essentially involves re-expressing the field directly in terms of sea-bosons. For example, we may rewrite the field in one dimension as follows:

$$\psi(r,t) \approx \left(\frac{\mathrm{e}^{-i\epsilon_{\mathrm{F}}t}}{\sqrt{L}} \sum_{p} \mathrm{e}^{\sum_q \mathrm{e}^{-iqr}[-\frac{\hat{p}\pi}{qL}\rho_q(t) - iX_{-q}(t)]} w_p \mathrm{e}^{ik_{\mathrm{F}}\hat{p}r} n_{\mathrm{F}}(p) \right). \tag{11}$$

The rapidly varying $\mathrm{e}^{-i\epsilon_{\mathrm{F}}t}$ has to be put in by hand, however it may be motivated by realizing that the $X_{q=0}$ term is conjugate to the total number of particles and this picks up a contribution similar to the one suggested upon time evolution with respect to the Hamiltonian of the system since, for example, the Hamiltonian of the free Fermi theory may be written as $H = N\epsilon_0 + \sum_{k,q} \frac{k\cdot q}{m} A_k^\dagger(q) A_k(q)$. If we set $\epsilon_0 = \epsilon_{\rm F}$ then we recover the factor suggested. We write $\rho_q = \sum_k [A_k(-q) + A_k^\dagger(q)] = \rho_{\rm R}(q) + \rho_{\rm L}(q)$, where $\rho_{\rm R}(q) = \sum_{k>0}[A_k(-q) + A_k^\dagger(q)]$ and $\rho_{\rm L}(q) = \sum_{k<0}[A_k(-q) + A_k^\dagger(q)]$ and $X_q = \frac{ik_{\rm F}q}{Nq^2}(\rho_{\rm R}(-q) - \rho_{\rm L}(-q))$. If $\hat{p} = +1$, then $[-\frac{\pi}{qL}\rho_q - iX_{-q}] = -\frac{2\pi}{qL}\rho_{\rm R}(q)$. If $\hat{p} = -1$, we have $[\frac{\pi}{qL}\rho_q - iX_{-q}] = \frac{2\pi}{qL}\rho_{\rm L}(q)$. Thus we have the familiar result for right and left movers. We may write $\psi(r,t) = \psi_{\rm R}(r,t) + \psi_{\rm L}(r,t)$ or,

$$\psi(r) \approx \left(\frac{1}{\sqrt{L}}\sum_{p>0} {\rm e}^{-\sum_q {\rm e}^{-iqr}\frac{2\pi}{qL}\rho_{\rm R}(q)} w_p {\rm e}^{ik_{\rm F}r} n_{\rm F}(p)\right) + \left(\frac{1}{\sqrt{L}}\sum_{p<0} {\rm e}^{\sum_q {\rm e}^{-iqr}\frac{2\pi}{qL}\rho_{\rm L}(q)} w_p {\rm e}^{-ik_{\rm F}r} n_{\rm F}(p)\right). \tag{12}$$

As usual we have $[\rho_{\rm R}(q), \rho_R(-q)] = \sum_{k>0}[A_k(-q), A_k^\dagger(-q)] + \sum_{k>0}[A_k^\dagger(q), A_k(q)] = -qL/2\pi$ and $[\rho_{\rm L}(q), \rho_{\rm L}(-q)] = qL/2\pi$. This expression is well-known to the traditional bosonizing community and is known to reproduce the exponents of the Luttinger liquid correctly. In passing, we note that the present formalism does not allow terms such as $\psi_{\rm R}^\dagger(x)\psi_{\rm L}(x')$, this being identically zero due to the singular complex number w_p. Thus in our formalism, the terms responsible for backward scattering come from the quadratic corrections on the right-hand side of eq. (3). Backward scattering is synonymous with large momentum transfer, which in turn means corrections to RPA, which then translates to quadratic corrections in eq. (3). Now we wish to compute the Green function of the Wigner crystal whose momentum distribution has been computed in an earlier work [5]. There we considered electrons on a circle interacting via a long-range interaction $V(x) = -(e^2/a^2)|x|$, where $|x|$ is the chord length. The ground state of this system was shown to be crystalline with lattice spacing $l_{\rm c} = \pi/k_{\rm F}$. It was shown that the momentum distribution of the Wigner crystal at zero temperature is given by

$$\bar{n}_p = \frac{1}{2}\left(1 + {\rm e}^{-\frac{m\omega_0}{k_{\rm F}^2 - p^2}}\right) n_{\rm F}(p) + \frac{1}{2}\left(1 + {\rm e}^{-\frac{m\omega_0}{p^2 - k_{\rm F}^2}}\right)(1 - n_{\rm F}(p)), \tag{13}$$

where $\omega_0 = \sqrt{\frac{2e^2k_{\rm F}}{\pi m a^2}}$. This was derived using the general formula for the momentum distribution in terms of sea-bosons,

$$\bar{n}_{\mathbf{p}} = n_{\rm F}(\mathbf{p})\frac{1}{2}(1 + {\rm e}^{-2\sum_{\mathbf{q}}\langle A^\dagger_{\mathbf{p}+\mathbf{q}/2}(\mathbf{q})A_{\mathbf{p}+\mathbf{q}/2}(\mathbf{q})\rangle}) + (1 - n_{\rm F}(\mathbf{p}))\frac{1}{2}(1 - {\rm e}^{-2\sum_{\mathbf{q}}\langle A^\dagger_{\mathbf{p}-\mathbf{q}/2}(\mathbf{q})A_{\mathbf{p}-\mathbf{q}/2}(\mathbf{q})\rangle}). \tag{14}$$

First we wish to generalize eq. (14) to a finite temperature. This is important since we know that even for a noninteracting system, the momentum distribution at absolute zero is discontinuous at $k = k_{\rm F}$ but is continuous at finite temperature. A naive approach that just performs thermodynamic averaging over the sea-boson occupation fails. We have to adopt a more subtle approach. One possibility is to express the sea-bosons in terms of the hydrodynamic variables and use the action in eq. (5). This can possibly be made to work out but is really not worth the effort. It is better to gain some intuition from this effort to guess the proper generalization. We propose the following generalization:

$$\bar{n}_{\mathbf{p}} = n_{\mathrm{F}}(\mathbf{p})\frac{1}{2}(1+\lambda(\mathbf{p})\mathrm{e}^{-2\sum_{\mathbf{q}}\langle\langle A^{\dagger}_{\mathbf{p}+\mathbf{q}/2}(\mathbf{q})A_{\mathbf{p}+\mathbf{q}/2}(\mathbf{q})\rangle\rangle}) \\ +(1-n_{\mathrm{F}}(\mathbf{p}))\frac{1}{2}(1-\lambda(\mathbf{p})\mathrm{e}^{-2\sum_{\mathbf{q}}\langle\langle A^{\dagger}_{\mathbf{p}-\mathbf{q}/2}(\mathbf{q})A_{\mathbf{p}-\mathbf{q}/2}(\mathbf{q})\rangle\rangle}). \tag{15}$$

Here $\lambda(\mathbf{p})$ contains the temperature information of the free theory only. The boson occupation in the exponent is defined as follows: $\langle\langle\cdots\rangle\rangle = \langle\cdots\rangle_{\beta} - \langle\cdots\rangle_{\beta,0}$. That is, the difference between the interacting theory at finite temperature and free theory at finite temperature. To calculate the finite temperature sea-boson occupation we have to invoke the hydrodynamic description. We first express the sea-bosons in terms of hydrodynamic variables, namely the velocity potential and density. It is given as follows: $A_{\mathbf{k}}(\mathbf{q}) = [A_{\mathbf{k}}(\mathbf{q}), A^{\dagger}_{\mathbf{k}}(\mathbf{q})](\frac{k_{\mathrm{F}}}{N|q|}\rho_{-\mathbf{q}} - iX_{\mathbf{q}})$. Here we have to make sure that only the s-wave contributes, in other words we ignore complications caused by the dot product $\mathbf{k}\cdot\mathbf{q}$ and replace it by its extremum value. This correspondence automatically reproduces the RPA level identities also valid only in the s-wave sense: $\sum_{\mathbf{k}}(A_{\mathbf{k}}(\mathbf{q}) + A^{\dagger}_{\mathbf{k}}(-\mathbf{q})) = \rho_{-\mathbf{q}}$ and $\sum_{\mathbf{k}}\frac{ik_{\mathrm{F}}}{N|q|}(A_{\mathbf{k}}(\mathbf{q}) - A^{\dagger}_{\mathbf{k}}(-\mathbf{q})) = X_{\mathbf{q}}$. We may also re-express these directly in terms of Fermi fields, $\rho_{\mathbf{q}} = \sum_{\mathbf{k}} c^{\dagger}_{\mathbf{k}+\mathbf{q}/2}c_{\mathbf{k}-\mathbf{q}/2}$ and $X_{\mathbf{q}} = \sum_{\mathbf{k}}\frac{ik_{\mathrm{F}}}{N|q|}sgn(\mathbf{k}\cdot\mathbf{q})c^{\dagger}_{\mathbf{k}-\mathbf{q}/2}c_{\mathbf{k}+\mathbf{q}/2}$. This is beneficial since we may now easily compute the correlation functions of these operators at finite temperature. In general, $\langle A^{\dagger}_{\mathbf{k}}(\mathbf{q})A_{\mathbf{k}}(\mathbf{q})\rangle = [A_{\mathbf{k}}(\mathbf{q}), A^{\dagger}_{\mathbf{k}}(\mathbf{q})](\frac{\pi^2}{q^2L^2}\langle\rho_{\mathbf{q}}\rho_{-\mathbf{q}}\rangle + \langle X_{-\mathbf{q}}X_{\mathbf{q}}\rangle - \frac{\pi}{|q|L})$. For a Luttinger liquid the zero temperature theory is

$$H = \sum_{k,q}\frac{k\cdot q}{m}A^{\dagger}_{k}(q)A_{k}(q) + \sum_{q}\frac{v_0}{L}[A(-q)A(q) + A^{\dagger}(q)A^{\dagger}(-q)] \\ = \sum_{q} v|q|d^{\dagger}_{\mathrm{c}}(q)d_{\mathrm{c}}(q), \tag{16}$$

where

$$d_{\mathrm{c}}(q) = \sqrt{\frac{2\pi}{|q|L}}\left(\frac{v_{\mathrm{F}}+v}{2v}\right)^{1/2}A(q) + \sqrt{\frac{2\pi}{|q|L}}\left(\frac{v_{\mathrm{F}}-v}{2v}\right)^{1/2}A^{\dagger}(-q), \tag{17}$$

$$d^{\dagger}_{\mathrm{c}}(-q) = \sqrt{\frac{2\pi}{|q|L}}\left(\frac{v_{\mathrm{F}}+v}{2v}\right)^{1/2}A^{\dagger}(-q) + \sqrt{\frac{2\pi}{|q|L}}\left(\frac{v_{\mathrm{F}}-v}{2v}\right)^{1/2}A(q), \tag{18}$$

where $v = \sqrt{v_{\mathrm{F}}^2 - \frac{v_0^2}{\pi^2}}$ and $[d_{\mathrm{c}}(q), d^{\dagger}_{\mathrm{c}}(q)] = 1$ and $\rho_{-q} = A(q) + A^{\dagger}(-q)$ and $X_q = \frac{i\pi}{|q|L}(A(q) - A^{\dagger}(-q))$ and $A(q) = \sum_k A_k(q)$ and $[A(q), A^{\dagger}(q)] = \frac{|q|L}{2\pi}$, $\rho_{-q} =$

$(d_{\rm c}(q)+d_{\rm c}^{\dagger}(-q))c_a$ and $X_q = (d_{\rm c}(q)-d_{\rm c}^{\dagger}(-q))c_b$, $c_a = \sqrt{\frac{|q|L}{2\pi}}((\frac{v_{\rm F}+v}{2v})^{1/2}-(\frac{v_{\rm F}-v}{2v})^{1/2})$ and $c_b = \sqrt{\frac{|q|L}{2\pi}}((\frac{v_{\rm F}+v}{2v})^{1/2}+(\frac{v_{\rm F}-v}{2v})^{1/2})\frac{i\pi}{|q|L}$. In other words, $\langle A_k^{\dagger}(q)A_k(q)\rangle = [A_k(q), A_k^{\dagger}(q)](\frac{v_{\rm F}}{v}-1)\frac{\pi}{|q|L}$. Substituting this into eq. (14) leads to a momentum distribution that has power-law singularities with anomalous exponent $\gamma = \frac{v_{\rm F}}{v}-1$. In general, we have to use the RPA-level action in eq. (5) to calculate the finite temperature expectation values namely, $\langle \rho_{\mathbf{q}}\rho_{-\mathbf{q}}\rangle$ and $\langle X_{\mathbf{q}}X_{-\mathbf{q}}\rangle$. Let us now use this method to find the proper generalization of eq. (13) to finite temperature. First, we have to find $\lambda(\mathbf{p})$. By making contact with the free theory we find $\lambda(\mathbf{p}) = [{\rm sgn}(k_{\rm F}-|\mathbf{p}|)]^{-1}\tanh[\frac{\beta}{2}(\mu-\epsilon_{\mathbf{p}})]$. When this procedure is implemented we obtain

$$\bar{n}_p = \frac{1}{2}\left(1+\lambda(p){\rm e}^{-\frac{m\omega_0}{k_{\rm F}^2-p^2}\coth(\frac{\beta\omega_0}{2})}\right)n_{\rm F}(p) + \frac{1}{2}\left(1-\lambda(p){\rm e}^{-\frac{m\omega_0}{p^2-k_{\rm F}^2}\coth(\frac{\beta\omega_0}{2})}\right)(1-n_{\rm F}(p)). \tag{19}$$

Therefore the essential singularity remains even at finite temperature. Finally we wish to calculate the dynamical propagator of the Wigner crystal. It seems that in this case even eq. (8) is not sufficient. We have to express the field operator in momentum space directly in terms of the sea-bosons. A partially correct formula was proposed in an earlier work [6]. Instead of searching for a rigorous approach, we directly make the following surmise for the propagators that is motivated by comparing with limiting cases. Define $c_{\mathbf{p},<} = n_{\rm F}(\mathbf{p})c_{\mathbf{p}}$ and $c_{\mathbf{p},>} = (1-n_{\rm F}(\mathbf{p}))c_{\mathbf{p}}$. Then

$$\langle c_{\mathbf{p},>}^{\dagger}(t')c_{\mathbf{p},>}(t)\rangle = (1-n_{\rm F}(\mathbf{p}))\frac{1}{2}(1-\lambda(\mathbf{p}) \times{\rm e}^{-2\sum_{\mathbf{q}}\langle\langle A_{\mathbf{p}-\mathbf{q}/2}^{\dagger}(\mathbf{q},t)A_{\mathbf{p}-\mathbf{q}/2}(\mathbf{q},t)\rangle\rangle}) \times{\rm e}^{-i\epsilon_{\rm F}(t-t')}E(|\mathbf{p}|-k_{\rm F},t-t'), \tag{20}$$

$$\langle c_{\mathbf{p},<}^{\dagger}(t')c_{\mathbf{p},<}(t)\rangle = n_{\rm F}(\mathbf{p})\frac{1}{2}(1+\lambda(\mathbf{p}) \times{\rm e}^{-2\sum_{\mathbf{q}}\langle\langle A_{\mathbf{p}+\mathbf{q}/2}^{\dagger}(\mathbf{q},t)A_{\mathbf{p}+\mathbf{q}/2}(\mathbf{q},t)\rangle\rangle}) \times{\rm e}^{-i\epsilon_{\rm F}(t-t')}E(|\mathbf{p}|-k_{\rm F},t'-t). \tag{21}$$

Here the envelope function has to be chosen with care. We make the following definition which we justify *a posteriori*.

$$E(q,t-t') = \frac{\int_0^{\infty}{\rm d}\omega W(q,\omega){\rm e}^{-i\omega(t-t')}}{\int_0^{\infty}{\rm d}\omega W(q,\omega)}, \tag{22}$$

where $W(q,\omega)$ is the spectral weight,

$$W(q,\omega) = {\rm Im}\left(\frac{1}{\epsilon(q,\omega-i\delta)}\right). \tag{23}$$

In one dimension, the spectral weight is a delta function at the collective mode. Hence in one dimension,

$$E(q, t-t') = \mathrm{e}^{-i\omega_{\mathrm{c}}(q)(t-t')}. \tag{24}$$

In more than one dimension we have both the particle–hole mode and collective mode. For example in the case of the jellium, the collective mode contribution occurs at the plasma frequency and this leads to a rapidly oscillating contribution which may be ignored. The important contribution comes from the particle–hole mode, which even though is not infinitely long-lived, makes a significant contribution. For the noninteracting theory in any number of dimensions this prescription gives us

$$E_{\mathrm{free}}(q, t-t') = \mathrm{e}^{-iv_{\mathrm{F}}|q|(t-t')}. \tag{25}$$

One may then use the Kubo–Martin–Schwinger (KMS) boundary conditions to evaluate $\langle c_{\mathbf{p}}(t)c^{\dagger}_{\mathbf{p}}(t')\rangle$ from these propagators. First we observe that this prescription reproduces the dynamical Green function of the free theory in any number of dimensions both at finite temperature and at zero temperature. We may use these to compute the dynamical propagator of the Luttinger liquid and see if these results agree with those of the more traditional approaches. Using the traditional method,

$$\langle \psi^{\dagger}_{\mathrm{R}}(x', t')\psi_{\mathrm{R}}(0, t)\rangle = \mathrm{e}^{-i\epsilon_{\mathrm{F}}(t-t')}\frac{\mathrm{e}^{-ik_{\mathrm{F}}x'}}{2\pi i[x' - v(t'-t)]}\left[\frac{1}{\Lambda|x' - v(t'-t)|}\right]^{\gamma}. \tag{26}$$

Taking the Fourier transform with respect to x' we obtain,

$$\begin{aligned}\langle c^{\dagger}_{p}(t')c_{p}(t)\rangle = {} & \mathrm{e}^{-i\epsilon_{\mathrm{F}}(t-t')}\mathrm{e}^{-i(|p|-k_{\mathrm{F}})v(t-t')} \\ & \times\frac{1}{2}\left[1 + \mathrm{sgn}(k_{\mathrm{F}} - |p|)\left(\frac{|k_{\mathrm{F}} - |p||}{\Lambda}\right)^{\gamma}\right].\end{aligned} \tag{27}$$

This is completely identical to the result using eqs (20) and (21) since $\omega(q) = v|q|$. This approach may seem quite ad-hoc but it would be very surprising indeed if a theory that reproduces all dynamical aspects of the free theory both at finite temperature and zero temperature correctly in any number of dimensions and also able to reproduce the very nontrivial dynamical propagator of the Luttinger liquid correctly in one dimension is not valid in general. Besides, for the jellium, the clever choice of the envelope function ensures that one-particle Green function is gapless even though the collective mode is gapped. Assuming that this is valid in general we may write down the dynamical propagator of the Wigner crystal as follows:

$$\begin{aligned}\langle c^{\dagger}_{p}(t')c_{p}(t)\rangle = {} & \frac{1}{2}\left(1 + \lambda(p)\mathrm{e}^{-\frac{m\omega_0}{k_{\mathrm{F}}^2 - p^2}\coth(\frac{\beta\omega_0}{2})}\right)n_{\mathrm{F}}(p)\mathrm{e}^{-i\epsilon_{\mathrm{F}}(t-t')}\mathrm{e}^{i\omega_0(t-t')} \\ & + \frac{1}{2}\left(1 - \lambda(p)\mathrm{e}^{-\frac{m\omega_0}{p^2 - k_{\mathrm{F}}^2}\coth(\frac{\beta\omega_0}{2})}\right) \\ & \times(1 - n_{\mathrm{F}}(p))\mathrm{e}^{-i\epsilon_{\mathrm{F}}(t-t')}\mathrm{e}^{-i\omega_0(t-t')},\end{aligned} \tag{28}$$

$$\langle c_p(t)c_p^\dagger(t')\rangle = \frac{1}{2}\left(1-\lambda(p)\mathrm{e}^{-\frac{m\omega_0}{k_\mathrm{F}^2-p^2}\coth(\frac{\beta\omega_0}{2})}\right)n_\mathrm{F}(p)\mathrm{e}^{-i\epsilon_\mathrm{F}(t-t')}\mathrm{e}^{i\omega_0(t-t')}$$
$$+\frac{1}{2}\left(1+\lambda(p)\mathrm{e}^{-\frac{m\omega_0}{p^2-k_\mathrm{F}^2}\coth(\frac{\beta\omega_0}{2})}\right)$$
$$\times(1-n_\mathrm{F}(p))\mathrm{e}^{-i\epsilon_\mathrm{F}(t-t')}\mathrm{e}^{-i\omega_0(t-t')}. \quad (29)$$

Note that the above formulas are independent of arbitrarily chosen momentum cutoffs and depend only on the microscopic parameters present in the original Hamiltonian with parabolic dispersion. This is in contrast with the conventional approaches to bosonization [7] where such cutoffs are mandated by the formalism. In this work, Schulz remarks that the time-dependent and temperature-dependent formulas for the propagators are complicated in his formalism and hence not too illuminating. This is in contrast with our 'momentum space bosonization' approach where the above general formulas are not only simple but also illuminating.

4. Conclusions

To conclude, we have computed the dynamical Green function of the Wigner crystal in one dimension whose momentum distribution exhibits essential singularities in momentum space. We have generalized the momentum distribution to finite temperature. We have written down a general formula for the field operator without any Klein factors in one dimension in terms of currents and densities that does not involve momentum cutoffs and applies directly to the Fermi gas with a parabolic dispersion rather than to its caricature namely the Luttinger model. Lastly, we have summarized all the developments in the subject made by the author and his collaborators in order to facilitate further developments.

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[16] It is worth noting that issues such as 'point splitting' [8] that occur in the traditional approaches to bosonization are due to peculiar nature of the Luttinger model that has right and left moving branches that extend all the way from minus infinity to plus infinity. We deal with the original Hamiltonian with a parabolic dispersion and with a finite Fermi momentum and mass of electron, hence we do not have to address such issues