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Generalization of quasi-exactly solvable and isospectral potentials

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Abstract. A unified approach in the light of supersymmetric quantum mechanics (SSQM) has been suggested for generating multidimensional quasi-exactly solvable (QES) potentials. This method provides a convenient means to construct isospectral potentials of derived potentials.

Keywords. SUSY algebras; case study.

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1. Introduction

Exact solutions of Schrödinger equations provide all important informations about the system concerned. But for physical systems, exactly solvable potentials are very few in number. Therefore, the quasi-exactly solvable (QES) potentials have received a lot of attention [1–3]. These QES models allow exact solutions only for a limited part of the discrete energies but not for the entire spectrum. Thus, these potentials fill up the gap between the exactly solvable and non-solvable potentials and help to understand many physical phenomena. Moreover, QES problem has its own inner mathematical beauty such as it can provide a good starting point for doing calculations perturbatively for complex systems.

On the other hand, supersymmetry inspired quantum mechanics proposed by Witten [4] is a very beautiful mathematical construct; it renders some very suggestive solutions to phenomenological questions such as the hierarchy problem. Supersymmetric quantum mechanics (SSQM) can provide an important testing ground for both physical and computational aspects of supersymmetry (SUSY) theories [4,5]. Moreover, a special attribute to the formal theory of SSQM is in its mathematical simplicity [6,7]. Therefore, it has been extensively used to analyse various physical processes in nuclear physics [8,9] and atomic physics [10,11].

Recently, the exact solutions for a single state for some singular potentials are obtained by using analytical methods [3] with some restrictions on the potential parameters. Singular potentials, specially Coulomb plus inverse power potentials and harmonic oscillator plus inverse power potentials have been extensively used in different branches of physics such as study of quark confinement problem in QCD [12] and false vacua in field theory [13], ion-atom scattering [14] and several interactions between the atoms [15] in atomic physics, interatomic interactions in molecular physics [16], polaron formation and other problem in solid-state physics [17,18], magnetic resonances between massive and massless spin-$\frac{1}{2}$ particle in low-energy physics [19] etc. In a recent work, Gönöul *et al* [20] have shown that two kinds of potentials as mentioned above can be linked together in N-dimensional space. Non-singular (polynomial) potentials are also very much fascinating nowadays because of their mathematical beauty like PT symmetry [21] and their application in different branches of physics [22]. Therefore, it will be very much interesting to develop a unified approach for constructing QES potentials of singular and non-singular type.

In SSQM, the shape-invariant potentials, the non-shape-invariant but exactly solvable potentials can be constructed and incorporated in the frame of the theorem of existence of the superpotential (W) [23]. Using this technique, we intend to derive a variety of generalized potentials in N-dimension by assuming different forms of the superpotentials for which Schrödinger equation yield exact solutions for a single state in each case. Then, we shall try to study the special cases of singular and non-singular potentials emerging from the generalized one for different choices of parameters. In this way, we shall search for a class of new QES potentials. Furthermore, the invariance of the Hamiltonian under the addition of a suitable function to the SUSY operator can be exploited to construct a family of isospectral potentials associated with our generalized QES potentials.

In this article, several aspects of SUSY algebras related to our problem are reviewed in §2 and applications of these algebras to construct generalized QES potentials and their family of isospectral potentials in N-dimensional space are discussed in §3. Finally, in §4, we will draw the conclusion.

2. SUSY algebras

Consider a particle of mass m moving in N-dimensional Euclidean space. The time-independent Schrödinger equation for any integral dimension is given by [24] (in units of $\hbar = 2m = 1$)

$$H\Phi = [-\nabla_N^2 + U_N]\Phi = \mathcal{E}\Phi. \tag{1}$$

Here, the wave function Φ belongs to the energy eigenvalue $\mathcal{E}$. The symbols ∇_N^2 and U_N stand for the Laplacian operator and potential energy respectively in N-dimensional space. Investigation of physical process based on eq. (1) is a well-studied problem and many authors proceed by using the standard central potential $U(r)$ in place of U_N. Here r represents the N-dimensional radius $[\sum_i^N x_i^2]^{1/2}$. Going over to the hyperspherical coordinate system with $N-1$ angular variable (θ_i) and one radial coordinate we can write

$$\Phi = \frac{\psi_l^n(r)}{r^{\frac{N-1}{2}}} Y_l^M(\theta_i), \tag{2}$$

where $Y_l^M(\theta_i)$ represents contributions from the hyperspherical harmonics that arise in higher dimensions. The eigenvalues and eigenfunctions for generalized angular momentum operators in hyperspherical coordinate are determined [25] using the results known from the factorization method [26]. However, from eqs (1) and (2) we have reduced radial Schrödinger equation for the lth partial wave on half-line as

$$\begin{aligned} H\psi_l^n(r) &= \left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{(l+\frac{N-1}{2})(l-1+\frac{N-1}{2})}{r^2} + V(r)\right]\psi_l^n(r) \\ &= \mathcal{E}_l^n \psi_l^n(r). \end{aligned} \tag{3}$$

Here the superscript n refers to a quantum number, the interpretation of which depends on the choice of $V(r)$.

In SSQM, the Hamiltonian corresponding to the reduced radial Schrödinger equation for lth partial wave can be written in the form

$$H = H_1 = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + U_1(r), \tag{4}$$

such that the binding energy $\mathcal{E}_{1,l}^{(0)}$ of the lowest bound state of H_1 is zero, i.e.

$$H_1 \psi_{1,l}^{(0)}(r) = 0. \tag{5}$$

The superscript (0) on reduced wave function ψ as well as $\mathcal{E}_{1,l}^{(0)}$ stands for the ground state wave function and energy while the subscript 1 merely indicates that the wave function $\psi_{1,l}^{(0)}(r)$ belongs to H_1. Here, we shall use analogous notations for the partner Hamiltonians. The effective potential in the partner Hamiltonian H_1 in eq. (4) consists of centrifugal as well as interaction term and can be written as

$$U_1(r) = \frac{(l+\frac{N-1}{2})(l-1+\frac{N-1}{2})}{r^2} + V_1(r). \tag{6}$$

The underlying idea of SSQM is to factorize H_1 in the form

$$H_1 = O^- O^+, \tag{7}$$

with the operators

$$O^{\pm} = \pm\frac{\mathrm{d}}{\mathrm{d}r} + W(r). \tag{8}$$

Here, the so-called superpotential $W(r)$ is related to the effective potential in eq. (6) by

$$U_1(r) = W^2(r) - \frac{\mathrm{d}W(r)}{\mathrm{d}r}. \tag{9}$$

The supersymmetric partner H_2 of H_1 is traditionally written as

$$H_2 = O^+O^- = -\frac{\mathrm{d}^2}{\mathrm{d}r^2} + U_2(r) \tag{10}$$

with the concomitant supersymmetric partner potential [5,6] as

$$U_2(r) = W^2(r) + \frac{\mathrm{d}W(r)}{\mathrm{d}r}. \tag{11}$$

Equation (7) through eq. (5) automatically guarantees that the ground state of H_1 has zero energy and provides the ground state wave function of H_1 as

$$\psi_{1,l}^{(0)}(r) = N_{1,l}^{(0)} \exp\left(-\int^r W(r')\mathrm{d}r'\right), \tag{12}$$

where $N_{1,l}^{(0)}$ is the normalization constant.

To construct a new class of potentials, two new operators are defined as

$$C^{\pm} = \pm\frac{\mathrm{d}}{\mathrm{d}r} + F(r), \tag{13}$$

with $F(r)$ to be determined for the requirement that eq. (10) also holds good for these new operators [27,28]. So, one can write

$$H_2 = O^+O^- = C^+C^-. \tag{14}$$

From this condition we obtain the Riccati differential equation

$$\frac{\mathrm{d}F(r)}{\mathrm{d}r} + F^2(r) = \frac{(l+\frac{N-1}{2})(l+1+\frac{N-1}{2})}{r^2} + V(r). \tag{15}$$

A particular solution of eq. (15) is $W(r)$, such that the general solution can be written in the form

$$F(r) = W(r) + Z(r). \tag{16}$$

With the help of eqs (6), (9) and (16), from eq. (15) we arrive at the equation

$$\frac{\mathrm{d}Z(r)}{\mathrm{d}r} + Z^2(r) + 2W(r)Z(r) = 0. \tag{17}$$

The solution of eq. (17) is given by

$$Z(r) = \frac{[\psi_{1,l}^{(0)}(r)]^2}{\lambda + \int_0^r [\psi_{1,l}^{(0)}(r')]^2\mathrm{d}r'}, \tag{18}$$

where λ is an arbitrary constant. In order to avoid problems with possible singularity, we impose that $\lambda > 0$ always for any type of problems with even or odd $(2l+N-1)$. With the help of the technique of Alves and Filho [27], we get a new Hamiltonian

$$\mathcal{H} = H_1 - 2\frac{\mathrm{d}Z(r)}{\mathrm{d}r} = C^- C^+. \tag{19}$$

This Hamiltonian can be regarded as a generalized version of H_1 and is characterized by a new family of potentials

$$\mathcal{U}(r) = U_1(r) - 2\frac{\mathrm{d}Z(r)}{\mathrm{d}r}. \tag{20}$$

Again, the spectra of H_2 coincide with the spectra of H_1 except for the missing ground state of the latter. By Mielnik [29] we now write

$$\mathcal{H}C^+ = C^+ H_1. \tag{21}$$

From eq. (21) it is clear that eigenvectors $\Phi_{2,l}$ and $C^+\Phi_{2,l}$ of H_2 and $\mathcal{H}$ belong to the same eigenvalues. Therefore, $\mathcal{H}$ and H_2 will be isospectral provided we can show that the lowest energy level of $\mathcal{H}$ lies at the ground state of H_1. To that end, we introduce a missing vector ϕ to the set $\{C^+\Phi_{2,l}; l = 1, 2, \ldots\}$ such that

$$\langle \phi | C^+ \Phi_{2,l} \rangle = 0, \quad l = 1, 2, 3, \ldots \tag{22}$$

This equation also implies that

$$\langle C^- \phi | \Phi_{2,l} \rangle = 0. \tag{23}$$

Since $\Phi_{2,l}$ is an eigenfunction of H_2, we arrive at the condition

$$C^- \phi = 0 \tag{24}$$

to determine $\phi(r)$ in the form

$$\phi(r) = \psi_{1,l}^{(0)}(r) \exp\left(- \int Z(r')\mathrm{d}r' \right). \tag{25}$$

From eqs (24) and (25) we get

$$\mathcal{H}\phi(r) = 0. \tag{26}$$

Equation (26) shows that $\phi(r)$, the eigenfunction of $\mathcal{H}$, belongs to the ground state eigenvalue of H_1.

3. Case study

We shall now use the aforementioned technique of supersymmetry-inspired quantum mechanics to generate several generalized potentials having the form of Coulomb plus singular and non-singular, harmonic oscillator plus singular and non-sigular, non-singular and finally singular plus non-singular potentials in N-dimensional space which are exactly solvable for a single state. In our construction process, eq. (9) has been used with a suitable choice of $W(r)$. Selection of $W(r)$ should be made in such a way that, the wave function $\psi_{1,l}^{(0)}(r)$ in eq. (12) is normalizable. The wave functions $\psi_{1,l}^{(0)}(r)$, of the derived potentials and their SUSY partners (U_2) and also the family of isospectral potentials ($\mathcal{U}$) have been derived. These information may turn out to be very instructive for the study of numerous aspects in various fields of physics. Such examples are in the study of loosely bound systems like halo nuclei, as well as, in scattering length and coupling parameter calculations [14].

3.1 *Coulomb plus singular and non-singular potentials*

Since the exponent of e in the Coulomb wave function contains a term linear in r, eq. (12) suggests us to choose the superpotential $W(r)$ to generate Coulomb plus singular or non-singular interactions as

$$W(r) = -\frac{l + \frac{N-1}{2}}{r} + \alpha + \beta \mathrm{d} r^{d-1}, \tag{27}$$

where the parameters α and β are to be determined by the demand of normalizability of $\psi_{1,l}^{(0)}(r)$ in eq. (12). Using this superpotential, eq. (9) gives the effective potential function $U_1(r)$ as

$$U_1(r) = \frac{(l + \frac{N-1}{2})(l - 1 + \frac{N-1}{2})}{r^2} - \frac{B}{r} + Cr^{2d-2} - Dr^{d-2} + Er^{d-1} + A, \tag{28}$$

which is exactly solvable at zero energy. Here, the coupling parameters are $B = 2\alpha(l + \frac{N-1}{2}), C = \beta^2 d^2, D = \beta d(2l + N + d - 2), E = 2\alpha\beta d$ and the symbol A represents the change of reference point of the energy. Hence it is possible to describe a large number of Coulomb-dominated interactions by suitable choice of α, β, d, l and N. For a given set of these parameters, the solution of the Schrödinger equation for the QES potential save the centrifugal term

$$V_1(r) = -\frac{B}{r} + Cr^{2d-2} - Dr^{d-2} + Er^{d-1} \tag{29}$$

is

$$\psi_{1,l}^{(0)}(r) = N_{1,l}^{(0)} r^{l+\frac{N-1}{2}} \exp(-\alpha r - \beta r^d), \tag{30}$$

with the ground state energy

$$\mathcal{E}_{1,l}^{(0)} = -A. \tag{31}$$

Interestingly, the results for Coulomb interaction can be recovered from eqs (29)–(31) by choosing $\beta = 0$ and $N = 3$.

In a similar manner, it is staightforward to get the expressions of wave functions and energy derived by Gönül *et al* [20] by choosing $d = -1$ in our expressions eqs (29)–(31). In table 1, we present the wave function and energy for some QES Coulomb-dominated interactions which are exactly solvable for ground state. From the expressions of wave functions presented in the table, it is clear that $\psi_{1,l}^{(0)}(r)$ satisfies the desired threshold $(Lt_{r\to 0}\psi_{1,l}^{(0)}(r) = 0)$ and asymptotic behaviour $(Lt_{r\to\infty}\psi_{1,l}^{(0)}(r) = 0)$ for $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$ and the normalization constant $N_{1,l}^{(0)}(1/\sqrt{\mathcal{N}_{1,l}^{(0)}})$ are found to be finite. The representative values of these for some d are presented in Appendix A. It is interesting to note that except for the case of $d = 1$, use of the analytic expressions for the coupling parameters E and D

Table 1. Ground state energies ($\mathcal{E}_{1,l}^{(0)}$) and reduced radial wave functions ($\psi_{1,l}^{(0)}(r)$) for Coulomb-dominated singular and non-singular interaction $V_1(r)$ derived from eqs (29)–(31) with $d = \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, -2, \pm 3$ and $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$.

Parameter	$V_1(r)$	$\psi_{1,l}^{(0)}(r) = N_{1,l}^{(0)} r^{l+\frac{N-1}{2}} \times$	$\mathcal{E}_{1,l}^{(0)}$
$d = \frac{1}{2}$	$-\frac{B-C}{r} + \frac{E}{\sqrt{r}} - \frac{D}{r^{3/2}}$	$\exp(-\alpha r - \beta\sqrt{r})$	$-\frac{E^2}{4D^2}(2l + N - \frac{3}{2})^2$
$d = -\frac{1}{2}$	$-\frac{B}{r} + \frac{C}{r^3} + \frac{E}{r^{3/2}} - \frac{D}{r^{5/2}}$	$\exp(-\alpha r - \frac{\beta}{\sqrt{r}})$	$-\frac{E^2}{4D^2}(2l + N - \frac{5}{2})^2$
$d = 1$	$-\frac{B}{r} - \frac{D}{r}$	$\exp(-\alpha r - \beta r)$	$-(\alpha + \beta)^2$
$d = -1$	$-\frac{B}{r} + \frac{E}{r^2} - \frac{D}{r^3} + \frac{C}{r^4}$	$\exp(-\alpha r - \frac{\beta}{r})$	$-\frac{E^2}{4D^2}(2l + N - 3)^2$
$d = \frac{3}{2}$	$-\frac{B}{r} + Cr + E\sqrt{r} - \frac{D}{\sqrt{r}}$	$\exp(-\alpha r - \beta r^{3/2})$	$-\frac{E^2}{4D^2}(2l + N - \frac{1}{2})^2$
$d = -\frac{3}{2}$	$-\frac{B}{r} + \frac{C}{r^5} - \frac{D}{r^{7/2}} + \frac{E}{r^{5/2}}$	$\exp(-\alpha r - \frac{\beta}{r^{3/2}})$	$-\frac{E^2}{4D^2}(2l + N - \frac{7}{2})^2$
$d = -2$	$-\frac{B}{r} + \frac{E}{r^3} - \frac{D}{r^4} + \frac{C}{r^6}$	$\exp(-\alpha r - \frac{\beta}{r^2})$	$-\frac{E^2}{4D^2}(2l + N - 4)^2$
$d = 3$	$-\frac{B}{r} + Cr^4 + Er^2 - Dr$	$\exp(-\alpha r - \beta r^3)$	$-\frac{E^2}{4D^2}(2l + N + 1)^2$
$d = -3$	$-\frac{B}{r} + \frac{C}{r^8} - \frac{D}{r^5} + \frac{E}{r^4}$	$\exp(-\alpha r - \frac{\beta}{r^3})$	$-\frac{E^2}{4D^2}(2l + N - 5)^2$

in eq. (31) gives the exact energy $\mathcal{E}_{1,l}^{(0)}$ independent of β. Now, it is pertinent to investigate whether $\psi_{1,l}^{(0)}(r)$ remains the eigenfunction belonging to the β-independent eigenvalue $\mathcal{E}_{1,l}^{(0)}(\alpha)$ inspite of the violation of threshold or asymptotic behaviour of $\psi_{1,l}^{(0)}(r)$ for $\mathrm{Re}(\beta) < 0$ and several choice of d. From our survey regarding this ambiguity, it reveals that (i) for $\beta < 0$ and $d < 0$, the depth of the interaction potentials $V_1(r)$ decreases as $|\beta|$ increases (figure 1), (ii) for the same domain of parameters β and d, the wave function diverges to ∞ sharply near the origin as $|\beta|$ and $|d|$ increase (figure 2), (iii) the behaviour of the wave function follows the inverted cup-shaped pattern of the ground state wave function in the interaction region with the energy given by eq. (31) (figure 3), (iv) for $\beta < 0$ and $d > 0$, the divergent part $\mathrm{e}^{|\beta| r^d}$ of $\psi_{1,l}^{(0)}(r)$ as $r \to \infty$ has been supressed by $\mathrm{e}^{-\alpha r} (\alpha > 0)$ for $d < 1$ and (v) figure 4 exhibits that although $\psi_{1,l}^{(0)}(r)$ diverges in the asymptotic region for $\beta < 0$ and $d > 1$, the wave function possesses the single maximum without node in the interaction region for the energy given in eq. (31). Furthermore, the square integrability of the wave function for the above choice of parameters assures the convergence of improper integral appearing in the normalization constants and determination of expectation values of several physical observables. From these observations, we may conclude that despite the violation of SUSY for $\beta < 0$ and $d < 0$, $\psi_{1,l}^{(0)}(r)$ may play the role of ground state energy wave function for the interaction potential $V_1(r)$ belonging to the β-independent energy eigenvalue $\mathcal{E}_{1,l}^{(0)}(\alpha)$ in eq. (31) for the range $-0.5 < \beta < 0, |d| \leq 1.5$.

The supersymmetric partner potential function $U_2(r)$ of the potential $U_1(r)$ is obtained from eq. (11) as

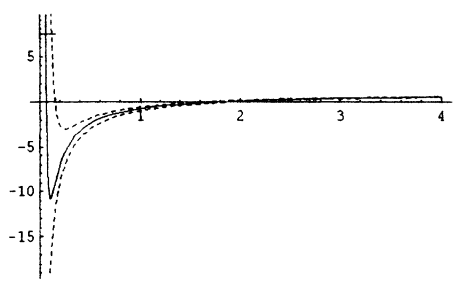

Figure 1. A plot of $V_1(r)$ in eq. (29) demonstrating the reduction of the depth of the potentials with $\beta = 0, -0.3, -0.5$ for the exponent $d = -0.5$. The x-axis denotes the variable r and the y-axis denotes the values of $V_1(r)$.

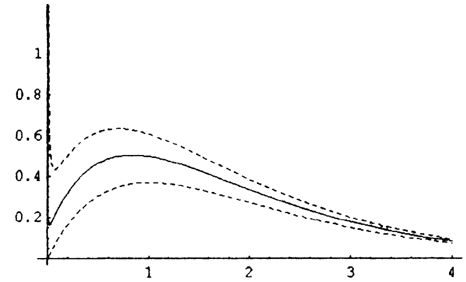

Figure 2. The graph for $\psi^{(0)}_{1,l}(r)$ in eq. (30) for $d = -0.5$ and $\beta = 0, -0.3, -0.5$ displaying the gradual change of the threshold behavior of the ground state wave function. The x-axis denotes the variable r and the y-axis denotes the values of $\psi^{(0)}_{1,l}(r)$.

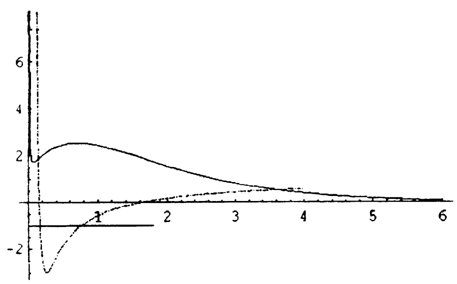

Figure 3. Figure displaying inverted cup shape of ground state wave function in the interaction region of the potential at the energy given by eq. (30) for $d = -0.5, \alpha = 1$ and $\beta = -0.5$ which supports the existence of bound state despite the violation of threshold behaviour. The x-axis denotes the variable r and the y-axis denotes the values of $\mathcal{E}^{(0)}_{1,l}, \psi^{(0)}_{1,l}(r), V_1(r)$.

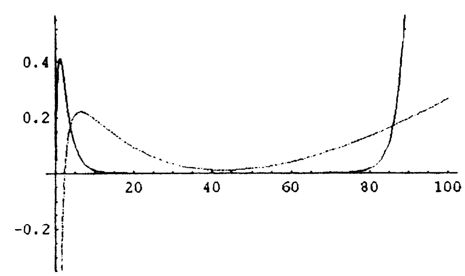

Figure 4. The graphs $\psi^{(0)}_{1,l}(r)$ and $V_1(r)$ for $d = 1.5, \alpha = 1$ and $\beta = -0.1$ demonstrate existence of the bound state in spite of the violation of asymptotic behaviour of the wave functions. The x-axis denotes the variable r and the y-axis denotes the values of $\psi^{(0)}_{1,l}(r), V_1(r)$.

$$U_2(r) = \frac{(l + \frac{N-1}{2})(l + 1 + \frac{N-1}{2})}{r^2} - \frac{B}{r} + Cr^{2d-2} - D' r^{d-2} + Er^{d-1} + A, \tag{32}$$

where $D' = \beta d(2l + N - d)$.

Differentiating eq. (18) with respect to r and with the help of eq. (30), a new family of isospectral potentials of QES potentials can be obtained from eq. (20) as

$$\mathcal{U}(r) = U_1(r) - 2\frac{r^{2l+N-1}\exp(-2\alpha r - 2\beta r^d)}{I^2(r)} \times\left[\left(\frac{(2l+N-1)}{r} - 2\alpha - 2\beta \mathrm{d} r^{d-1}\right)I(r) - r^{2l+N-1}\exp(-2\alpha r - 2\beta r^d)\right], \tag{33}$$

where the integration

$$I(r) = \lambda + \int_0^r r'^{2l+N-1}\exp(-2\alpha r' - 2\beta r'^d)\mathrm{d}r'. \tag{34}$$

Expanding $\mathrm{e}^{-2\beta r'^d}$ to a series and with the help of integral representation [30]

$$\int_0^u x^{\nu-1}(u-x)^{\mu-1}\mathrm{e}^{\gamma x^n}\mathrm{d}x = B(\mu,\nu)u^{\mu+\nu-1} \times {}_nF_n\left(\frac{\nu}{n}, \frac{\nu+1}{n}, ..., \frac{\nu+n-1}{n}; \frac{\mu+\nu}{n}, \frac{\mu+\nu+1}{n}, ..., \frac{\mu+\nu+n-1}{n}; \gamma u^n\right),$$
$$\mathrm{Re}\ \mu > 0, \quad \mathrm{Re}\nu > 0, n = 2, 3, ...; \tag{35}$$

we obtain

$$I(r) = \lambda + \sum_0^\infty \frac{(-2\beta)^k}{\Gamma(k+1)}\frac{\Gamma(2l+N+kd)}{\Gamma(2l+N+kd+1)}r^{2l+N+kd} \times {}_1F_1(2l+N+kd; 2l+N+kd+1; -2\alpha r), \tag{36}$$

where $\Gamma(\cdot)$ and ${}_nF_n(\cdot)$ are gamma function and confluent hypergeometric function respectively.

For the Coulomb potential ($\beta = 0$), we obtain the family of isospectral potentials from eqs (35) and (36) as

$$\mathcal{U}^{\mathcal{C}}(r) = U_1^C(r) - 2\frac{r^{2l+N-1}\ \exp(-2\alpha r)}{I_C^2(r)} \times\left[\left(\frac{(2l+N-1)}{r} - 2\alpha\right)I_C(r) - r^{2l+N-1}\ \exp(-2\alpha r)\right], \tag{37}$$

where the integration

$$I_C(r) = \lambda + \frac{\Gamma(2l+N)}{\Gamma(2l+N+1)}r^{2l+N}{}_1F_1(2l+N; 2l+N+1; -2\alpha r). \tag{38}$$

3.2 *Harmonic oscillator plus singular and non-singular potentials*

The study of harmonic oscillator plus singular and non-singular potentials have been desirable to understand several physical phenomena like structural phase transitions, polaron formation in solids etc. For this case, we consider a superpotential

$$W(r) = -\frac{l + \frac{N-1}{2}}{r} + 2\alpha r + \beta \mathrm{d} r^{d-1}, \tag{39}$$

where the parameters α and β are also to be determined by the demand of normalizability of $\psi_{1,l}^{(0)}(r)$ in eq. (12). Using this superpotential, we can write the effective potential function $U_1(r)$ as

$$U_1(r) = \frac{(l + \frac{N-1}{2})(l - 1 + \frac{N-1}{2})}{r^2} + Br^2 + Cr^{2d-2} - Dr^{d-2} + Er^d + A, \tag{40}$$

which is also exactly solvable at zero energy. Here, the coupling parameters are $B = 4\alpha^2, C = \beta^2 d^2, D = \beta d(2l + N + d - 2), E = 4\alpha\beta d$ and shift of the reference point of energy A is given by $A = -2\alpha(2l + N)$. As before, for a given set of parameters $\mathrm{Re}(\alpha)$, $\mathrm{Re}(\beta), d, l$ and N, the ground state solution of the Schrödinger equation for the interaction

$$V_1(r) = Br^2 + Cr^{2d-2} - Dr^{d-2} + Er^d \tag{41}$$

is

$$\psi_{1,l}^{(0)}(r) = N_{1,l}^{(0)} r^{l + \frac{N-1}{2}} \exp(-\alpha r^2 - \beta r^d), \tag{42}$$

with the corresponding ground state energy

$$\mathcal{E}_{1,l}^{(0)} = \frac{E}{2D}(2l + N)(2l + N + d - 2). \tag{43}$$

It is easy to check that the choice $\beta = 0$ in (41)–(43) reproduces the harmonic oscillator potential $V_1(r) = 4\alpha^2 r^2$ with the ground state energy

$$\mathcal{E}_{1,l}^{(0)\mathrm{H.O.}} = 2\alpha(2l + N) \tag{44}$$

and wave function as

$$\psi_{1,l}^{(0)\mathrm{H.O.}}(r) = N_{1,l}^{(0)} r^{l + \frac{N-1}{2}} \exp(-\alpha r^2). \tag{45}$$

Here, the parameter B can be written in terms of other parameters as

$$B = \frac{E^2}{4C}; \quad C, E > 0 \tag{46}$$

and for $d = -1$ we can recover the results of ref. [3]. For other choice of d values, one can construct a number of harmonic oscillators plus different types of singular

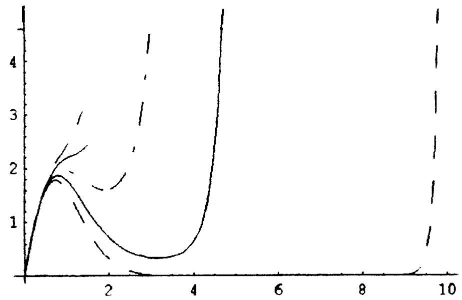

Figure 5. $\psi_{1,l}^{(0)}(r)$ for $d = 3, \alpha = 1$ and $\beta = -0.1, -0.5(-0.1)$. The x-axis denotes the variable r and the y-axis denotes the values of $\psi_{1,l}^{(0)}(r)$.

Table 2. Ground state energies ($\mathcal{E}_{1,l}^{(0)}$) and reduced radial wave functions ($\psi_{1,l}^{(0)}(r)$) for harmonic oscillator plus singular and non-singular interaction $V_1(r)$ derived from eqs (41)–(43) with $d = \pm\frac{1}{2}, \pm 1, -2, 3$ and $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$.

Parameter	$V_1(r) =$	$\psi_{1,l}^{(0)}(r) = N_{1,l}^{(0)} r^{l+\frac{N-1}{2}} \times$	$\mathcal{E}_{1,l}^{(0)}$
$d = \frac{1}{2}$	$Br^2 + E\sqrt{r} + \frac{C}{r} - \frac{D}{r^{3/2}}$	$\exp(-\alpha r^2 - \beta\sqrt{r})$	$\frac{E}{2D}(2l+N)(2l+N-\frac{3}{2})$
$d = -\frac{1}{2}$	$Br^2 + \frac{E}{\sqrt{r}} + \frac{C}{r^3} - \frac{D}{r^{5/2}}$	$\exp(-\alpha r^2 - \frac{\beta}{\sqrt{r}})$	$\frac{E}{2D}(2l+N)(2l+N-\frac{5}{2})$
$d = 1$	$Br^2 + Er - \frac{D}{r}$	$\exp(-\alpha r^2 - \beta r)$	$\frac{E}{2D}(2l+N)(2l+N-1) - C$
$d = -1$	$Br^2 + \frac{E}{r} + -\frac{D}{r^3} + \frac{C}{r^4}$	$\exp(-\alpha r^2 - \frac{\beta}{r})$	$\frac{E}{2D}(2l+N)(2l+N-3)$
$d = -2$	$Br^2 + \frac{E}{r^2} - \frac{D}{r^4} + \frac{C}{r^6}$	$\exp(-\alpha r^2 - \frac{\beta}{r^2})$	$\frac{E}{2D}(2l+N)(2l+N-4)$
$d = 3$	$Br^2 + Er^3 - Dr + Cr^4$	$\exp(-\alpha r^2 - \beta r^3)$	$\frac{E}{2D}(2l+N)(2l+N+1)$

and non-singular potentials from eq. (41). Analogous to our previous study, it is observed that for $\beta < 0$ and $d > 0$ the singular part $\mathrm{e}^{|\beta| r^d}$ of $\psi_{1,l}^{(0)}(r)$ has been compensated by the strongly decaying harmonic oscillator part $\mathrm{e}^{-\alpha r^2} (\alpha < 0)$ for $0 < d < 2$. Consequently, the wave function behaves regular near the origin and in the asymptotic region for $0 < d < 2$ and $-1 < \beta < 0$. However, for $d > 2$, the wave function behaves like the ground state in a wide region for finite part of r for $-0.1 < \beta < 0$ and increases to ∞ as $r \to \infty$ as shown in figure 5. But this feature gradually worsen as $|\beta|$ increases. Therefore, the physically acceptable domain of parameters α, β and d for the harmonic plus singular and non-singular potentials should be considered as $\alpha > 0$. Table 2 consists of the QES potentials and their ground state wave functions and the ground state energy. The normalization constants of the ground state wave function for $d = \pm\frac{1}{2}, \pm 1, -2, 3$ are given in Appendix B.

The supersymmetric partner potential function $U_2(r)$ of the potential $U_1(r)$ is obtained as

$$U_2(r) = \frac{(l+\frac{N-1}{2})(l+1+\frac{N-1}{2})}{r^2} + Br^2 + Cr^{2d-2}$$
$$-D'r^{d-2} + Er^d + A, \tag{47}$$

where $D' = \beta d(2l + N - d)$.

Following the same mathematical artifice, we obtain a family of isospectral potential harmonic oscillator-dominated interactions as

$$\mathcal{U}(r) = U_1(r) - 2\frac{r^{2l+N-1}\exp(-2\alpha r^2 - 2\beta r^d)}{I^2(r)}$$
$$\times\left[\left(\frac{2l+N-1}{r} - 4\alpha r - 2\beta \mathrm{d}r^{d-1}\right)I(r)\right.$$
$$\left. - r^{2l+N-1}\exp(-2\alpha r^2 - 2\beta r^d)\right], \tag{48}$$

where the value of the integration

$$I(r) = \lambda + \sum_{k=0}^{\infty}\frac{(-2\beta)^k}{\Gamma(k+1)}\frac{\Gamma(2l+N+kd)}{\Gamma(2l+N+kd+1)}r^{2l+N+kd}$$
$$\times {}_1F_1\left(l+\frac{N+kd}{2}; l+\frac{N+kd+2}{2}; -2\alpha r^2\right). \tag{49}$$

For harmonic oscillator ($\beta = 0$), the isospectral potential is

$$\mathcal{U}^{\mathrm{H.O.}}(r) = U_1^{\mathrm{H.O.}}(r) - 2\frac{r^{2l+N-1}\exp(-2\alpha r^2)}{I^2_{\mathrm{H.O.}}(r)}$$
$$\times\left[\left(\frac{(2l+N-1)}{r} - 4\alpha r\right)I_{\mathrm{H.O.}}(r) - r^{2l+N-1}\exp(-2\alpha r^2)\right], \tag{50}$$

where the value of the integration $I_{\mathrm{H.O.}}$ is

$$I_{\mathrm{H.O.}}(r) = \lambda + \frac{\Gamma(2l+N)}{\Gamma(2l+N+1)}r^{2l+N}\,{}_1F_1\left(l+\frac{N}{2}; l+\frac{N+2}{2}; -2\alpha r^2\right). \tag{51}$$

3.3 *Non-singular potential*

In this subsection, we shall consider the non-singular potentials mainly of the form of polynomials. These kinds of potentials have drawn attention to the study of PT-symmetry and pseudo-harmonicity of the Hamiltonian [21,22].

Type A:

Here, we consider a superpotential

Table 3. Ground state energies ($\mathcal{E}_{1,l}^{(0)}$) and reduced radial wave functions ($\psi_{1,l}^{(0)}(r)$) for non-singular interaction $V_1(r)$ derived from eqs (54) and (55) with $d = \frac{5}{2}, 3, \frac{7}{2}, 4, 5$ and $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$.

Parameter	$V_1(r) =$	$\psi_{1,l}^{(0)}(r) = N_{1,l}^{(0)} r^{l+\frac{N-1}{2}} \times$	$\mathcal{E}_{1,l}^{(0)}$
$d = \frac{5}{2}$	$Ar^8 + Br^{11/2} + Cr^3 - D\sqrt{r}$	$\exp(-\alpha r^5 - \beta r^{5/2})$	0
$d = 3$	$Ar^{10} + Br^7 + Cr^4 - Dr$	$\exp(-\alpha r^6 - \beta r^3)$	0
$d = \frac{7}{2}$	$Ar^{12} + Br^{17/2} + Cr^5 - Dr^{3/2}$	$\exp(-\alpha r^7 - \beta r^{7/2})$	0
$d = 4$	$Ar^{14} + Br^{10} + Cr^6 - Dr^2$	$\exp(-\alpha r^8 - \beta r^4)$	0
$d = 5$	$Ar^{18} + Br^{13} + Cr^8 - Dr^3$	$\exp(-\alpha r^{10} - \beta r^5)$	0

$$W(r) = -\frac{l + \frac{N-1}{2}}{r} + 2\alpha \mathrm{d} r^{2d-1} + \beta \mathrm{d} r^{d-1}, \tag{52}$$

where the parameters d, α and β are to be determined to match the interactions appropriately. Using this superpotential, we obtain the effective potential function $U_1(r)$ as

$$U_1(r) = \frac{(l + \frac{N-1}{2})(l - 1 + \frac{N-1}{2})}{r^2} + Ar^{4d-2} + Br^{3d-2} + Cr^{2d-2} - Dr^{d-2}, \tag{53}$$

which is also exactly solvable at zero energy. Here, the coupling parameters are $A = 4\alpha^2 d^2, B = 4\alpha\beta d^2, C = \beta^2 d^2 - 2\alpha d(2l + N + 2d - 2), D = \beta d(2l + N + d - 2)$. For a given set of parameters α, β, d, l and N, the solution representing ground state ($\mathcal{E}_{1,l}^{(0)} = 0$) wave function for the Schrödinger equation with the interaction

$$V_1(r) = Ar^{4d-2} + Br^{3d-2} + Cr^{2d-2} - Dr^{d-2} \tag{54}$$

is

$$\psi_{1,l}^{(0)}(r) = N_{1,l}^{(0)} r^{l+\frac{N-1}{2}} \exp(-\alpha r^{2d} - \beta r^d). \tag{55}$$

For the several choices of the exponent d, one can construct a number of non-singular potentials from eq. (54). The physically meaningful domain of the parameters α (or β) for these interactions are found to be $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$. Table 3 consists of the representative QES potentials, ground state wave functions and the ground state energy for $d = \frac{5}{2}, 3, \frac{7}{2}, 4, 5$. As before, the normalization constants of $\psi_{1,l}^{(0)}(r)$ for the above choices of d are presented in Appendix C.

The supersymmetric partner potential function $U_2(r)$ of the potential $U_1(r)$ is

$$U_2(r) = \frac{(l + \frac{N-1}{2})(l + 1 + \frac{N-1}{2})}{r^2} + Ar^{4d-2} + Br^{3d-2} + C'r^{2d-2} - D'r^{d-2}, \tag{56}$$

where $C' = \beta^2 d^2 - 2\alpha d(2l + N - 2d), D' = \beta d(2l + N - d)$.

Following the same procedure as discussed in the case of Coulomb-dominated interaction, we obtain a new family of isospectral potentials for these QES non-singular potentials as

$$\mathcal{U}(r) = U_1(r) - 2\frac{r^{2l+N-1}\exp(-2\alpha r^{2d} - 2\beta r^d)}{I^2(r)} \times \left[\left(\frac{2l+N-1}{r} - 4\alpha\,\mathrm{d}r^{2d-1} - 2\beta \mathrm{d}r^{d-1}\right)I(r) - r^{2l+N-1}\exp(-2\alpha r^{2d} - 2\beta r^d)\right], \quad (57)$$

where the value of the integration

$$I(r) = \lambda + \sum_{k=0}^{\infty}\frac{(-2\alpha)^k}{\Gamma(k+1)}\frac{\Gamma(2l+N+2kd)}{\Gamma(2l+N+2kd+1)}r^{2l+N+2kd} \times {}_dF_d\left(\frac{2l+N+2kd}{d}, \frac{2l+N+2kd+1}{d}, ..., \frac{2l+N+2kd+d-1}{d}; \frac{2l+N+2kd+1}{d}, \frac{2l+N+2kd+2}{d}, ..., \frac{2l+N+2kd+d}{d}; -2\beta r^d\right). \quad (58)$$

Type B:

We consider a superpotential as

$$W(r) = -\frac{l+\frac{N-1}{2}}{r} + \alpha\,\mathrm{d}r^{d-1} + \beta(d-1)r^{d-2}. \quad (59)$$

Using this superpotential, we get the effective potential function $U_1(r)$ as

$$U_1(r) = \frac{(l+\frac{N-1}{2})(l-1+\frac{N-1}{2})}{r^2} + Ar^{2d-2} + Br^{2d-3} + Cr^{2d-4} - Dr^{d-2} - Er^{d-3}, \quad (60)$$

which is exactly solvable at zero energy. Here, the coupling parameters are $A = \alpha^2 d^2, B = 2\alpha\beta d(d-1), C = \beta^2(d-1)^2, D = \alpha d(2l+N+d-2)$, and $E = \beta(d-1)(2l+N+d-3)$. So for a given set of parameters $\mathrm{Re}(\alpha)$, $\mathrm{Re}(\beta), d, l$ and N, $\psi_{1,l}^{(0)}(r)$ for the interaction

$$V_1(r) = Ar^{2d-2} + Br^{2d-3} + Cr^{2d-4} - Dr^{d-2} - Er^{d-3} \quad (61)$$

is

$$\psi_{1,l}^{(0)}(r) = N_{1,l}^{(0)} r^{l+\frac{N-1}{2}}\exp(-\alpha r^d - \beta r^{d-1}). \quad (62)$$

For the ground state wave function given in eq. (62) the normalization constants are found to be a combination of special functions whose representatives are presented in Appendix C for $d = 4$ and 5. The supersymmetric partner potential function $U_2(r)$ of the potential $U_1(r)$ is

$$U_2(r) = \frac{(l+\frac{N-1}{2})(l+1+\frac{N-1}{2})}{r^2} + Ar^{2d-2} + Br^{2d-3} + Cr^{2d-4} - D'r^{d-2} - E'r^{d-3}, \tag{63}$$

where $D' = \alpha d(2l+N-d)$ and $E' = \beta(d-1)(2l+N-d+1)$.

The family of isospectral potentials for these QES potentials is found to be

$$\mathcal{U}(r) = U_1(r) - 2\frac{r^{2l+N-1}\exp(-2\alpha r^d - 2\beta r^{d-1})}{I^2(r)} \times \left[\left(\frac{2l+N-1}{r} - 2\alpha\,\mathrm{d}r^{d-1} - 2\beta(d-1)r^{d-2}\right)I(r) - r^{2l+N-1}\exp(-2\alpha r^d - 2\beta r^{d-1})\right], \tag{64}$$

where the integration

$$I(r) = \lambda + \sum_{k=0}^{\infty}\frac{(-2\beta)^k}{\Gamma(k+1)}\frac{\Gamma(2l+N+k(d-1))}{\Gamma(2l+N+k(d-1)+1)}r^{2l+N+k(d-1)} \times {}_dF_d\left(\frac{2l+N+k(d-1)}{d}, \frac{2l+N+k(d-1)+1}{d}, \ldots, \frac{2l+N+k(d-1)+d-1}{d}; \frac{2l+N+k(d-1)+1}{d}, \frac{2l+N+k(d-1)+2}{d}, \ldots, \frac{2l+N+k(d-1)+d}{d}; -2\alpha r^d\right). \tag{65}$$

Type C:

We next consider another choice of the superpotential

$$W(r) = -\frac{l+\frac{N-1}{2}}{r} + \alpha\,\mathrm{d}r^{d-1} + \beta(d-2)r^{d-3}. \tag{66}$$

Using this superpotential, we get the effective potential function $U_1(r)$ as

$$U_1(r) = \frac{(l+\frac{N-1}{2})(l-1+\frac{N-1}{2})}{r^2} + Ar^{2d-2} + Br^{2d-4} + Cr^{2d-6} - Dr^{d-2} - Er^{d-4}, \tag{67}$$

which is exactly solvable at zero energy. Here, the coupling parameters are $A = \alpha^2 d^2, B = 2\alpha\beta d(d-2), C = \beta^2(d-2)^2, D = \alpha d(2l+N+d-2)$, and $E = \beta(d-2)$

$(2l+N+d-4)$. For a given set of parameters Re(α), Re$(\beta), d, l$ and N, the ground state ($\mathcal{E}_{1,l}^{(0)}=0$) solution of the Schrödinger equation for the interaction

$$V_1(r) = Ar^{2d-2} + Br^{2d-4} + Cr^{2d-6} - Dr^{d-2} - Er^{d-4} \tag{68}$$

is

$$\psi_{1,l}^{(0)} = N_{1,l}^{(0)} r^{l+\frac{N-1}{2}} \exp(-\alpha r^d - \beta r^{d-2}) \tag{69}$$

with the ground state energy. For the different d, Re$(\alpha) > 0$ and Re$(\beta) > 0$, one can construct a number of different types of non-singular potentials from eq. (68). The normalization constants for eq. (69) are given in Appendix C for $d = 5$ and 6.

The supersymmetric partner potential function $U_2(r)$ of the potential $U_1(r)$ is

$$U_2(r) = \frac{(l+\frac{N-1}{2})(l+1+\frac{N-1}{2})}{r^2} + Ar^{2d-2} + Br^{2d-4} + Cr^{2d-6} - D'r^{d-2} - E'r^{d-4}, \tag{70}$$

where $D' = \alpha d(2l+N-d)$ and $E' = \beta(d-2)(2l+N-d+2)$.

From eq. (20) we get a new family of isospectral potentials of these QES potentials as

$$\mathcal{U}(r) = U_1(r) - 2\frac{r^{2l+N-1}\exp(-2\alpha r^d - 2\beta r^{d-2})}{I^2(r)} \times \left[\left(\frac{2l+N-1}{r} - 2\alpha\,\mathrm{d}r^{d-1} - 2\beta(d-2)r^{d-3}\right)I(r) - r^{2l+N-1}\exp(-2\alpha r^d - 2\beta r^{d-2})\right], \tag{71}$$

where the integration

$$I(r) = \lambda + \sum_{k=0}^{\infty} \frac{(-2\beta)^k}{\Gamma(k+1)} \frac{\Gamma(2l+N+k(d-2))}{\Gamma(2l+N+k(d-2)+1)} r^{2l+N+k(d-2)} \times {}_dF_d\left(\frac{2l+N+k(d-2)}{d}, \frac{2l+N+k(d-2)+1}{d}, \ldots, \frac{2l+N+k(d-2)+d-1}{d}; \frac{2l+N+k(d-2)+1}{d}, \frac{2l+N+k(d-2)+2}{d}, \ldots, \frac{2l+N+k(d-2)+d}{d}; -2\alpha r^d\right). \tag{72}$$

3.4 *Singular plus non-singular potentials*

Here, we shall construct a generalized potential consisting of both singular and non-singular parts. For this, we consider a superpotential

$$W(r) = -\frac{l + \frac{N-1}{2}}{r} + \alpha\, \mathrm{d}r^{d-1} - \beta\, \mathrm{d}r^{-d-1}. \tag{73}$$

Using this superpotential, we get the effective potential function $U_1(r)$ as

$$U_1(r) = \frac{(l + \frac{N-1}{2})(l - 1 + \frac{N-1}{2}) - E}{r^2} + Ar^{2d-2} - Br^{d-2} + \frac{C}{r^{2d+2}} + \frac{D}{r^{d+2}}, \tag{74}$$

which is exactly solvable at zero energy. Here the coupling parameters are $A = \alpha^2 d^2, B = \alpha d(2l+N+d-2), C = \beta^2 d^2, D = \beta d(2l+N-d-2), E = 2\alpha\beta d^2$. Hence for a given set of parameters $\mathrm{Re}(\alpha)$, $\mathrm{Re}(\beta), d, l$ and N, the ground state ($\mathcal{E}_{1,l}^{(0)} = 0$) solution of the Schrödinger equation for the interaction

$$V_1(r) = Ar^{2d-2} - Br^{d-2} + \frac{C}{r^{2d+2}} + \frac{D}{r^{d+2}} - \frac{E}{r^2} \tag{75}$$

is

$$\psi_{1,l}^{(0)}(r) = N_{1,l}^{(0)} r^{l+\frac{N-1}{2}} \exp(-\alpha r^d - \beta r^{-d}). \tag{76}$$

For different values of d, one can construct different types of singular and non-singular potentials from eq. (75). Here, we also avoid the unphysical choices of the parameters. The physically meaningful domain of the parameters α (and β), for these interactions are found to be $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$. Table 4 consists of the QES potentials, their ground state wave functions and the ground state energy. The normalization constants of the ground state wave function for different values of d are given in Appendix D.

The supersymmetric partner potential function $U_2(r)$ of the potential $U_1(r)$ is

$$U_2(r) = \frac{(l + \frac{N-1}{2})(l + 1 + \frac{N-1}{2}) + E}{r^2} + Ar^{2d-2} + B'r^{d-2} + \frac{C}{r^{2d+2}} + \frac{D'}{r^{d+2}}, \tag{77}$$

where $B' = -\alpha d(2l+N-d), D' = \beta d(2l+N+d)$. We get a new family of isospectral potentials for these QES potentials as

$$\mathcal{U}(r) = U_1(r) - 2\frac{r^{2l+N-1}\exp(-2\alpha r^d - 2\beta r^{-d})}{I^2(r)} \times \left[\left(\frac{2l+N-1}{r} - 2\alpha\ \mathrm{d}r^{d-1} + 2\beta\ \mathrm{d}r^{-d-1}\right) \times I(r) - r^{2l+N-1}\exp(-2\alpha r^d - 2\beta r^{-d})\right], \tag{78}$$

where the integration

Table 4. Ground state energies ($\mathcal{E}_{1,l}^{(0)}$) and reduced radial wave functions ($\psi_{1,l}^{(0)}(r)$) for non-singular interaction $V_1(r)$ derived from eqs (75) and (80) with $d = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, 3, 4, 5$ and $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$.

Parameter	$V_1(r) =$	$\psi_{1,l}^{(0)}(r) = N_{1,l}^{(0)} r^{l+\frac{N-1}{2}} \times$	$\mathcal{E}_{1,l}^{(0)}$
$d = \frac{1}{2}$	$\frac{A}{r} - \frac{E}{r^2} + \frac{C}{r^3} - \frac{B}{r^{3/2}} + \frac{D}{r^{5/2}}$	$\exp(-\alpha\sqrt{r} - \frac{\beta}{\sqrt{r}})$	0
$d = \frac{3}{2}$	$Ar - \frac{E}{r^2} + \frac{C}{r^5} - \frac{B}{\sqrt{r}} + \frac{D}{r^{7/2}}$	$\exp(-\alpha r^{3/2} - \frac{\beta}{r^{3/2}})$	0
$d = \frac{5}{2}$	$Ar^3 - B\sqrt{r} + \frac{C}{r^7} + \frac{D}{r^{9/2}} - \frac{E}{r^2}$	$\exp(-\alpha r^{5/2} - \frac{\beta}{r^{5/2}})$	0
$d = 3$	$Ar^4 - Br + \frac{C}{r^8} + \frac{D}{r^5} - \frac{E}{r^2}$	$\exp(-\alpha r^3 - \frac{\beta}{r^3})$	0
$d = 4$	$Ar^6 - Br^2 + \frac{C}{r^{10}} + \frac{D}{r^6} - \frac{E}{r^2}$	$\exp(-\alpha r^4 - \frac{\beta}{r^4})$	0
$d = 5$	$Ar^8 - Br^3 + \frac{C}{r^{12}} + \frac{D}{r^7} - \frac{E}{r^2}$	$\exp(-\alpha r^5 - \frac{\beta}{r^5})$	0

$$
\begin{aligned}
I(r) = \lambda + \sum_{k=0}^{\infty} \frac{(-2\beta)^k}{\Gamma(k+1)} \frac{\Gamma(2l+N-kd)}{\Gamma(2l+N-kd+1)} r^{2l+N-kd} \\
\times {}_dF_d\bigg(\frac{2l+N-kd}{d}, \frac{2l+N-kd+1}{d}, ..., \frac{2l+N-kd+d-1}{d}; \\
\frac{2l+N-kd+1}{d}, \frac{2l+N-kd+2}{d}, ..., \frac{2l+N-kd+d}{d}; -2\alpha r^d \bigg).
\end{aligned}
\tag{79}
$$

4. Conclusion

In the present work, we invoked the fundamental principle of SSQM to construct generalized QES potentials of the Coulomb-dominated singular plus non-singular, harmonic oscillator plus singular and non-singular, non-singular and finally singular types which are found to be useful for the description of various physical processes [31,32]. The elegance of our approach is its ability to study large classes of potentials which are exactly solvable for single state. Here, we attain added realism and sophistication by dealing with higher dimensional Schrödinger equation so that the results can easily be applied to any required lower dimension ($N > 1$). Over and above, the dimensional variable N may be treated as a perturbation parameter ($\frac{1}{N}$) for the perturbative calculation of other states for these QES potentials. We have further constructed a family of new QES potentials for each case using the concept of isospectral potentials of SSQM. These new QES isospectral potentials could be helpful if one tries to fit some other properties of a system with the same energy eigenvalue.

Recently, Tkachuk and Fityo [33] have developed a simple approach for constructing QES potentials with two known energies and their corresponding wave functions. So one of the interesting possibility is to find out the higher excited states and corresponding energy of our derived QES potentials. Our future investigation will be on this problem.

Appendix A

The normalization constant $N_{1,l}^{(0)}$ can be calculated in principle from the normalization condition

$$\int_0^\infty |\psi_{1,l}^{(0)}|^2 \, \mathrm{d}r = 1 \tag{A1}$$

which yields

$$N_{1,l}^0 = \sqrt{\frac{1}{\mathcal{N}_{1,l}^0}},$$

where

$$\mathcal{N}_{1,l}^{(0)} = \int_0^\infty \exp\biggl(-2\int^r W(r')\mathrm{d}r'\biggr)\, \mathrm{d}r.$$

Coulomb plus singular and non-singular potentials

The normalization constant of the ground state wave function for Coulomb plus singular and non-singular potentials for different values of d are given below. For $d = \frac{1}{2}$, $\mathrm{Re}(2l+N) > 0$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$

$$\mathcal{N}_{1,l}^0 = 2^{-(2l+N)}\alpha^{-(2l+N+\frac{1}{2})}\Biggl(\sqrt{\alpha}\Gamma(2l+N)_1F_1\biggl(2l+N, \frac{1}{2}, \frac{\beta^2}{2\alpha}\biggr)$$
$$-\sqrt{2}\beta\Gamma\left(2l+N+\frac{1}{2}\right){}_1F_1\biggl(2l+N+\frac{1}{2}, \frac{3}{2}, \frac{\beta^2}{2\alpha}\biggr)\Biggr).$$

For $d = -\frac{1}{2}$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$,

$$\mathcal{N}_{1,l}^0 = 2^{-(2l+N)}\biggl(\alpha^{-(2l+N)}\Gamma(2l+N)_0F_2\Bigl((\cdot), (1/2, 1-2l-N), -2\alpha\beta^2\Bigr)$$
$$-2\sqrt{2}\alpha^{(1/2-2l-N)}\beta\Gamma(-1/2+2l+N)_0F_2$$
$$\times\Bigl((\cdot), (3/2, 3/2-2l-N), -2\alpha\beta^2\Bigr)$$
$$+2^{(1+6l+3N)}\left(\frac{1}{\beta}\right)^{-2(2l+N)}\Gamma(-2(2l+N))$$
$$\times{}_0F_2\Bigl((\cdot), (1/2+2l+N, 1+2l+N), -2\alpha\beta^2\Bigr)\biggr).$$

The negative gamma function in calculating the normalization for $d = -1/2$ can be overcome by putting $N = 3$ and $l = 0, 1, 2, \ldots$ before evaluation of the integration.

The result of the integral contains Meijer G function which gives finite normalization for several values of $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\beta) > 0$. It is verified that the results from this formula is in full agreement with the results obtained from numerical integration for the same integral.

For $d = 1$, $\mathrm{Re}(2l + N) > 0$, $\mathrm{Re}(\alpha + \beta) > 0$,

$$\mathcal{N}^0_{1,l} = (2\alpha + 2\beta)^{-(2l+N)}\Gamma(2l + N).$$

For $d = -1$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$,

$$\mathcal{N}^0_{1,l} = 2\left(\frac{\alpha}{\beta}\right)^{-(l+\frac{N}{2})} K_{-(2l+N)}(4\sqrt{\alpha\beta}).$$

For $d = \frac{3}{2}$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$,

$$\begin{aligned}\mathcal{N}^0_{1,l} &= \frac{1}{3}2^{-\frac{2}{3}(2l+N)}\beta^{-\frac{2}{3}(2+2l+N)} \\ &\times\Bigg(2\beta^{4/3}\Gamma\left(\frac{2}{3}(2l+N)\right){}_2F_2\Bigg((2l/3+N/3, 1/2+2l/3+N/3), \\ &\left(\frac{1}{3}, 2/3\right) - 8/27\frac{\alpha^3}{\beta^2}\Bigg) + 2^{1/3}\alpha\Bigg(-2\beta^{2/3}\Gamma\left(\frac{2}{3}(1+2l+N)\right) \\ &\times{}_2F_2\Bigg((1/3+2l/3+N/3, 5/6+2l/3+N/3), (2/3, 4/3), \\ &-\frac{8}{27}\frac{\alpha^3}{\beta^2}\Bigg) + 2^{1/3}\alpha\Gamma\left(\frac{2}{3}(2+2l+N)\right) \\ &\times{}_2F_2\Bigg(\left(\frac{2}{3}+\frac{2l}{3}+\frac{N}{3}, \frac{7}{6}+\frac{2l}{3}+\frac{N}{3}\right), \left(\frac{4}{3}, \frac{5}{3}\right), -\frac{8}{27}\frac{\alpha^3}{\beta^2}\Bigg)\Bigg)\Bigg).\end{aligned}$$

For $d = -\frac{3}{2}$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$,

$$\begin{aligned}\mathcal{N}^0_{1,l} = &\frac{2}{3}(2\beta)^{\frac{4l}{3}+\frac{2N}{3}}\Gamma(-(4l/3+2N/3))_0F_4\Bigg((\cdot), (1/3, 2/3, 1/2+2l/3+N/3, \\ &1+2l/3+N/3), -\frac{8}{27}\alpha^3\beta^2\Bigg) + (2\alpha)^{-2l-N}\Gamma(2l+N) \\ &\times{}_0F_4\Bigg((\cdot), (1/2, 1/3-2l/3-N/3, 2/3-2l/3-N/3, 1-2l/3-N/3), \\ &-\frac{8}{27}\alpha^3\beta^2\Bigg) - \frac{1}{3}2^{\frac{8}{3}+\frac{4l}{3}+\frac{2N}{3}}\alpha\left(\frac{1}{\beta}\right)^{-2/3-4l/3-2N/3}\Gamma(-(2/3+4l/3+2N/3)) \\ &\times{}_0F_4\Bigg((\cdot), (2/3, 4/3, 5/6+2l/3+N/3, 4/3+2l/3+N/3), -\frac{8}{27}\alpha^3\beta^2\Bigg) \\ &+\frac{1}{3}2^{\frac{10}{3}+\frac{4l}{3}+\frac{2N}{3}}\alpha^2\left(\frac{1}{\beta}\right)^{-4/3-4l/3-2N/3}\Gamma(-(4/3+4l/3+2N/3))\end{aligned}$$

$$\times {}_0F_4\bigg((\cdot),(4/3,5/3,7/6+2l/3+N/3,5/3+2l/3+N/3),-\frac{8}{27}\alpha^3\beta^2\bigg)$$
$$-\frac{1}{\pi}\bigg(2^{3/2-2l-N}3^{-2+2l+N}\alpha^{3/2-2l-N}\beta\Gamma(-1/2+2l/3+N/3)$$
$$\times\Gamma(-1/6+2l/3+N/3)\Gamma(1/6+2l/3+N/3)_0F_4\bigg((\cdot),(3/2,5/6-2l/3$$
$$-N/3,7/6-2l/3-N/3,3/2-2l/3-N/3),-\frac{8}{27}\alpha^3\beta^2\bigg)\bigg).$$

For $d=-2$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$,

$$\mathcal{N}^0_{1,l}=2^{-1+l+N/2}\left(\frac{1}{\beta}\right)^{-l-N/2}\Gamma\left(-l-\frac{N}{2}\right){}_0F_2((\cdot),(1/2,1+l+N/2),-2\alpha^2\beta)$$
$$-2^{1/2+l+N/2}\alpha\left(\frac{1}{\beta}\right)^{-1/2-l-N/2}\Gamma\left(-l/2-l-\frac{N}{2}\right)$$
$$\times{}_0F_2((\cdot),(3/2,3/2+l+N/2),-2\alpha^2\beta)$$
$$+2^{-2l-N}\alpha^{-2l-N}\Gamma(2l+N){}_0F_2((\cdot),(1/2-l-N/2,1-l-N/2),-2\alpha^2\beta).$$

For $d=3$, $\mathrm{Re}(2l+N)>0$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$,

$$\mathcal{N}^0_{1,l}=\frac{1}{3}2^{-\frac{1}{3}(2l+N)}\beta^{-\frac{1}{3}(2+2l+N)}\bigg(\beta^{\frac{2}{3}}\Gamma\left(\frac{1}{3}(2l+N)\right)$$
$$\times{}_1F_2\bigg((2l/3+N/3),(1/3,2/3),-\frac{4}{27}\frac{\alpha^3}{\beta}\bigg)$$
$$+2^{1/3}\alpha\bigg(-(2\beta)^{1/3}\Gamma\left(\frac{1}{3}(1+2l+N)\right)$$
$$\times{}_1F_2\bigg((1/3+2l/3+N/3),(2/3,4/3),-\frac{4}{27}\frac{\alpha^3}{\beta}\bigg)$$
$$+\alpha\Gamma\left(\frac{1}{3}(2+2l+N)\right){}_1F_2\bigg((2/3+2l/3+N/3),(4/3,5/3),-\frac{4}{27}\frac{\alpha^3}{\beta}\bigg)\bigg)\bigg).$$

For $d=-3$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$,

$$\mathcal{N}^0_{1,l}=\frac{1}{3}(2\alpha)^{-2l-N}\left(\frac{1}{\beta}\right)^{-2/3-2l/3-N/3}\bigg(2^{8l/3+4N/3}\alpha^{2l+N}\beta^{-2/3}\Gamma(-2l/3-N/3)$$
$$\times{}_0F_3\bigg((\cdot),(1/3,2/3,1+2l/3+N/3),\frac{16}{27}\alpha^3\beta\bigg)$$
$$-2^{4/3+8l/3+4N/3}\alpha^{1+2l+N}\beta^{-1/3}\Gamma(-1/3-2l/3-N/3)$$

$$\times {}_0F_3\left((\cdot),(2/3,4/3,4/3+2l/3+N/3),\frac{16}{27}\alpha^3\beta\right)$$
$$+2^{5/3+8l/3+4N/3}\alpha^{2+2l+N}\Gamma(-2/3-2l/3-N/3)$$
$$\times {}_0F_3\left((\cdot),(4/3,5/3,5/3+2l/3+N/3),\frac{16}{27}\alpha^3\beta\right)$$
$$+3\beta^{-2/3-2l/3-N/3}\Gamma(2l+N)$$
$$\times {}_0F_3\left((\cdot),(1/3-2l/3-N/3,2/3-2l/3-N/3,1-2l/3-N/3),\frac{16}{27}\alpha^3\beta\right)\Bigg).$$

Appendix B

Harmonic oscillator plus non-singular and singular potentials

Normalization constant for different values of d of the ground state wave function

$$\psi^{(0)}_{1,l}(r)=N^0_{1,l}r^{l+\frac{N-1}{2}}\exp(-\alpha r^2-\beta r^d)$$

for harmonic oscillator plus non-singular and singular potentials are given below.

For $d=1/2$, $\mathrm{Re}(2l+N)>0$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$,

$$\mathcal{N}^0_{1,l}=\frac{1}{3}2^{-\frac{1}{2}(3+2l+N)}\alpha^{-\frac{1}{4}(3+4l+2N)}\Bigg(3\sqrt{2}\alpha^{3/4}\Gamma(l+N/2)$$
$$\times {}_1F_3\left((l+N/2),(1/4,1/2,3/4),\frac{\beta^4}{32\alpha}\right)-2\beta\Bigg(32^{1/4}\sqrt{\alpha}\Gamma(1/4+l+N/2)$$
$$\times {}_1F_3\left((1/4+l+N/2),(1/2,3/4,5/4),\frac{\beta^4}{3\times 2\alpha}\right)$$
$$+\beta\Bigg(-3\alpha^{1/4}\Gamma(1/2+l+N/2){}_1F_3\left((1/2+l+N/2),(3/4,5/4,3/2),\frac{\beta^4}{32\alpha}\right)$$
$$+2^{3/4}\beta\Gamma(3/4+l+N/2){}_1F_3\left((3/4+l+N/2),(5/4,3/2,7/4),\frac{\beta^4}{32\alpha}\right)\Bigg)\Bigg)\Bigg).$$

For $d=-\frac{1}{2}$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$,

$$\mathcal{N}^0_{1,l}=2^{-(1+l+\frac{N}{2})}\alpha^{-(l+\frac{N}{2})}\Gamma\left(l+\frac{N}{2}\right)$$
$$\times {}_0F_4\left((\cdot),(l/4,1/2,3/4,1-l-N/2),-\frac{\alpha\beta^4}{8}\right)$$
$$-2^{(1/4-l-\frac{N}{2})}\alpha^{(1/4-l-\frac{N}{2})}\beta\Gamma\left(-1/4+l+\frac{N}{2}\right)$$

$$\times {}_0F_4\left((\cdot),(1/2,3/4,5/4,5/4-l-N/2),-\frac{\alpha\beta^4}{8}\right)$$
$$+2^{(1/2-l-\frac{N}{2})}\alpha^{(1/2-l-\frac{N}{2})}\beta^2\Gamma\left(-1/2+l+\frac{N}{2}\right)$$
$$\times {}_0F_4\left((\cdot),(3/4,5/4,3/2,3/2-l-N/2),-\frac{\alpha\beta^4}{8}\right)$$
$$-\frac{1}{3}2^{(7/4-l-\frac{N}{2})}\alpha^{(3/4-l-\frac{N}{2})}\beta^3\Gamma\left(-3/4+l+\frac{N}{2}\right)$$
$$\times {}_0F_4\left((\cdot),(5/4,3/2,7/4,7/4-l-N/2),-\frac{\alpha\beta^4}{8}\right)$$
$$+2^{(1+4l+2N)}\left(\frac{1}{\beta}\right)^{-4l-2N}\Gamma(-4l-2N)$$
$$\times {}_0F_4\left((\cdot),(1/4+l+N/2,1/2+l+N/2,3/4 +l+N/2,1+l+N/2),-\frac{\alpha\beta^4}{8}\right).$$

The negative gamma function can be overcome like Coulomb case as before.

For $d=1$, $\mathrm{Re}(2l+N)>0$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$,

$$\mathcal{N}^0_{1,l}=2^{-\frac{1}{2}(3+2l+N)}\alpha^{-\frac{1}{2}(l+2l+N)}$$
$$\times\left(\sqrt{2\alpha}\Gamma(l+N/2){}_1F_1\left(l+N/2,1/2,\frac{\beta^2}{2\alpha}\right)\right.$$
$$\left.-2\beta\Gamma(1/2+l+N/2){}_1F_1\left(1/2+l+N/2,3/2,\frac{\beta^2}{2\alpha}\right)\right).$$

For $d=-1$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$,

$$\mathcal{N}^0_{1,l}=2^{-(1+l+\frac{N}{2})}\left(\alpha^{-(l+\frac{N}{2})}\Gamma\left(l+\frac{N}{2}\right)\right.$$
$$\times {}_0F_2\left((\cdot),(1/2,1-l-N/2),-2\alpha\beta^2\right)$$
$$-2\sqrt{2}\alpha^{\frac{1}{2}(1-2l-N)}\beta\Gamma\left(-1/2+l+\frac{N}{2}\right)$$
$$\times {}_0F_2\left((\cdot),(3/2,3/2-l-N/2),-2\alpha\beta^2\right)$$
$$+2^{(1+3l+3N/2)}\left(\frac{1}{\beta}\right)^{-2l-N}\Gamma(-2l-N)$$
$$\left.\times {}_0F_2\left((\cdot),(1/2+l+N/2,1+l+N/2),-2\alpha\beta^2\right)\right).$$

For $d=-2$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$,

$$\mathcal{N}_{1,l}^{0}=\left(\frac{\alpha}{\beta}\right)^{-\frac{1}{4}(2l+N)}K_{l+\frac{N}{2}}(4\sqrt{\alpha\beta}).$$

For $d=3$, $\mathrm{Re}(2l+N)>0$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$,

$$\begin{aligned}\mathcal{N}_{1,l}^{0}=&\frac{1}{3}2^{-\frac{1}{3}(1+2l+N)}\beta^{-\frac{1}{3}(4+2l+N)}\Bigg(2^{1/3}\beta^{4/3}\Gamma\left(\frac{1}{3}(2l+N)\right)\\&\times{}_2F_2\left((l/3+N/6,1/2+l/3+N/6),(1/3,2/3),-\frac{8\alpha^3}{27\beta^2}\right)\\&+\alpha\Bigg(-2^{2/3}\beta^{2/3}\Gamma\left(\frac{1}{3}(2+2l+N)\right)\\&\times{}_2F_2\left((1/3+l/3+N/6,5/6+l/3+N/6),(2/3,4/3),-\frac{8\alpha^3}{27\beta^2}\right)\\&+\alpha\Gamma\left(\frac{1}{3}(4+2l+N)\right)\ {}_2F_2\Bigg((2/3+l/3+N/6,7/6+l/3+N/6),\\&(4/3,5/3),-\frac{8\alpha^3}{27\beta^2}\Bigg)\Bigg)\Bigg).\end{aligned}$$

Appendix C

Non-singular potential

Type A:

Normalization constant for the ground state wave function

$$\psi_{1,l}^{(0)}(r)=N_{1,l}^{0}r^{l+\frac{N-1}{2}}\exp(-\alpha r^{2d}-\beta r^{d})$$

for different values of d.

For $d=5/2$, $\mathrm{Re}(2l+N)>0$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$,

$$\begin{aligned}\mathcal{N}_{1,l}^{0}=&\frac{1}{5}2^{-\frac{1}{5}(2l+N)}\alpha^{-\frac{1}{10}(5+4l+2N)}\\&\times\left(\sqrt{\alpha}\Gamma\left(\frac{1}{5}(2l+N)\right){}_1F_1\left(\frac{1}{5}(2l+N),1/2,\frac{\beta^2}{2\alpha}\right)\right.\\&\left.-\sqrt{2}\beta\Gamma\left(\frac{1}{10}(5+4l+2N)\right){}_1F_1\left(\frac{1}{10}(5+4l+2N),3/2,\frac{\beta^2}{2\alpha}\right)\right).\end{aligned}$$

For $d=3$, $\mathrm{Re}(2l+N)>0$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$,

$$\mathcal{N}^0_{1,l} = \frac{1}{3} 2^{-\frac{1}{6}(9+2l+N)} \alpha^{-\frac{1}{6}(3+2l+N)} \left(\sqrt{2\alpha}\Gamma\left(\frac{1}{6}(2l+N)\right) \right.$$
$$\times {}_1F_1\left(\frac{1}{6}(2l+N), 1/2, \frac{\beta^2}{2\alpha}\right) - 2\beta\Gamma\left(\frac{1}{6}(3+2l+N)\right)$$
$$\left. \times {}_1F_1\left(\frac{1}{6}(3+2l+N), 3/2, \frac{\beta^2}{2\alpha}\right)\right).$$

For $d = 7/2$, $\mathrm{Re}(2l+N) > 0$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$,

$$\mathcal{N}^0_{1,l} = \frac{1}{7} 2^{-\frac{1}{7}(2l+N)} \alpha^{-\frac{1}{14}(7+4l+2N)}$$
$$\times \left(\sqrt{\alpha}\Gamma\left(\frac{1}{7}(2l+N)\right) {}_1F_1\left(\frac{1}{7}(2l+N), 1/2, \frac{\beta^2}{2\alpha}\right) \right.$$
$$\left. -\sqrt{2}\beta\Gamma\left(\frac{1}{14}(7+4l+2N)\right) {}_1F_1\left(\frac{1}{14}(7+4l+2N), 3/2, \frac{\beta^2}{2\alpha}\right)\right).$$

For $d = 4$, $\mathrm{Re}(2l+N) > 0$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$,

$$\mathcal{N}^0_{1,l} = 2^{-\frac{1}{8}(28+2l+N)} \alpha^{-\frac{1}{8}(4+2l+N)}$$
$$\times \left(\sqrt{2\alpha}\Gamma\left(\frac{1}{8}(2l+N)\right) {}_1F_1\left(\frac{1}{8}(2l+N), 1/2, \frac{\beta^2}{2\alpha}\right) \right.$$
$$\left. -2\beta\Gamma\left(\frac{1}{8}(4+2l+N)\right) {}_1F_1\left(\frac{1}{8}(4+2l+N), 3/2, \frac{\beta^2}{2\alpha}\right)\right).$$

For $d = 5$, $\mathrm{Re}(2l+N) > 0$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$,

$$\mathcal{N}^0_{1,l} = \frac{1}{5} 2^{-\frac{1}{10}(15+2l+N)} \alpha^{-\frac{1}{10}(5+2l+N)}$$
$$\times \left(\sqrt{2\alpha}\Gamma\left(\frac{1}{10}(2l+N)\right) {}_1F_1\left(\frac{1}{10}(2l+N), 1/2, \frac{\beta^2}{2\alpha}\right) \right.$$
$$\left. -2\beta\Gamma\left(\frac{1}{10}(5+2l+N)\right) {}_1F_1\left(\frac{1}{10}(5+2l+N), 3/2, \frac{\beta^2}{2\alpha}\right)\right).$$

Type B:

Normalization constant for the ground state wave function

$$\psi^{(0)}_{1,l}(r) = N^0_{1,l} r^{l+\frac{N-1}{2}} \exp(-\alpha r^d - \beta r^{d-1})$$

for different values of d are given below.

For $d = 4$, $\mathrm{Re}(2l+N) > 0$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$,

$$\mathcal{N}^0_{1,l} = 2^{-(2+l/2+N/4)}\alpha^{-(l/2+N/4)}\Gamma(l/2+N/4)$$
$$\times {}_3F_3\bigg((l/6+N/12, 1/3+l/6+N/12, 2/3+l/6+N/12), (1/4, 1/2, 3/4),$$
$$\frac{27\beta^4}{128\alpha^3}\bigg) - \frac{1}{\pi}\bigg(2^{-11/4-l/2-N/4}3^{1/4+l/2+N/4}\alpha^{-3/4-l/2-N/4}\beta$$
$$\times\Gamma(1/4+l/6+N/12)\Gamma(7/12+l/6+N/12)\Gamma(11/12+l/6+N/12)$$
$$\times {}_3F_3\bigg((1/4+l/6+N/12, 7/12+l/6+N/12, 11/12+l/6+N/12),$$
$$(1/2, 3/4, 5/4), \frac{27\beta^4}{128\alpha^3}\bigg)\bigg) + \frac{1}{\pi}\bigg(2^{-7/2-l/2-N/4}3^{1+l/2+N/4}\alpha^{-3/2-l/2-N/4}\beta^2$$
$$\times\Gamma(1/2+l/6+N/12)\Gamma(5/6+l/6+N/12)\Gamma(7/6+l/6+N/12)$$
$$\times {}_3F_3\bigg((1/2+l/6+N/12, 5/6+l/6+N/12, 7/6+l/6+N/12),$$
$$(3/4, 5/4, 3/2), \frac{27\beta^4}{128\alpha^3}\bigg)\bigg) - \frac{1}{\pi}\bigg(2^{-13/4-l/2-N/4}3^{3/4+l/2+N/4}\alpha^{-9/4-l/2-N/4}\beta^3$$
$$\times\Gamma(3/4+l/6+N/12)\Gamma(13/12+l/6+N/12)\Gamma(17/12+l/6+N/12)$$
$$\times {}_3F_3\bigg((3/4+l/6+N/12, 13/12+l/6+N/12, 17/12+l/6+N/12),$$
$$(5/4, 3/2, 7/4), \frac{27\beta^4}{128\alpha^3}\bigg)\bigg).$$

For $d = 5$, $\mathrm{Re}(2l+N) > 0$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$,

$$\mathcal{N}^0_{1,l} = \frac{1}{5}2^{-2l/5-N/5}\alpha^{-(2l/5+N/5)}\Gamma(2l/5+N/5)$$
$$\times {}_4F_4\bigg((l/10+N/20, 1/4+l/10+N/20, 1/2+l/10$$
$$+N/20, 3/4+l/10+N/20), (1/5, 2/5, 3/5, 4/5), -\frac{512\beta^5}{3125\alpha^4}\bigg)$$
$$-\frac{1}{10}\bigg(2^{6/5-2l/5-N/5}\alpha^{-(4/5+2l/5+N/5)}\beta\Gamma(4/5+2l/5+N/5)$$
$$\times {}_4F_4\bigg((1/5+l/10+N/20, 9/20+l/10+N/20,$$
$$7/10+l/10+N/20, 19/20+l/10+N/20), (2/5, 3/5, 4/5, 6/5),$$
$$-\frac{512\beta^5}{3125\alpha^4}\bigg)\bigg) + \frac{1}{10}\bigg(2^{2/5-2l/5-N/5}\alpha^{-(8/5+2l/5+N/5)}\beta^2\Gamma(8/5+2l/5+N/5)$$
$$\times {}_4F_4\bigg((2/5+l/10+N/20, 13/20+l/10+N/20, 9/10+l/10+N/20,$$

$$23/20 + l/10 + N/20), (3/5, 4/5, 6/5, 7/5), -\frac{512\beta^5}{3125\alpha^4}\Big)\Big)$$

$$-\frac{1}{15} 2^{2/5-2l/5-N/5}\alpha^{-(12/5+2l/5+N/5)}\beta^3$$

$$\times\Gamma(12/5 + 2l/5 + N/5)\ {}_4F_4\Big((3/5 + l/10 + N/20, 17/20 + l/10 + N/20,$$

$$11/10 + l/10 + N/20, 27/20 + l/10 + N/20), (4/5, 6/5, 7/5, 8/5), -\frac{512\beta^5}{3125\alpha^4}\Big)$$

$$+\frac{1}{15} 2^{11/5-2l/5-N/5}\alpha^{-(16/5+2l/5+N/5)}\beta^4\Gamma(16/5 + 2l/5 + N/5)$$

$$\times {}_4F_4\Big((4/5 + l/10 + N/20, 21/20 + l/10 + N/20, 13/10 + l/10 + N/20,$$

$$31/20 + l/10 + N/20), (6/5, 7/5, 8/5, 9/5), -\frac{512\beta^5}{3125\alpha^4}\Big).$$

Type C:

Normalization constant for the ground state wave function

$$\psi^{(0)}_{1,l}(r) = N^0_{1,l} r^{l+\frac{N-1}{2}} \exp(-\alpha r^d - \beta r^{d-2})$$

for different values of d are given below.

For $d = 5$, $\mathrm{Re}(2l + N) > 0$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$,

$$\mathcal{N}^0_{1,l} = \frac{1}{5} 2^{-2l/5-N/5}\alpha^{-(2l/5+N/5)}\Gamma(2l/5 + N/5)$$

$$\times {}_3F_4\Big((2l/15 + N/15, 1/3 + 2l/15 + N/15, 2/3 + 2l/15 + N/15),$$

$$(1/5, 2/5, 3/5, 4/5), -\frac{108\beta^5}{3125\alpha^3}\Big)$$

$$-\frac{1}{10\pi}\Big(2^{2/5-2l/5-N/5}3^{1/10+2l/5+N/5}\alpha^{-(3/5+2l/5+N/5)}\beta$$

$$\times\Gamma(1/5 + 2l/15 + N/15)\Gamma(8/15 + 2l/15 + N/15)\Gamma(13/15 + 2l/15 + N/15)$$

$$\times {}_3F_4\Big((1/5 + 2l/15 + N/15, 8/15 + 2l/15 + N/15, 13/15 + 2l/15 + N/15),$$

$$(2/5, 3/5, 4/5, 6/5), -\frac{108\beta^5}{3125\alpha^3}\Big)\Big)$$

$$+\frac{1}{10\pi}\Big(2^{-1/5-2l/5-N/5}3^{7/10+2l/5+N/5}\alpha^{-(6/5+2l/5+N/5)}\beta^2$$

$$\times\Gamma(2/5 + 2l/15 + N/15)\Gamma(11/15 + 2l/15 + N/15)\Gamma(16/15 + 2l/15 + N/15)$$

$$\times {}_3F_4\Big((2/5 + 2l/15 + N/15, 11/15 + 2l/15 + N/15, 16/15 + 2l/15 + N/15),$$

$$(3/5,4/5,6/5,7/5),-\frac{108\beta^5}{3125\alpha^3}\bigg)\bigg)$$

$$-\frac{1}{5\pi}\bigg(2^{-4/5-2l/5-N/5}3^{3/10+2l/5+N/5}\alpha^{-(9/5+2l/5+N/5)}\beta^3$$

$$\times\Gamma(3/5+2l/15+N/15)\Gamma(14/15+2l/15+N/15)\Gamma(19/15+2l/15+N/15)$$

$$\times{}_3F_4\bigg((3/5+2l/15+N/15,14/15+2l/15+N/15,19/15+2l/15+N/15),$$

$$(4/5,6/5,7/5,8/5),-\frac{108\beta^5}{3125\alpha^3}\bigg)\bigg)+\frac{1}{5\pi}\bigg(2^{-12/5-2l/5-N/5}3^{9/10+2l/5+N/5}$$

$$\times\alpha^{-(12/5+2l/5+N/5)}\beta^4\Gamma(4/5+2l/15+N/15)\Gamma(17/15+2l/15+N/15)$$

$$\times\Gamma(22/15+2l/15+N/15){}_3F_4\bigg((4/5+2l/15+N/15,17/15+2l/15+N/15,$$

$$22/15+2l/15+N/15),(6/5,7/5,8/5,9/5),-\frac{108\beta^5}{3125\alpha^3}\bigg)\bigg).$$

For $d=6$, $\mathrm{Re}(2l+N)>0$, $\mathrm{Re}(\alpha)>0$, $\mathrm{Re}(\beta)>0$,

$$\mathcal{N}_{1,l}^0=\frac{1}{3}2^{-\frac{1}{6}(8+2l+N)}\alpha^{-\frac{1}{6}(8+2l+N)}\bigg(2^{1/3}\alpha^{4/3}\Gamma\left(\frac{1}{6}(2l+N)\right)$$

$$\times{}_2F_2\bigg((l/6+N/12,1/2+l/6+N/12),(1/3,2/3),-\frac{8\beta^3}{27\alpha^2}\bigg)$$

$$+\beta\bigg(-2^{2/3}\alpha^{2/3}\Gamma\left(\frac{1}{6}(4+2l+N)\right)$$

$$\times{}_2F_2\bigg((1/3+l/6+N/12,5/6+l/6+N/12),$$

$$(2/3,4/3),-\frac{8\beta^3}{27\alpha^2}\bigg)+\beta\Gamma\left(\frac{1}{6}(8+2l+N)\right){}_2F_2\bigg((2/3+l/6+N/12,$$

$$7/6+l/6+N/12),(4/3,5/3),-\frac{8\beta^3}{27\alpha^2}\bigg)\bigg)\bigg).$$

Appendix D

Singular plus non-singular potential

Normalization constant for the ground state wave function

$$\psi_{1,l}^{(0)}(r)=N_{1,l}^0r^{l+\frac{N-1}{2}}\exp\left(-\alpha r^d-\frac{\beta}{r^d}\right)$$

for different values of d are given below.

For $d = 1/2$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$,

$$\mathcal{N}^0_{1,l} = 4\left(\frac{\alpha}{\beta}\right)^{-(2l+N)} K_{2(2l+N)}\left(4\sqrt{\alpha\beta}\right).$$

For $d = 3/2$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$,

$$\mathcal{N}^0_{1,l} = \frac{4}{3}\left(\frac{\alpha}{\beta}\right)^{-\frac{1}{3}(2l+N)} K_{\frac{2}{3}(2l+N)}\left(4\sqrt{\alpha\beta}\right).$$

For $d = 5/2$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$,

$$\mathcal{N}^0_{1,l} = \frac{4}{5}\left(\frac{\alpha}{\beta}\right)^{-\frac{1}{5}(2l+N)} K_{\frac{2}{5}(2l+N)}\left(4\sqrt{\alpha\beta}\right).$$

For $d = 3$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$,

$$\mathcal{N}^0_{1,l} = \frac{2}{3}\left(\frac{\alpha}{\beta}\right)^{-\frac{1}{6}(2l+N)} K_{\frac{1}{3}(2l+N)}\left(4\sqrt{\alpha\beta}\right).$$

For $d = 4$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$,

$$\mathcal{N}^0_{1,l} = \frac{1}{2}\left(\frac{\alpha}{\beta}\right)^{-\frac{1}{8}(2l+N)} K_{\frac{1}{4}(2l+N)}\left(4\sqrt{\alpha\beta}\right).$$

For $d = 5$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$,

$$\mathcal{N}^0_{1,l} = \frac{2}{5}\left(\frac{\alpha}{\beta}\right)^{-\frac{1}{10}(2l+N)} K_{\frac{1}{5}(2l+N)}\left(4\sqrt{\alpha\beta}\right).$$

References

[1] A V Turbiner and A G Ushveridze, *Phys. Lett.* **A126**, 181 (1987)
A V Turbiner, *Zh. Eksp. Teor. Fiz.* **94**, 33 (1988)
A V Turbiner, *Funct. Anal.* **22**, 92 (1988)
A V Turbiner, *Commun. Math. Phys.* **118**, 467 (1988)
A V Turbiner, *Contemp. Math.* **160**, 263 (1994)
A G Ushveridze, *Quasi-exactly solvable models in quantum mechanics* (Institute of Physics Publishing, Bristol, 1994)
[2] C M Bender and G V Dunne, *J. Math. Phys.* **36**, 6 (1996)
A Khare and B P Mandal, *J. Math. Phys.* **39**, 5789 (1998)
A Ganguli, *J. Math. Phys.* **43**, 5310 (2002)
B Bagchi and A Ganguli, *J. Math. Phys.* **36**, L161 (2003)

[3] A O Barut, M Berroundo and G Garcia-Calderon, *J. Math. Phys.* **21**, 1851 (1980)
S Özcelik and M Simsek, *Phys. Lett.* **A152**, 145 (1995)
S Özcelik, *Tr. J. Phys.* **20**, 1233 (1996)
E Papp, *Phys. Lett.* **A157**, 192 (1991)
[4] E Witten, *Nucl. Phys.* **B185**, 513 (1981)
[5] R W Haymaker and A R P Rau, *Am. J. Phys.* **54**, 928 (1986)
R Dutt, A H Khare and U Sukhatme, *Am. J. Phys.* **56**, 163 (1988)
F Cooper, A Khare and U Sukhatme, *Phys. Rep.* **251**, 267 (1995)
[6] C V Sukumar, *J. Phys.* **A18**, L577 (1985)
C V Sukumar, *J. Phys.* **A18**, 2937 (1985)
C V Sukumar, *J. Phys.* **A18**, 2917 (1985)
M de Crombrugghe and V Rittenberg, *Ann. Phys. (NY)* **181**, 99 (1983)
A Kostelecky, M M Nieto and D R Truax, *Phys. Rev.* **32**, 2627 (1985)
[7] M M Nieto, *Phys. Lett.* **145**, 208 (1984)
[8] F Iachello, *Phys. Rev. Lett.* **44**, 772 (1980)
A B Balantekin, I Bars and F Iachello, *Phys. Rev. Lett.* **47**, 19 (1981)
[9] T Sil, R Dutt and Y P Varshni, *Phys. Rev.* **C50**, 2458 (1994)
P K Bera, T K Nandi and B Talukdar, *J. Phys.* **A26**, L1073 (1993)
[10] V A Kostelecky and M M Nieto, *Phys. Rev. Lett.* **53**, 2285 (1984)
[11] B Talukdar, U Das, C Bhattacharyya and P K Bera, *J. Phys.* **A25**, 4072 (1992)
[12] M M Nieto and L M Simons, *Am. J. Phys.* **47**, 634 (1979)
R L Hall, *Phys. Rev.* **A39**, 5500 (1989)
S Godfrey and J Napolitano, *Rev. Mod. Phys.* **71**, 1411 (1999)
[13] S Coleman, *Aspects of symmetry selected Erice lectures* (Cambridge University Press, Cambridge, 1988)
[14] G F Gribekun and V V Flambeum, *Phys. Rev.* **A48**, 546 (1993)
R Szmytkowski, *J. Phys.* **A28**, 7333 (1995)
[15] E Vogt and G H Wannier, *Phys. Rev.* **95**, 1190 (1954)
D R Bates and I Esterman, *Advances in atomic and molecular physics* (Academic, New York, 1970)
[16] M Kaplus and R N Porter, *Atoms and molecular physics* (Cambridge University Press, Cambridge, 1970)
R J Le Roy and R B Bernstein, *J. Chem. Phys.* **52**, 3869 (1970)
R J Le Roy and W Lam, *Chem. Phys. Lett.* **71**, 544 (1980)
[17] D Emin and T Holstein, *Phys. Rev. Lett.* **36**, 323 (1976)
D Emin and T Holstein, *Phys. Today* **35**, 34 (1982)
[18] A M Sherry and M Kumar, *J. Chem. Solids* **52**, 1145 (1991)
[19] A O Barut, *J. Math. Phys.* **21**, 568 (1980)
[20] B Gönül, O Özer, M Kocak, D Tuteu and Y Cancelik, *J. Phys.* **A34**, 8271 (2001)
[21] C M Bender and S Boettcher, *Phys. Rev. Lett.* **80**, 5243 (1998); *J. Phys.* **A31**, L273 (1998)
M Znojil, arXiv quant-ph/9911116 (2000)
[22] M Znojil, D Yanovich and V P Gerdt, *J. Phys.* **A36**, 6531 (2003)
R L Hall and N Saad, *J. Phys.* **A38**, 4909 (1997)
R E Carndall and M H Reno, *J. Math. Phys.* **23**, 64 (1982)
[23] C X Chuan, *J. Phys.* **A24**, L1165 (1991)
[24] D R Herrick, *J. Math. Phys.* **16**, 281 (1975)
D R Herrick and F H Stillinger, *Phys. Rev.* **A11**, 42 (1975)
M M Nieto, *Am. J. Phys.* **47**, 1067 (1979)
[25] J D Louck, *J. Mol. Spectrosc.* **4**, 298 (1960)

[26] L Infeld and T D Hull, *Rev. Mod. Phys.* **23**, 21 (1951)
[27] N A Alves and E D Filho, *J. Phys. A: Math. Gen.* **21**, 3215 (1988)
[28] D J C Fernandez, *Lett. Math. Phys.* **8**, 337 (1984)
[29] B Mielnik, *J. Math. Phys.* **25**, 3387 (1984)
[30] I S Gradshteyn and I M Ryzhik, *Tables of integrals, series, and products* (Academic Press, New York, 1980)
[31] E Garrido, D V Fedorov and A S Jenson, *Nucl. Phys.* **A617**, 153 (1997)
[32] E Nielsen, D V Fedorov and A S Jenson, *J. Phys.* **B31**, 4085 (1998)
[33] V M Tkachuk and T V Fityo, *Phys. Lett.* **A309**, 351 (2003)
V M Tkachuk, *J. Phys.* **A34**, 6339 (2001)