

Ray space ‘Riccati’ evolution and geometric phases for N -level quantum systems

S CHATURVEDI¹, E ERCOLESSI^{2,*}, G MARMO³, G MORANDI⁴,
N MUKUNDA⁵ and R SIMON⁶

¹School of Physics, University of Hyderabad, Hyderabad 500 046, India

²Physics Department, University of Bologna, CNISM and INFN, 46 v.Irnerio, I-40126, Bologna, Italy

³Dipartimento di Scienze Fisiche, University of Napoli and INFN, v.Cinzia, I-80126, Napoli, Italy

⁴Physics Department, University of Bologna, CNISM and INFN, 6/2 v.le Berti Pichat, I-40127, Bologna, Italy

⁵Centre for High Energy Physics, Indian Institute of Science, Bangalore 560 012, India

⁶The Institute of Mathematical Sciences, C.I.T Campus, Chennai 600 113, India

*Corresponding author. E-mail: ercolessi@bo.infn.it

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Abstract. We present a simple derivation of the matrix Riccati equations governing the reduced dynamics as one descends from the group $U(N)$ describing the Schrödinger evolution of an N -level quantum system to the various coset spaces and Grassmanian manifolds associated with it. The special case pertaining to the geometric phase in N -level systems is described in detail. Further, we show how the matrix Riccati equation thus obtained can be reformulated as an equation describing Hamiltonian evolution in a classical phase space and establish correspondences between the two descriptions.

Keywords. Quantum dynamics; Riccati equations; geometric phase.

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1. Introduction

There is currently considerable interest in studying various properties of quantum systems with finite-dimensional state spaces, for instance in understanding entanglement in the context of quantum information and quantum computation [1]. Perhaps surprisingly, there seem to be many things yet to be learnt in such systems. From the standpoint of fundamental quantum mechanics as well, there has been longstanding interest in, for example, extending the concept of Wigner distribution to finite dimensions and its role in quantum state estimation [2].

For two-level systems, $N = 2$, it is well-known that the Poincaré–Bloch sphere S^2 provides an excellent realization and practical tool for dealing with pure (as

well as mixed) quantum states [3]. Some efforts to extend the Poincaré sphere construction to $N \geq 3$ also exist [4]. In another direction, there has been a systematic programme of ‘unitary integration’ methods [5] and also towards generalizing the Poincaré sphere concept to higher dimensions [6] leading to methods of projecting quantum dynamics at the N -level Hilbert space governed by a Schrödinger equation to various ‘base spaces’ of lower dimension. From a mathematical point of view, the central ideas and structures underlying these ‘reductions’ are similar to those known and extensively developed, over a long period of time, in the context of classical dynamical systems with the objective of seeking dynamical superpositions in nonlinear evolution equations [7]. In the work of Uskov and Rau [6], one begins with the group $\text{SU}(N)$ acting on the state space of an N -level system, the Schrödinger unitary evolution operator being also an element of $\text{SU}(N)$. For any partition: $N = n_1 + n_2$, one has the subgroup $\text{SU}(n_1) \times \text{SU}(n_2)$ and the coset space $\text{SU}(N)/\text{SU}(n_1) \times \text{SU}(n_2)$, which is a Grassmanian manifold and which functions as the base manifold of a fiber bundle, the total space being $\text{SU}(N)$. Using matrices outside of $\text{SU}(N)$ generated by nilpotent matrices in a well-chosen manner, a parametrization of the base manifold by a set of $n_1 \cdot n_2$ independent complex coordinates is set up. It is then shown that the original Schrödinger evolution projects down to a system of matrix Riccati equations for these base-space coordinates, and some connections to geometric phases [8] are indicated. While the work of Uskov and Rau has much in common with more mathematical treatments [9], it lays particular emphasis on highlighting the essentials through nothing more complicated than matrix algebra.

Much in the spirit of [6], the aim of the present work is to give an alternative treatment of this problem of reduction of Schrödinger evolution to various base spaces in a manner that works throughout within the unitary group $\mathbb{U}(N)$ intrinsic and natural to quantum mechanics, and to obtain the Riccati equations avoiding the use of ‘nilpotent generators’. Following this, in the case $n_1 = N - 1, n_2 = 1$ appropriate for discussing evolution of pure states by the Schrödinger equation, it is shown that the base space can be conveniently viewed as a classical phase space, and the Riccati equations expressing quantum dynamics then appear as classical Hamiltonian equations with a suitable Hamiltonian function. This framework is then used to describe in detail the structure of pure state geometric phases for the original N -level system: one sees the extent to which such phases can be expressed in purely classical terms, and also particularly clearly that they are ray-space quantities.

A brief summary of the present work is as follows: In §2 we derive the Riccati equations associated with various coset spaces of the group $\mathbb{U}(N)$ governing the Schrödinger evolution of an N level quantum system without resorting to nilpotent operators. In §3 we apply the general formalism of §2 to a special case appropriate to the geometric phase. Here we also show how the resulting equations can be reformulated in purely classical terms and compare the two perspectives on the geometric phase. Section 4 contains concluding remarks.

2. N -Level system dynamics, coset spaces, Riccati equations

Consider an N -level quantum system with a Hermitian Hamiltonian matrix $H(t)$ which may be time-dependent. The unitary evolution operator $U(t)$ is an element of $\mathbb{G} = \mathbb{U}(N)$ obeying the equation:

$$i\dot{U}(t) = H(t)U(t), \quad U(t_0) = \mathbb{I}. \quad (1)$$

When H is time-independent, one easily sees that the solutions of this equation are given by the one-parameter group generated by iH acting on \mathbb{G} from the left.

The right action of \mathbb{G} on itself acts transitively on the family of solutions. Therefore, all the solutions are simply obtained by acting on the subgroup generated by iH . This circumstance gives rise to the nonlinear superposition rule present in Lie–Scheffers systems [7].

As discussed below, when we consider ‘decompositions’ of \mathbb{G} , we may relate our equations of motion with other evolution equations on the space defined by the decomposition, the ‘reduced space’. On this reduced space the evolution equations are in general nonlinear and acquire the form of Riccati-type equation. Many ‘reduction procedures’ are available in the literature, see for instance [10] for some applications. What we consider here are specific instances applied to relevant physical situations which arise in quantum mechanics.

To ‘decompose’ the unitary group \mathbb{G} , let us consider any partition: $N = n_1 + n_2$ and the subgroup:

$$\mathbb{H} = \mathbb{U}(n_1) \times \mathbb{U}(n_2) \subset \mathbb{G}, \quad (2)$$

with the factor $\mathbb{U}(n_1)$ acting on the first n_1 dimensions, $\mathbb{U}(n_2)$ on the rest. The idea is to describe the coset space \mathbb{G}/\mathbb{H} ‘nicely’ and obtain from eq. (1) by ‘projection’ an evolution equation on it. Write a general matrix in \mathbb{G} in block form as

$$U = \begin{vmatrix} A & B \\ C & D \end{vmatrix}, \quad (3)$$

where A, B, C, D are $n_1 \times n_1, n_1 \times n_2, n_2 \times n_1, n_2 \times n_2$ dimensional matrices respectively. The condition $U^\dagger U = \mathbb{I}$ becomes

$$A^\dagger A + C^\dagger C = \mathbb{I}, \quad A^\dagger B + C^\dagger D = 0, \quad D^\dagger D + B^\dagger B = \mathbb{I}. \quad (4)$$

Under right multiplication by an element

$$\begin{vmatrix} U_1 & \mathbf{0} \\ \mathbf{0} & U_2 \end{vmatrix}$$

in \mathbb{H} , the changes in A, B, C, D are

$$A \rightarrow AU_1, \quad B \rightarrow BU_2, \quad C \rightarrow CU_1, \quad D \rightarrow DU_2. \quad (5)$$

So, in case A and D are non-singular, CA^{-1} and BD^{-1} are invariant, i.e. they are constants over each coset, and so are CA^\dagger and BD^\dagger .

Let us now limit ourselves to the subset of \mathbb{G} in which A, D are both non-singular. Then one can get a nice coset representative as follows: using the freedom of right

multiplication by elements in \mathbb{H} , we can use the polar decomposition of A and D to bring them both to Hermitian positive-definite forms:

$$\text{coset representative} = U_0 = \begin{vmatrix} A_0 & B_0 \\ C_0 & D_0 \end{vmatrix}, \quad A_0^\dagger = A_0 > 0, \quad D_0^\dagger = D_0 > 0. \quad (6)$$

The coset invariant mentioned above can then be defined as an $n_1 \times n_2$ complex rectangular matrix:

$$Z = B_0 D_0^{-1}. \quad (7)$$

We can now use eq. (4) to express A_0, B_0, C_0, D_0 in terms of Z :

$$A_0 = \Gamma_1^{-1/2}, \quad B_0 = Z \Gamma_2^{-1/2}, \quad C_0 = -Z^\dagger \Gamma_1^{-1/2}, \quad D_0 = \Gamma_2^{-1/2}, \quad (8)$$

where $\Gamma_1 = \mathbb{I} + ZZ^\dagger$ and $\Gamma_2 = \mathbb{I} + Z^\dagger Z$ are $n_1 \times n_1$ and $n_2 \times n_2$ positive Hermitian matrices intertwined with each other through Z : $\Gamma_1 Z = Z \Gamma_2$. Thus, in the region of \mathbb{G} defined above, a general matrix U in eq. (3) can be written as

$$U = \begin{vmatrix} A_0 & B_0 \\ C_0 & D_0 \end{vmatrix} \begin{vmatrix} U_1 & \mathbf{0} \\ \mathbf{0} & U_2 \end{vmatrix}, \quad (9)$$

giving $A = A_0 U_1, B = B_0 U_2, C = C_0 U_1, D = D_0 U_2$. Using this in the evolution equation (1) and writing $H(t)$ in block form:

$$H(t) = \begin{vmatrix} H_1(t) & V(t) \\ V(t)^\dagger & H_2(t) \end{vmatrix}, \quad (10)$$

we get

$$i \frac{d}{dt} \left(\begin{vmatrix} A_0 & B_0 \\ C_0 & D_0 \end{vmatrix} \begin{vmatrix} U_1 & \mathbf{0} \\ \mathbf{0} & U_2 \end{vmatrix} \right) = \begin{vmatrix} H_1 & V \\ V^\dagger & H_2 \end{vmatrix} \begin{vmatrix} A_0 & B_0 \\ C_0 & D_0 \end{vmatrix} \begin{vmatrix} U_1 & \mathbf{0} \\ \mathbf{0} & U_2 \end{vmatrix},$$

i.e.

$$i \begin{vmatrix} \dot{A}_0 & \dot{B}_0 \\ \dot{C}_0 & \dot{D}_0 \end{vmatrix} = \begin{vmatrix} H_1 & V \\ V^\dagger & H_2 \end{vmatrix} \begin{vmatrix} A_0 & B_0 \\ C_0 & D_0 \end{vmatrix} - i \begin{vmatrix} A_0 & B_0 \\ C_0 & D_0 \end{vmatrix} \begin{vmatrix} \dot{U}_1 U_1^{-1} & \mathbf{0} \\ \mathbf{0} & \dot{U}_2 U_2^{-1} \end{vmatrix},$$

which yields

$$\begin{aligned} i \dot{A}_0 &= H_1 A_0 + V C_0 - i A_0 \dot{U}_1 U_1^{-1}, \\ i \dot{B}_0 &= H_1 B_0 + V D_0 - i B_0 \dot{U}_2 U_2^{-1}, \\ i \dot{C}_0 &= V^\dagger A_0 + H_2 C_0 - i C_0 \dot{U}_1 U_1^{-1}, \\ i \dot{D}_0 &= V^\dagger B_0 + H_2 D_0 - i D_0 \dot{U}_1 U_1^{-1}. \end{aligned} \quad (11)$$

Using the second and fourth of these we get an ‘autonomous’ equation for Z :

$$\begin{aligned}
 i\dot{Z} &= i\dot{B}_0 D_0^{-1} - iB_0 \dot{D}_0^{-1} \dot{D} D_0^{-1} \\
 &= H_1 Z + V - iB_0 \dot{U}_2 U_2^{-1} D_0^{-1} - Z(V^\dagger Z + H_2 - iD_0 \dot{U}_2 U_2^{-1} D_0^{-1}) \\
 &= V + H_1 Z - ZH_2 - ZV^\dagger Z,
 \end{aligned} \tag{12}$$

where, for brevity, the t -dependences of V, H_1, H_2, V^\dagger have been omitted.

This is a matrix Riccati equation for evolution on the coset space. The complete quantum evolution (1) involves also equations for \dot{U}_1 and \dot{U}_2 . This derivation seems simpler and more direct than others [6], and shows the appearance of the Riccati structure rather clearly, including the result of the projection.

3. Connections to the geometric phase

To connect to the geometric phase, we limit to the choices $n_1 = N - 1$ and $n_2 = 1$. So in the Hamiltonian matrix $H(t)$ of (10), H_1 is $(N - 1) \times (N - 1)$, V is an $(N - 1)$ -component column vector, and H_2 is a single real quantity. We recall here some notations:

- $\mathcal{H}^{(N)}$: complex N -dimensional Hilbert space consisting of vectors $\psi, \psi' \dots$
- \mathcal{B}_N : unit sphere in $\mathcal{H}^{(N)}$ of real dimension $(2N - 1)$ identified as a coset space $U(N)/U(N - 1) = G/H_0$, $H_0 = U(N - 1)$.
- \mathcal{R}_N : space of unit rays, of real dimension $2(N - 1)$ identified as a quotient and coset space $\mathcal{B}_N/U(1) \simeq G/H = U(N)/U(N - 1) \times U(1)$; $H = H_0 \times U(1) = U(N - 1) \times U(1)$.

We now consider four aspects:

3.1 (Local) coordinates over \mathcal{R}_N

Let us limit ourselves to that part of $\mathcal{H}^{(N)}, \mathcal{B}_N$ in which the last, N -th component of $\psi \in \mathcal{H}^{(N)}$ is non-zero. Then, for a vector $\psi \in \mathcal{B}_N$ and its image $\rho \in \mathcal{R}_N$ we can say

$$\begin{aligned}
 \psi &= \frac{1}{\gamma^{1/2}} e^{i\alpha} \begin{vmatrix} z \\ 1 \end{vmatrix}, \quad z \in \mathcal{H}^{(N-1)}, \quad \gamma = 1 + z^\dagger z; \quad \psi = e^{i\alpha} \psi_0(z), \\
 \rho &= \psi \psi^\dagger = \psi_0(z) \psi_0(z)^\dagger, \quad \psi_0(z) = \frac{1}{\gamma^{1/2}} \begin{vmatrix} z \\ 1 \end{vmatrix}.
 \end{aligned} \tag{13}$$

So $z \in \mathcal{H}^{(N-1)}$ becomes a system of coordinates for the base space $\mathcal{R}_N = \mathcal{B}_N/U(1) \simeq G/H = U(N)/U(N - 1) \times U(1)$. When useful we will later write

$$z_r = q_r + ip_r, \quad r = 1, 2, \dots, N - 1, \tag{14}$$

so q_r, p_r are $2(N - 1)$ real independent local coordinates on ray space.

3.2 Schrödinger equation on $\mathcal{H}^{(N)}$ to Riccati equation on \mathcal{R}_N

Consider the Schrödinger equation for a vector $\psi(t) \in \mathcal{H}^{(N)}$:

$$i\dot{\psi}(t) = H(t)\psi(t) = \begin{vmatrix} H_1 & V \\ V^\dagger & H_2 \end{vmatrix} \psi(t). \quad (15)$$

Separating the upper $(N - 1)$ components of ψ denoted by ξ from the N th one η :

$$\psi = \begin{vmatrix} \xi \\ \eta \end{vmatrix}, \quad (16)$$

eq. (15) may be rewritten as

$$i\dot{\xi} = H_1\xi + V\eta, \quad i\dot{\eta} = V^\dagger\xi + H_2\eta. \quad (17)$$

If we now set: $z = \xi/\eta$, in analogy to eq. (7), we get its evolution equation:

$$\begin{aligned} i\dot{z} &= i\dot{\xi}/\eta - i\xi\dot{\eta}/\eta^2 \\ &= V + H_1z - zH_2 - zV^\dagger z. \end{aligned} \quad (18)$$

This is again a Riccati equation, like (12), but now z is an $(N - 1)$ -component column vector, as is V , and H_2 is a single real variable. Now we have obtained the Riccati structure for a single solution of the Schrödinger equation, not using the entire unitary evolution operator. So the quantum-mechanical ray space evolution is given by the nonlinear Riccati equation (18).

3.3 Reformulation in classical phase space form

We now show that eq. (18) can be written in completely classical form on \mathcal{R}_N regarded as a phase space. We start from eq. (13) and define first a one-form θ_0 and then a symplectic two-form ω_0 on \mathcal{R}_N as follows:

$$\begin{aligned} \theta_0 &= -i\psi_0(z)^\dagger d\psi_0(z) \\ &= -\frac{i}{\sqrt{\gamma}} \begin{vmatrix} z^\dagger & 1 \end{vmatrix} \left\{ \gamma^{-1/2} \begin{vmatrix} dz \\ 0 \end{vmatrix} + \begin{vmatrix} z \\ 1 \end{vmatrix} d\gamma^{-1/2} \right\} \\ &= \frac{1}{\gamma} \text{Im } z^\dagger dz \\ &= \frac{1}{\gamma} (q_r dp_r - p_r dq_r) \end{aligned} \quad (19)$$

and

$$\begin{aligned} \omega_0 &= d\theta_0 = -id\psi_0(z)^\dagger \wedge d\psi_0(z) \\ &= \frac{2}{\gamma} dq_r \wedge dp_r - \frac{2}{\gamma^2} (q_r dq_r + p_r dp_r) \wedge (q_s dp_s - p_s dq_s) \\ &= 2 \begin{vmatrix} dq_r & dp_r \end{vmatrix} \wedge \begin{vmatrix} L_{rs} & M_{rs} \\ -M_{rs} & L_{rs} \end{vmatrix} \begin{vmatrix} dq_s \\ dp_s \end{vmatrix}, \end{aligned} \quad (20)$$

where

$$\begin{aligned} L_{rs} &= -L_{sr} = (q_r p_s - q_s p_r)/2\gamma^2, \\ M_{rs} &= M_{sr} = \frac{1}{2\gamma} \delta_{rs} - (q_r q_s + p_r p_s)/2\gamma^2. \end{aligned} \quad (21)$$

The two-form ω_0 is indeed non-singular, and the inverse of the $2(N-1) \times 2(N-1)$ matrix above can be computed as

$$\begin{vmatrix} \mathbf{L} & \mathbf{M} \\ -\mathbf{M} & \mathbf{L} \end{vmatrix}^{-1} = \begin{vmatrix} \mathbf{X} & \mathbf{Y} \\ -\mathbf{Y} & \mathbf{X} \end{vmatrix}, \quad (22)$$

where

$$\begin{aligned} X_{rs} &= -X_{sr} = 2\gamma(q_r p_s - q_s p_r), \\ Y_{rs} &= Y_{sr} = -2\gamma(\delta_{rs} + q_r q_s + p_r p_s), \end{aligned} \quad (23)$$

so we can read off the fundamental PB 's on ray space in $q-p$ and $z-z^*$ forms:

$$\begin{aligned} \{q_r, q_s\} &= \{p_r, p_s\} = \gamma(q_r p_s - q_s p_r)/2 = X_{rs}/4, \\ \{q_r, p_s\} &= -\gamma(\delta_{rs} + q_r q_s + p_r p_s)/2 = Y_{rs}/4, \\ \{z_r, z_s^*\} &= i\gamma(\delta_{rs} + z_r z_s^*), \quad \{z_r, z_s\} = \{z_r^*, z_s^*\} = 0. \end{aligned} \quad (24)$$

From here the following useful PB 's can be obtained:

$$\begin{aligned} \{z_r, \gamma\} &= i\gamma^2 z_r, \quad \{z_r, \gamma^{-1}\} = -iz_r, \\ \{z_r, z_s^*/\gamma\} &= i\delta_{rs}, \quad \{z_r, f(z)/\gamma\} = -iz_r f(z). \end{aligned} \quad (25)$$

Here $f(z)$ is analytic in the z_r 's. Using all this, we can show that the quantum-mechanical Riccati evolution equation (18) can be written as a purely classical Hamiltonian evolution on \mathcal{R}_N with the PB 's given in eq. (24) as

$$\begin{aligned} \dot{z} &= -i(V + H_1 z - z H_2 - z V^\dagger z) \\ &= \{z, \mathcal{H}(z, z^\dagger)\}, \end{aligned} \quad (26)$$

where

$$\begin{aligned} \mathcal{H}(z, z^\dagger) &= -(H_2 + z^\dagger V + V^\dagger z + z^\dagger H_1 z)/\gamma \\ &= -\frac{1}{\gamma} \begin{vmatrix} z^\dagger & 1 \\ H_1 & V \\ V^\dagger & H_2 \end{vmatrix} \begin{vmatrix} z \\ 1 \end{vmatrix} \\ &= -\psi_0(z)^\dagger H \psi_0(z). \end{aligned} \quad (27)$$

The motion of z in \mathcal{R}_N , induced by unitary Schrödinger evolution in $\mathcal{H}^{(N)}$, is thus a continuous classical canonical transformation.

3.4 Geometric phase and its ‘classical’ aspects

We consider Schrödinger evolution as given by eq. (15), with no appeal to the adiabatic approximation. The Riccati equation (18) for z , as we have seen, is a consequence of eq. (15), and from it we find for γ :

$$\dot{\gamma}(t) = i\gamma(t)(V^\dagger z(t) - z(t)^\dagger V), \tag{28}$$

where in general $V = V(t)$. Comparing eqs (13) and (16) gives

$$\xi = e^{i\alpha} z/\sqrt{\gamma}, \quad \eta = e^{i\alpha}/\sqrt{\gamma}. \tag{29}$$

Using these and eq. (28) in the $\dot{\eta}$ equation of motion in eq. (17) gives the evolution equation for the phase $\alpha(t)$:

$$\dot{\alpha} = -H_2 - \frac{1}{2}(V^\dagger z + z^\dagger V). \tag{30}$$

(Both eqs (28) and (30) appear in [6]). In summary, the Schrödinger equation (15) for $\psi(t)$ amounts to an autonomous Riccati equation (18) for z , re-expressed in eq. (25) in classical phase space form, and equation of motion (30) for $\alpha(t)$ where the right-hand side is α independent.

If now $\psi(t)$ is any (not necessarily cyclic) solution of the Schrödinger equation (15) between the given times t_1 and t_2 , from general theory it is known that the geometric phase is the difference of two terms, a total phase and a dynamical phase:

$$\begin{aligned} \varphi_{\text{geom}} &= \varphi_{\text{tot}} - \varphi_{\text{dyn}}, \\ \varphi_{\text{tot}} &= \arg(\psi(t_1), \psi(t_2)) \\ &= \alpha(t_2) - \alpha(t_1) + \arg(1 + z^\dagger(t_1)z(t_2)), \\ \varphi_{\text{dyn}} &= \text{Im} \int_{t_1}^{t_2} dt \psi(t)^\dagger \frac{d\psi(t)}{dt} \\ &= - \int_{t_1}^{t_2} dt \psi_0(z(t))^\dagger H(t) \psi_0(z(t)) \\ &= \int_{t_1}^{t_2} dt \mathcal{H}(z(t), z(t)^*). \end{aligned} \tag{31}$$

The first part is calculable using eq. (30) and depends only on ray space quantities; the second part is the time integral of the classical Hamiltonian along the ray space trajectory $z(t)$. Thus the geometric phase is seen to involve only ray space quantities, as it must, and is expressed as far as possible in terms of the classical Hamiltonian $\mathcal{H}(z, z^*)$ via the dynamical phase. In the special case of cyclic evolution, $z(t_1) = z(t_2)$ and the total phase simplifies to just $\alpha(t_2) - \alpha(t_1)$.

If we adopt the kinematic approach in which there is no use of the Schrödinger equation, the results are similar except for a shifting of terms. We now consider directly a (smooth) parametrized curve \mathcal{C} in \mathcal{B}_N with image C in \mathcal{R}_N :

$$\begin{aligned} \mathcal{C} &= \{\psi(s) = e^{i\alpha(s)} \psi_0(z(s)) | s_1 \leq s \leq s_2\} \subset \mathcal{B}_N, \\ C &= \{\rho(s) = \psi_0(z(s)) \psi_0(z(s))^\dagger | s_1 \leq s \leq s_2\} \subset \mathcal{R}_N. \end{aligned} \tag{32}$$

The latter is essentially a curve in the space of z :

$$C = \{z(s) | s_1 \leq s \leq s_2\}. \quad (33)$$

Then the geometric phase is again the difference as usual:

$$\begin{aligned} \varphi_{\text{geom}}[C] &= \varphi_{\text{tot}}[C] - \varphi_{\text{dyn}}[C], \\ \varphi_{\text{tot}}[C] &= \arg(\psi(s_1), \psi(s_2)) \\ &= \alpha(s_2) - \alpha(s_1) + \arg(1 + z(s_1)^\dagger z(s_2)), \\ \varphi_{\text{dyn}}[C] &= \text{Im} \int_{s_1}^{s_2} ds \psi(s)^\dagger \frac{d}{ds} \psi(s) \\ &= \text{Im} \int_{s_1}^{s_2} ds \psi(s)^\dagger \{i\dot{\alpha}(s)\psi(s) + e^{i\alpha(s)} \frac{d}{ds} \psi_0(z(s))\} \\ &= \alpha(s_2) - \alpha(s_1) + \text{Im} \int_{s_1}^{s_2} \psi_0(z(s))^\dagger d\psi_0(z(s)) \\ &\quad \text{along } C \\ &= \alpha(s_2) - \alpha(s_1) + \int_C \theta_0. \end{aligned} \quad (34)$$

Here we used eq. (19). Therefore again we have a ray space quantity:

$$\varphi_{\text{geom}}[C] = \arg(1 + z(s_1)^\dagger z(s_2)) - \int_C \theta_0. \quad (35)$$

If C is a closed loop, then $z(s_1) = z(s_2)$, and the geometric phase is a purely classical ‘symplectic area’:

$$\partial C = 0 : \varphi_{\text{geom}}[C] = - \int_C \theta_0 = - \iint_S \omega_0, \quad (36)$$

where S is any two-surface with $\partial S = C$.

4. Concluding remarks

In the present work, we have attempted to highlight the connections between the following in the quantum evolution of N -level systems:

1. Appearance of Riccati equations in as direct a way as possible to describe ‘unitary’ evolution in the coset space $\mathbb{G}/\mathbb{U}(n_1) \times \mathbb{U}(n_2)$, where $\mathbb{G} = \mathbb{U}(N)$ and $n_1 + n_2 = N$.
2. In the case $n_1 = N - 1, n_2 = 1$: to recast the Schrödinger equation for ψ in \mathcal{H}^N as a Riccati equation on the ray space \mathcal{R}_N plus an equation (30) for the overall phase α ‘driven’ by this ray-space evolution; to express the ray-space evolution as a classical Hamiltonian evolution with PB ’s (24) and Hamiltonian function $\mathcal{H}(z, z^*)$; to study the structure of the GP in both Schrödinger evolution and kinematic case, bring in the classical symplectic structure as far as possible, and show that the GP is always a ray-space quantity.

We hope the approach to Riccati equations developed in the present work, will find useful applications in problems involving ‘reductions’ as illustrated here with geometric phases as an example and will motivate the reader to look up more mathematically sophisticated and comprehensive literature that exists on the subject. It will be interesting to consider the situation in which \mathbb{G} is the unitary group associated with a composite Hilbert space of dimension $N = nm$, with the Hilbert spaces of the subsystems being of dimension n and m respectively. The subgroup \mathbb{H} will be the tensor product of $\mathbb{U}(n)$ and $\mathbb{U}(m)$, the so-called group of local transformations. In this way the reduced space will be related to the set of entangled states of the composite system and this may shed new light on the classification problem of separable versus entangled states.

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