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Horizons in 2+1-dimensional collapse of particles

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Abstract. A simple, geometrical construction is given for three-dimensional spacetimes with negative cosmological constant that contain two particles colliding head-on. Depending on parameters like particle masses and distance, the combined geometry will be that of a particle, or of a black hole. In the black hole case the horizon is calculated. It is found that the horizon typically starts at a point and spreads into a closed curve with corners, which propagate along spacelike caustics and disappear as the horizon passes the particles.

Keywords. Black holes; gravitational collapse; horizon.

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1. Introduction

Event horizons are generated by null geodesics. They are therefore natural candidates for applying a time development equation of the Raychaudhuri type. But in the present context we focus on the non-smooth regions of horizons and other features that we describe by geometrical construction, a kind of virtual use of the Raychaudhuri equation. We hope that our material is nevertheless of interest to readers of this volume.

The presence of an event horizon is the defining characteristic of a black hole. When a black hole is formed, typically by gravitational collapse, an event horizon starts at some stage of the time development. It then spreads out increasing its area until it encompasses all the dynamical features, and eventually becomes stationary. The final black hole is then one of the small family of 'hairless' types. The most active and interesting period in the life of a black hole is the time near the formation of the horizon. Because the horizon is a global property of the spacetime and cannot be characterized locally, this interesting period is typically studied only in numerically generated spacetime regions that extend over a long time.

In two space- and one time-dimensions (2+1 D) the situation is much simplified for several reasons. There is a simple kind of matter that can collapse gravitationally and form black holes, namely point particles. The geometry of spacetimes with such collapsing matter is known exactly, at least in principle. It is also known that there are no gravitational waves emitted in the collapse that may allow the final black hole state to be reached only asymptotically in time. Thus, in the collapse of 2+1 D particles, the formative stage of the horizon lasts only a finite time and is over as soon as the horizon has passed all the particles. These simplifying features in a 2+1 D spacetime of course make it somewhat unrealistic as a model for 3+1 D collapse, but one would still expect many of the lessons one learns in 2+1 dimensions to survive in some form in 3+1 dimensions.

In this paper we focus on the horizon formation when two particles in 2+1 D collapse head-on. In order to form a true black hole, the particles must have sufficient mass–energy and there must be a negative cosmological constant. We first recall the equations of motion for test particles, and then discuss a model process for the case of a vanishing cosmological constant. The changes in the horizon's behavior during the active period for negative cosmological constant are shown to be relatively minor. We also mention the collapse of a particle into a black hole and the case of more than two collapsing particles.

2. Motion of test particles in anti-de Sitter space

A 2+1-dimensional (spinless) point particle is a spacetime with an angle deficit δ about the worldline of the particle. It involves identification by a finite rotation (by the deficit angle), where the fixed point set (axis) is the particle's worldline. For particles with general locations, this construction is possible only in homogeneous and isotropic spacetimes in which case the particle's worldline is necessarily a geodesic. We can construct such a spacetime from a given background in the following way. Choose a surface (usually totally geodesic time-like) containing the particle geodesic and rotate it about that geodesic by δ, remove the wedge volume swept out, and identify its boundaries by the rotation [1]. In the case of several particles (and masses δ not so large as to make the spacetime close up), there is an outer region where the wedges extend outwards from the particles and an inner region between the particles where the background spacetime is unchanged by the cut-and-paste. Within this inner region, the particles therefore move with respect to each other like geodesic test particles. In flat space ($\Lambda = 0$) this motion is simply the usual constant velocity motion that occurs when there is no interaction between the particles. In anti-de Sitter (adS) space ($\Lambda < 0$) the motion of such geodesics is not at constant relative velocity, but can be easily derived as follows.

We can choose the geodesic of one particle as an origin. In 'Schwarzschild' coordinates [2] the AdS metric (for unit negative curvature, $\Lambda = -1$) is

$$\mathrm{d}s^2 = -(1+q^2)\mathrm{d}t^2 + \frac{\mathrm{d}q^2}{1+q^2} + q^2\mathrm{d}\Omega^2. \tag{1}$$

Here $\mathrm{d}\Omega^2$ is the metric on the unit sphere; in 2+1 dimensions we simply have $\mathrm{d}\Omega^2 = \mathrm{d}\phi^2$. For geodesics with velocity $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$ the time- and rotational symmetries of (1) imply the conserved quantities

$$E = u_t = (1+q^2)\frac{\mathrm{d}t}{\mathrm{d}\tau} \qquad \text{and} \qquad L = u_\phi = q^2\frac{\mathrm{d}\phi}{\mathrm{d}\tau},$$

where E and L are the energy and angular momentum per unit mass. In terms of these quantities the normalization condition that τ be proper time becomes

$$-\frac{E^2}{1+q^2} + \frac{1}{1+q^2}\left(\frac{\mathrm{d}q}{\mathrm{d}\tau}\right)^2 + \frac{L^2}{q^2} = -1$$

or

$$\left(\frac{\mathrm{d}q}{\mathrm{d}\tau}\right)^2 + q^2 + \frac{L^2}{q^2} = E^2 + L^2 - 1.$$

But this is the same as the 'radial' conservation of energy equation for a simple harmonic oscillator of unit mass, unit spring constant, angular momentum L and energy $E^2 + L^2 - 1$, in terms of the proper time τ. Since the angular equation $L = q^2\mathrm{d}\phi/\mathrm{d}\tau$ is also analogous to that of the 2- (or higher-) dimensional harmonic oscillator, the motion in q, ϕ coordinates, or in rectangular coordinates $x = q\cos\phi$, $y = q\sin\phi$, is that of a harmonic oscillator: x and y depend harmonically on τ, the orbits are ellipses, and all geodesics are periodic with period 2π.

This means, in particular, that two particles that will collide head-on at some instant in the future, reach a maximum distance from each other, and their spacetime is time-symmetric about this instant. Without loss of generality, we may therefore assume that the initial state of such a colliding-particle-spacetime is time-symmetric, i.e. a two-dimensional space-like surface of constant negative curvature and vanishing extrinsic curvature (totally geodesic hyperbolic 2-space H^2).

3. Minkowski space model

Although a flat spacetime does not allow horizons, there are null surfaces that share many of the properties of AdS horizons. In locally Minkowski space, the time-symmetric initial state of two particles, with zero relative velocity of the particles, corresponds to a static spacetime. We cut out two wedges, each having its edge at the location of a particle, and choose them to be symmetrically oriented with respect to the particles (figure 1a). The line joining the initial particles on the initial space-like surface (or the plane joining their two geodesics in spacetime) then divides the space into two congruent halves, and the full space is obtained by 'doubling' one of the halves, that is gluing two copies along the edges (figure 1b). This is rather like a Melitta coffee filter as it comes out of the box and before it has been spread into a 'cone'. The subsequent figures are understood to be doubled in this way. In the region outside the particles the initial surface (and all other surfaces $t = \text{const.}$) is exactly conical; we will call it outer-conical, the 2D analog of 3D asymptotically flat.

From the figure and the angle sum of the plane triangle at the bottom of figure 1b (dotted), it follows immediately that the angle deficit δ_{T} measured at large distances for the equivalent single particle is the sum of the angle deficits of the two constituent particles. So the particle masses simply add, as expected when there is no gravitational interaction energy. The location of the single particle (tip of the

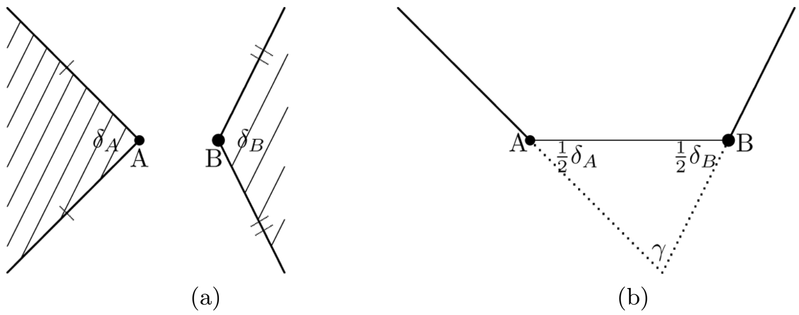
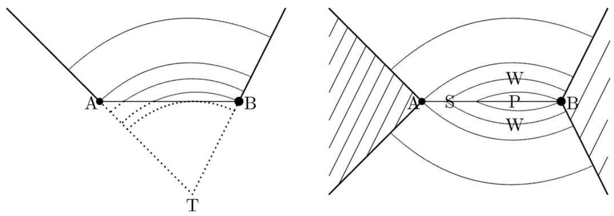


Figure 1. (**a**) A less massive and a more massive static particle, A and B, on the $t = 0$ space-like surface of three-dimensional Minkowski space. The cross-hatched wedges are removed from the space, and the boundaries with equal number of strokes are identified. The deficit angles δ_{A}, δ_{B} measure the particle masses. (**b**) Half of figure 1a is sufficient to represent the space if it is understood that it represents two layers that are glued together at the boundary – the result of folding figure a along the line AB and identifying the boundaries as indicated. The total angle deficit $\delta_{\mathrm{T}} = 2\pi - 2\gamma$.

cone) is of course outside the two-particle spacetime. Only in the limit of small masses is it approximately at the center of mass between them.

Such a spacetime, of course, cannot represent a black hole, since null geodesics from all events eventually reach infinity. However, one can consider a congruence of null geodesics, and the wave-fronts normal to them at each Minkowski time, that reach infinity in the same way that the wave-front of a black hole's horizon reaches infinity: as a smooth curve of constant curvature without self-intersection. We will call such a congruence a pseudohorizon. Although in flat space such a wavefront can be arbitrarily translated, such is not the case in a conical space. The smooth wave-front having any (sufficiently small) constant curvature in an outer-conical space is *unique*. The center of these wave-fonts is the tip of the cone if the space is truly conical. For outer-conical spaces we can construct the true cone that analytically continues the outer parts. The wave-fronts that represent the pseudohorizon at different times are then simply those parts of circles centered at the tip of this true cone that lie in the actual space. In the half-figure of the static two-particle space this tip T is easily constructed by extending the slanted borders to a point, and the circles representing the horizon are drawn about this center (figure 2a). Figure 2b shows both halves joined at the line between the particles so that successive stages of the complete pseudohorizon can be seen. It consists of constant curvature wave-front sections W that propagate at the speed of light, interrupted by points of singularities S that travel at faster-than-light speed toward the particles. New generators enter the pseudohorizon along this space-like line of singularities. As the pseudohorizon crosses a particle it becomes a smooth wave-front (note that the last wave-fronts intersect the boundary of the wedges at right angles). It is clear the the pseudohorizon always has the topology of a circle and starts at a kind of 'center of mass' P between the particles, or at one of the particles if that particle has an angle deficit of more than π.



(a) (b)

Figure 2. Construction of pseudohorizon for two static particles in flat space. (**a**) A wave-front that will be smooth and of constant curvature at late times starts at the tip T of the cone that is the continuation of the two-particle space's outer region. (**b**) Figure 2a doubled so that the full shape of the wave-fronts can be seen.

4. Two particles in anti-de Sitter space

Anti-de Sitter space, like Minkowski space, can be represented in static coordinates, eq. (1), as a time sequence of three-dimensional homogeneous spaces, which however have constant negative curvature. A point particle, centred at the origin, can be constructed as in Minkowski space by removing a time-like wedge of angle δ and identifying, for example $\phi = 0$ and $\phi = 2\pi - \delta$. If we define a rescaled angle $\varphi = (1 - \delta/2\pi)^{-1}\phi$, with the usual 2π periodicity, and a rescaled radial measure $r = (1 - \delta/2\pi)q$, the metric (1) becomes

$$\mathrm{d}s^2 = -(r^2 - m)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2 - m} + r^2\,\mathrm{d}\varphi^2, \tag{2}$$

with the parameter $m = -(1 - \delta/2\pi)^2 < 0$.

These space-like surfaces are conveniently represented by Poincaré disks, yielding what has been called the 'sausage model' of AdS spacetime. In terms of the polar coordinates ρ, ϕ of the disk (related to q by $q = \frac{2\rho}{1-\rho^2}$, $\rho \leq 1$) the spacetime metric has the 'sausage coordinate' [3] form

$$\mathrm{d}s^2 = -\left(\frac{1+\rho^2}{1-\rho^2}\right)^2 \mathrm{d}t^2 + \frac{4}{(1-\rho^2)^2}(\mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\phi^2).$$

Our Minkowski space construction involved mainly straight lines, planes, and circles. The analogous objects in AdS space are geodesics, totally geodesic surfaces, and constant curvature curves. Geodesics and constant non-zero curvature curves are represented on the Poincaré disk by circles perpendicular and oblique to the boundary of the disk, respectively. Totally geodesic surfaces intersect the Poincré disks in geodesics, and in the time direction they execute the periodic motion described in §2.

A two-particle spacetime modeled on AdS space will in general be dynamic. We also saw, that this is so even in the test particle limit. We also saw, that for particles with angular momentum $L = 0$, there is always a time-symmetric moment. We will orient our sausage model so that this moment is one of the space-like coordinate surfaces, say $t = 0$. As in Minkowski space, we cut the spacetime into

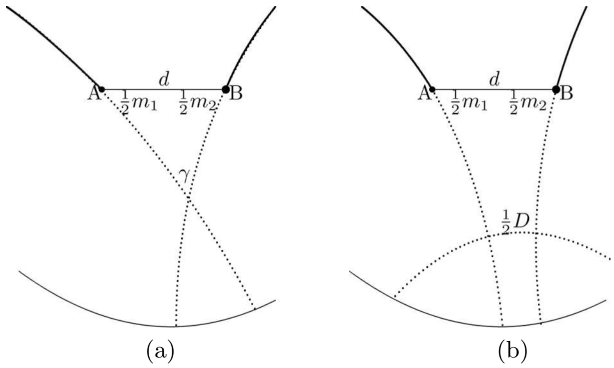


Figure 3. Two particles in anti-de Sitter space at the moment of their maximum separation d, shown on the Poincaré disk. The deficit angles are labeled by the corresponding masses. The disk is larger than the figure; only its bottom boundary is shown. (**a**) In the outer region the geometry is that of a single mass $M = 2\pi - 2\gamma$, which is more than the sum of the individual masses, corresponding to an increase in interaction energy as the masses are separated. (**b**) For larger particle masses or larger separation the outer geometry is no longer that of a single particle, but that of a black hole, characterized by its horizon circumference D.

two congruent halves along the geodesics joining the particles. The initial state of one half will look like figure 3a, the AdS analog of figure 1b, provided the particles' masses are not too large. (An advantage of the Poincaré disk representation is its conformal nature, so that angle deficits are shown faithfully. In the alternative 'Klein disk' representation, the figure at time symmetry would be indistinguishable from a Minkowski space figure, except for the disk boundary, because geodesics appear as straight lines on the Klein disk, but angles are distorted.) The equivalent single particle has no location in the two-particle spacetime, but its angle deficit can be defined from the behavior of the geometry at large distances. Hyperbolic trigonometry on the bottom triangle of figure 3a again shows that the mass M of the composite particle is given in terms of the masses m_1, m_2 of the constituents and their separation d by

$$\cos M = \cos m_1 \cos m_2 + \sin m_1 \sin m_2 \cosh d. \tag{3}$$

The presence of the factor $\cosh d$ may be viewed as the effect of the gravitational interaction energy between the particles.

The time development in terms of sausage time will show the approach of the particles to the centre of the figure and to each other, as expected from their geodesic motion in the space between them. They meet there at $t = \pi/2$, and the further time development cannot be ascertained without a law that determines the result of this collision (though in the literature some results are considered more natural than others [4]). Equation (3) in terms of angles is of course valid on each Poincaré slice of the 'sausage', but m_1 and m_2 are no longer the (constant) rest masses of the particles, containing a contribution from the kinetic energy. Thus the total mass M of the system remains unchanged as d changes. At large distances the system always looks like one static particle and so there is no horizon and no black hole formation.

It is, however, possible to choose the initial masses and the distance between them in such a way that eq. (3) cannot be fulfilled by a real M because the RHS is greater than 1. In this case the boundary geodesics that reach infinity in the figure (and which determine M by their angle of intersection, if they do intersect) are ultraparallel (figure 3b). The metric in the outer region, in terms of an angle coordinate with the usual 2π periodicity and a 'Schwarzschild' radial coordinate that gives the circumference of circles $r =$ const. the usual value $2\pi r$, takes the form (3) with a positive m. This is the BTZ metric [5] for a non-rotating black hole, and m is its mass measured asymptotically. (By contrast, the particle masses m_1 and m_2 determined by deficit angles are measured locally.) Analogous to eq. (3) we have [6]

$$\cosh D = -\cos m_1 \cos m_2 + \sin m_1 \sin m_2 \cosh d, \tag{4}$$

where $D = 2\pi m$ is the circumference of the black hole throat at $r = m$. This is twice the minimum distance between the ultraparallel boundaries in figure 2b. Thus the problem, so difficult in 3+1 dimensions, to determine whether the given initial data will lead to black hole formation, is easily solved by considering the asymptotic dependence of the initial geometry and determining whether the total BTZ mass m is positive or negative.

If the metric in the outer region is that of a black hole, the initial position of the horizon is at the throat of the single black hole described by eq. (4). Since the outer region covers only a part of this equivalent black hole, its throat will generally not be part of the initial space, but as before we can show it by a kind of analytic continuation as in figure 3b. On later time slices, which are superimposed on the initial slice in figure 4, the horizon expands and the particles fall toward each other. The horizon enters the real spacetime at a point on the geodesic between the particles and spreads out with generally two singular points on that geodesic, which become smooth when the horizon crosses the particles. Thus this behaviour is qualitatively similar to that of the Minkowski space (figure 2a). A major difference is that the size of the horizon becomes constant once it has passed all the particles.

5. Generalizations

The starting 'point' of the horizon is not just the point on the line between the particles where it first appears in sausage time. Every point on that line contributes at some time a pair of generators (one for each half space) to the horizon. Because the horizon moves faster than the speed of light along this line, the line of the horizon's origin in spacetime is space-like. In a suitably different time slicing the horizon may therefore appear simultaneously everywhere along the line, and then spread out in all directions away from it [7], or it may start at one of the masses and have a singularity at only one point, which then runs to the other mass. (In fact, this is the case in sausage time if one particle's mass is sufficiently larger than the other's.)

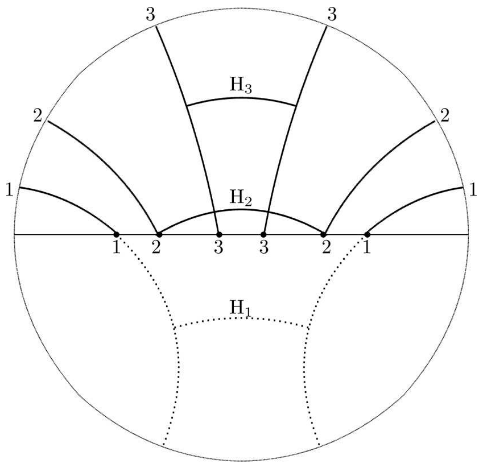


Figure 4. Three time slices in sausage coordinates of two-particle collapse and associated horizon, superimposed. The outer circle represents infinity. Only the half space is shown, the complete configuration at each time is obtained by reflection about the horizontal line and identifying the heavy curves that go to infinity. The two equal-mass particles are indicated by black dots at successive times 1, 2, 3. The initial, time-symmetric configuration is represented by the curves between by the four points labeled 1. There is no horizon initially, but the outer geometry, when continued inward without particle singularities, is shown by the dotted curves and would have a horizon at H_1. At the time labeled 2, the particles have approached each other, and the horizon H_2 has already propagated into the actual space and just reached the particles. In the third configuration, the particles are closer to collision, and the horizon H_3 surrounds the particles, propagating outward toward infinity. The horizon's circumference remains constant after the position H_2, though the Poincaré disk representation does not show its true size.

In our figures we can replace the geodesics that intersect at one or both of the particles by ultraparallel ones. Such initial states can be described as a particle falling into a black hole, or two black holes merging. For two black holes, the horizon of the final single black hole is always present on an initially time-symmetric surface [8]. For a black hole and a particle it is at least partially exposed. In the latter case the initial horizon is not that of the black hole alone, but it has a singularity and is thus prepared to become smooth by swallowing the particle. For two black holes (or the more extreme case of a black hole and a particle) the horizon's future time development is simply that of the resulting single black hole, with no trace of it having arisen from a collapse. However, in the past of the time-symmetric initial surface, the horizon consisted of two separate circular parts, each of which had singularities that moved toward each other and cancelled after they merged into a single circle.

If there are more than two particles collapsing to form a black hole, the main qualitative difference is the pattern formed by the horizon singularities. Let us follow the smooth, constant curvature horizon backwards in time from some late

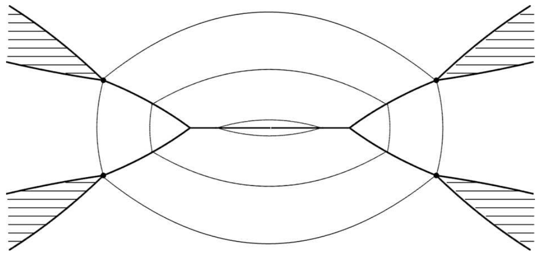


Figure 5. Development of the horizon in the collapse of four particles. The picture for each of the three times at which the horizon is shown was enlarged so that the particles appear at constant positions, but it is schematic only in certain respects. The heavy, V-shaped lines in the outer parts are to be identified as in previous figures to create the angle deficits. The inner heavy lines show the paths of the singular points. The lighter curves are stages of the horizon up to the time when it reaches the particles.

time. It contracts and remains smooth at increasing curvature until it crosses a particle. After crossing it acquires a discontinuity equal to the particle's angle deficit. Such discontinuities propagate and increase inward along geodesics from each particle. The lines of discontinuity meet in pairs and merge until a single line with two singularities propagating toward each other that eventually annihilate. Thus the branching tree pattern of the singularities tell the essential story of the horizon's development. An example for the case of four particles is shown in figure 5.

If the point particles are replaced by finite but concentrated matter distributions, the singular points of the horizon on space-like surfaces will be replaced by concentrations of high extrinsic curvature, running along a similar tree pattern.

In the case that the particles have non-zero angular momentum, they never meet and instead follow periodic orbits in the space between them. With appropriate initial conditions, they can nevertheless form a black hole, if one follows the usual custom in 2+1 D AdS geometries to regard as singular only the region of certain smooth time-like curves. This region has a gap through which the particles can pass at closest approach, and then separate again, albeit into another Universe, as is the case for the analytic extension of the Kerr geometry. A case of black holes on circular orbits has been discussed by DeDeo and Gott [9], and similar configurations have been recognized as a rotating BTZ wormhole by Holst and Matschull [10]. It will be interesting to explore the tree pattern of the horizon singularities in such cases.

References

[1] The analogous construction using a space-like geodesic and Lorentz boost is sometimes referred to as a tachyon, but it is equally appropriate to interpret it as the singularity inside a black hole, at least in spacetimes where such black holes exist

[2] Here the radial coordinate q is defined so that the circumference of a circle $q =$ const. is $2\pi q$. These coordinates are related to another common form of the AdS metric, $\mathrm{d}s^2 = -\cosh^2 r\,\mathrm{d}t^2 + \mathrm{d}r^2 + \sinh^2 r\,\mathrm{d}\Omega^2$ by the substitution $q = \sinh r$

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