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# String theory and cosmological singularities

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**Abstract.** In general relativity space-like or null singularities are common: they imply that 'time' can have a beginning or end. Well-known examples are singularities inside black holes and initial or final singularities in expanding or contracting universes. In recent times, string theory is providing new perspectives of such singularities which may lead to an understanding of these in the standard framework of time evolution in quantum mechanics. In this article, we describe some of these approaches.

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## 1. Introduction

More than fifty years ago, Raychaudhuri initiated what is arguably one of the most exciting developments in the classical theory of gravity. The culmination of this development was the formulation and proof of singularity theorems – and taught us that the theory of general relativity in fact precisely predicts the limits of its validity. These theorems showed that singularities appear under rather general conditions and led to the question: what are we supposed to do when the space-time has a singularity? While it has been clear for a long time that the very notions of space and time needs revision near these singularities where quantum effects of gravity become important, it is still not clear what structure could replace space-time at the fundamental level. In recent years, however, string theory is beginning to provide a concrete and convincing framework for a quantitative analysis near a class of such singularities.

Some kinds of singularities are easier to understand than others. Time-like singularities are regions of space which exist for all times – and very often they do not appear mysterious once one understands that the singular nature of the space-time is caused by some object sitting there. A proper microscopic description of this object would lead to a resolution of such singularities. *Space-like* or *null* singularities are much more difficult to understand. These singularities are not located at some point – one cannot look around and *see* that they are there. Rather they just *happen*. A well-known example is the singularity inside a neutral black hole

– all observers who have crossed the horizon will encounter this singularity in finite proper time. They cannot 'look ahead' and see that there is a singularity and skirt around them. Another example is the Big Bang singularity of an expanding universe. This, too, just *happened* at some time in the past.

In string theory, the basic degrees of freedom are one-dimensional extended (rather than point-like) objects in the regime where the string coupling constant is small. The very fact that strings have an extension leads to a resolution of some kinds of singularities – a classic example is an *orbifold* singularity. Because of a finite extension, strings can happily propagate on orbifold space-times, while the time evolution of particles is necessarily singular. This in fact can – and does – happen at the *classical* level. There are other time-like singularities which can be cured in string theory at the quantum level – this too is reasonable since after all string theory is a consistent quantum mechanical description of gravity.

In this article I will describe some recent attempts to understand space-like singularities. One reason why it is difficult to understand a space-like singularity is that this signifies a 'beginning' or 'end' of time. If we believe that the laws of quantum mechanics can be applied to gravity, this is a troublesome concept. Even though the notion of time in a quantum theory of gravity is rather subtle, no one has been able to make sense of a situation where time just stops. Note that this has nothing to do with the fact that such boundaries of time are often associated with the fact that space-time curvatures diverge at these points. Consider for example normal flat space-time and we simply cut out one half of it by putting in a boundary at some time $t = 0$. This is a trivial example of a geodesically incomplete space-time. The quantum dynamics of some field in this space-time is problematic – one has to impose some final state conditions, and there is no reason why the standard Hamiltonian cannot evolve the system beyond $t = 0$. The problem becomes more pronounced if in addition the space-time has curvature singularities at initial or final times.

The key to this understanding is the fact that in string theory dynamical space-time is not fundamental, but an *emergent* concept. Over the past decade we have been able to understand the underlying structure from which space-time emerges in *certain circumstances*. Surprisingly this fundamental structure is a gauge theory without dynamical gravity. And even more suprisingly it turns out that this gauge theory lives in a *lower* number of dimensions. Thus a theory without gravity (like a gauge theory) can encode all the details of a theory with dynamical gravity which however lives in higher number of dimensions – a phenomenon which is now called *holography*.

In this article I will discuss this approach to cosmological singularities which has been reviewed in [1,2]. Instead of trying to compile an exhaustive list of references, I have cited a few review articles which should be regarded as a guide to the original literature in the field. This is by no means the only approach to space-like and null singularities in string theory. Early attempts to address the question in perturbative string theory is reviewed in [3]. Attempts to study perturbative strings in exact time-dependent backgrounds and the problems encountered therein are reviewed in [4] and [2]. More recently, there has been some progress in understanding how a phase of 'tachyon condensation' can replace space-like singularities in worldsheet formulations of string theory. This is reviewed in [5].

## 2. Faces of holography

The key idea evolved from the classic work of 't Hooft in the 1970s. By mid 1970s it was clear that the correct theory for strong interactions is QCD – a non-abelian gauge theory with gauge group $SU(3)$ coupled to quarks in the fundamental representation. However the quarks are confined into mesons and baryons. It was soon realized that this happens because in QCD electric fields cannot spread out – rather they form flux tubes between quarks and antiquarks, forming mesons; or a flux tube closing onto itself forming a glueball; or three flux tubes emerging from three quarks of different colors joining at a vertex, forming a baryon. This was a satisfying picture of confinement. It qualitatively explained why experimentally hadrons appeared to behave like strings – which gave rise to string theory in the first place. This begs the question: what is the coupling constant of strings, $g_s$ in terms of QCD quantities? This should be a dimensionless parameter, and QCD does not have any free dimensionless parameter! In a landmark paper in 1974, 't Hooft argued that this dimensionless parameter can be discovered if one generalizes the gauge group in QCD from $SU(3)$ to $SU(N)$ and expand the theory in a power series expansion in $1/N$, with the combination $g_{\rm YM}^2 N$ regarded to be $O(1)$. Each term in this expansion is a sum of an infinite number of Feynman diagrams which may be thought of tiling a two-dimensional surface. If this surface has $h$ handles and $B$ boundaries the overall power of $N$ is simply $N^\chi$, where $\chi$ is the Euler characteristic $\chi = 2 - 2h - B$. This is precisely how a string perturbation theory ought to look like. A typical amplitude in string theory may be written as a sum over two-dimensional surface which represents the worldsheet of strings with each surface weighted by a factor of $g_s^{-\chi}$. Thus the string coupling constant is precisely $1/N$. In the limit $N \to \infty$ with $g_{\rm YM}^2 N$ fixed these strings are weakly coupled. A useful review of large-$N$ may be found in [6].

Following the work of 't Hooft it was in fact realized that not just gauge theories, but any theory whose fields are in the adjoint representation of a large group like $SU(N)$ should give rise to a string theory in the large-$N$ limit. However the precise details of this string theory turned out to be quite elusive.

Around the same time of 't Hooft's work, Yoneya, Scherk and Schwarz proposed that certain supersymmetric versions of string theory contain gravity and reduce to general relativity at low energies. The celebrated work of Green and Schwarz in 1984 indicated that such string theories in fact form quantum mechanically consistent ultraviolet completions of the supersymmetric generalization of general relativity, supergravity [7]. This indicated that if one can find descriptions of these string theories in terms of large-$N$ gauge theories, the latter may provide a non-perturbative description of gravity itself.

Developments which took place throughout the late 1980s and 1990s led to concrete realizations of this idea – and led to a surprise. The underlying gauge theory lives in lower number of dimensions. This is now referred to as *holography* and ties up beautifully with an idea which emerged from black hole thermodynamics.

The first concrete formulation of holography was achieved in the theory of closed strings in $1+1$ dimensions [10–12]. Here the holographic theory is the gauged quantum mechanics of a single hermitian matrix – which is a gauge theory in $0+1$ dimensions. This is the first time it was realized that *space-time is an emergent* *concept.* At the same time it became clear that space-time itself is an approximate concept which breaks down in suitable circumstances. The closed string theory contains gravity (though there is no dynamical graviton) and gravitational interactions are encoded in the gauge theory in a subtle and interesting way.

The discovery of duality symmetry (for a review see [13]) and D-branes (for an introduction see e.g. [14]) led to another realization of holography. The low energy dynamics of a stack of $N$ $p$-dimensional D-branes in string theory is a $U(N)$ gauge theory living on the $p+1$ dimensional world-volume. However these objects produce gravitational backgrounds. This indicates that gravitational phenomena in the *entire* space-time should be describable in this gauge theory. In fact the appropriate limit of the gauge theory is precisely 't Hooft's large-$N$ limit. Perhaps the most striking result of this development has been the statistical explanation of black hole thermodynamics and Hawking radiation. A certain class of black holes appear as stacks of a large number of D-branes wrapped in internal directions and appear as states in the gauge theory. In certain cases, their degeneracies can be counted reliably. This leads to a statistical entropy which is in precise agreement with the results of Bekenstein and Hawking in semiclassical gravity. Furthermore, Hawking radiation of such black holes may be understood as usual quantum mechanical decay of excited states in this gauge theory and the decay rate is in precise agreement with the semiclassical luminosity. For reviews of black holes in string theory, see [15,16].

This connection between gauge theory and gravity is in fact a manifestation of a basic property of string theory which has been known since its inception: open–closed duality. In string theory processes involving open strings can also be viewed as processes involving closed strings. Now the low energy limit of open string theory is a gauge theory, while the low energy limit of closed strings contain gravity – so there should be connection here. The dynamics of D-branes are described by open strings which live on the brane – the gauge theory in the previous paragraph is in fact the low energy limit of this theory of open strings. It has been recently realized that the gauge theory – gravity connection uncovered in $1+1$ dimensions is also a result of open–closed duality. The matrix of gauged matrix quantum mechanics in fact represents the degrees of freedom of a bunch of D-particles in this theory. In the region near horizons of certain black holes, the theory of open strings can be truncated to its low energy limit – which leads to a precise connection between this gauge theory and gravity. This connection is known as the AdS/CFT correspondence since these near-horizon geometries are asymptotically anti de-Sitter space-times and the holographic gauge theory which may be thought of as living on the boundary is some deformation of a conformally invariant field theory. A useful review of this dual correspondence is [17]. Pretty much like the two-dimensional example, an extra space dimension is generated in the gauge theory, and at the same time the theory secretly contains dynamical gravity. In the AdS/CFT correspondence, the additional dimension arises out of the *renormalization group scale* of the quantum field theory. The RG equations of the field theory are essentially Raychaudhuri equation in the bulk (for connections of Raychaudhuri equation with AdS holography, see e.g. [18,19].

The open–closed duality applied to D-branes implies a rather different relation between gauge theories and closed string theories which are called 'matrix theories'. This follows from the fact that even though critical string theories live in 10 space-time dimensions, these are actually 11-dimensional theories in disguise. This 11-dimensional theory is called M-theory. String theories appear as Kaluza–Klein reductions of M-theory on a circle, the momentum along the circle appearing as usual as a charge in the 10-dimensional theory – this charge is in fact the charge associated with D0 branes. Now suppose we consider M-theory in an infinite boosted reference frame along a compact direction. In this frame the only states which survive are those which carry a quantized positive momentum $J$ along the direction of boost. From the point of view of string theory, these are states which carry a large D0 brane charge $J$. Now, boosting along a compact spatial circle leads to a compact *null* circle. It turns out that one can go to a regime where the effective theory is the $U(J)$-gauged matrix quantum mechanics which describe the low energy limit of the theory of D0 branes. Furthermore, if additional dimensions are compact this matrix quantum mechanics effectively becomes low dimensional supersymmetric Yang–Mills theories on compact spaces. This Yang–Mills theory then describes string theory and hence gravity in 10-dimensional space-time. Like the previous two examples, space dimensions are 'manufactured' in the gauge theory. The reason why this gauge-string connection is different from the previous two examples is that here we do not need a 't Hooft limit. In fact a strong version of the matrix theory conjecture works for any finite $J$. A review of matrix theory is [20].

In these examples, a gauge theory leads to gravitational theories in higher number of *space* dimensions. The *time* was built in. Nevertheless, as we will see below, the notion of time perceived by the open strings, i.e. the gauge theory can be quite different from the notion of time perceived by closed strings which emerge out of the fundamental theory. There are in fact versions of matrix theory in which even time is manufactured – the holographic theory is simply a random matrix theory. For a review, see [21].

Except in the two-dimensional string theory example, the gauge-gravity correspondence remain conjectures rather than proven relationships. The common theme in all these realizations of holography is that the quantum dynamics of the system is unambiguously defined in the holographic gauge theory. The higher dimensional space-time and gravitational dynamics in it is always an *effective* description in a certain regime of parameters. By the same token, usual *perturbative* closed string theory would appear only in a certain corner of the parameter space. A 'proof' of the conjecture would involve a demonstration that in this corner of parameter space, the gauge theory indeed reproduces closed string perturbation theory. This is the sense in which there is such a proof for the two-dimensional model. It is confusing to ask what a 'proof' would mean at the non-perturbative level – since we do not know of an independent way to define closed strings non-perturbatively. In fact, it seems reasonable to take the open string theory as a fundamental definition of the theory itself. This theory has no dynamical space-time, no notion of geodesic incompleteness and so on – but does have a notion of time evolution of states. In a certain approximation, the theory can be re-interpreted as a theory of perturbative closed strings. In general, there is no such interpretation and therefore no notion of dynamical space-time.

This is the key point in our discussion of singularities. These holographic descriptions lead to a proposal for the structure which replaces space-time near singularities. *If there are situations where the fundamental dynamics defined in terms of the gauge theory makes sense in regions where the space-time appears singular one has the possibility that these singularities are simply problems with interpretation.*

## 3. Cosmologies in two-dimensional string theory

The first concrete relations between large-$N$ gauge theories and string theories appeared in the late 1980s in the study of toy models of strings in small number of dimensions. The best understood model of this type is the description of string theory in one space and one time dimension [10–12]. The corresponding large-$N$ theory is quantum mechanics of a single $N \times N$ hermitian matrix in a certain double scaling limit where one takes $N \to \infty$ and the coupling of the matrix quantum mechanics $G \to G_{\rm c}$ with a certain combination of $N\mu_{\rm F}$ held constant, where $\mu_{\rm F}$ is a function of $(G - G_{\rm c})$. Recently it has been realized that this should be in fact a gauge theory. In the double scaling limit the action is given by

$$S = \int {\rm d}t \, \frac{1}{2}[(D_t M)^2 + M^2] \tag{1}$$

which has a gauge symmetry with gauge group $U(N)$. The covariant derivative $D_t = \partial_t + iA_t$ involves the gauge field $A_t$. In $0+1$ dimensions a gauge field can be set to zero by a gauge transformation and this results in a constraint which implies that all physical states must be *singlets* under $U(N)$.

In the singlet sector, the entire dynamics is encoded in the eigenvalues of the matrix, $\lambda_i(t)$. However the change of variables from the matrix to the eigenvalues results into a well-known measure factor which makes the wave function into a slater determinant. The eigenvalues may be therefore thought to be locations of $N$ non-interacting fermions living in an inverted harmonic oscillator potential, with a single particle Hamiltonian given by

$$H = \frac{1}{2}[p^2 - x^2]. \tag{2}$$

Equivalently, the theory may be re-written in terms of the density of eigenvalues of the matrix $\partial_x \varphi(x,t)$ which is defined by

$$\partial_x \varphi(x,t) = \frac{1}{N} {\rm tr} \delta(M(t) - x \cdot I). \tag{3}$$

This field may be regarded as a bosonization of the fermions.

At the classical level the action of the collective field is given by

$$S = N^2 \int {\rm d}x \, {\rm d}t \, \left[ \frac{1}{2} \frac{(\partial_t \varphi)^2}{(\partial_x \varphi)} - \frac{\pi^2}{6} (\partial_x \varphi)^3 - \left( \mu - \frac{1}{2} x^2 \right) \partial_x \varphi \right]. \tag{4}$$

The dimensionful parameter $\mu$ is a Lagrange multiplier which ensures that the total number of eigenvalues is fixed.

### 3.1 *Physics near the ground state*

This is of course a field theory in $1+1$ dimension, the spatial dimension arising out of the space of eigenvalues. What is remarkable is the fact that this two-dimensional theory in fact represents a massless particle with a relativistic dispersion relation – even though there was no such relativistic invariance to begin with. It is clear from (4) that for large $N$ we can expand the field around the ground state classical solution which is

$$\partial_x \varphi_0 = \frac{1}{\pi}\sqrt{x^2 - 2\mu}\partial_t \varphi_0 = 0, \quad |x| > \sqrt{2\mu} \tag{5}$$

and vanishes for $|x| < \sqrt{2\mu}$. The quadratic part of the action for fluctuations

$$\eta(x,t) = \varphi(x,t) - \varphi_0(x) \tag{6}$$

is given by

$$S_\eta^{(2)} = \frac{1}{2}\int \mathrm{d}t\ \mathrm{d}x\ \sqrt{g} g^{\mu\nu}\partial_\mu \eta \partial_\nu \eta, \tag{7}$$

where the two-dimensional metric is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{\mathrm{d}x^2}{x^2 - 2\mu}. \tag{8}$$

The fluctuation obeys boundary conditions

$$\eta(\pm\sqrt{2\mu}, t) = 0 \tag{9}$$

so that $x = \pm\sqrt{2\mu}$ acts as a *mirror*. The classical interaction Hamiltonian is purely cubic when expressed in terms of the fluctuation field $\eta$ and its canonically conjugate momentum $\Pi_\eta$,

$$H_\eta^{(3)} = \int \mathrm{d}x \left[\frac{1}{2}\Pi_\eta^2 \partial_x \eta + \frac{\pi^2}{6}(\partial_x \eta)^3\right]. \tag{10}$$

The global nature of the semiclassical space-time which is generated is transparent if we make a change of variables to 'Minkowskian' coordinates $(\tau, \sigma)$,

$$t = \tau, \quad x = \pm\sqrt{2\mu}\cosh\sigma. \tag{11}$$

In these coordinates

$$\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \mathrm{d}\sigma^2 \tag{12}$$

and we have *two* fields $\eta_\pm(\tau, \sigma)$.

The interaction Hamiltonian in these coordinates becomes

$$H_3 = \int \mathrm{d}\sigma \frac{1}{2\sinh^2\sigma}\left[\frac{1}{2}\tilde{\Pi}_\eta^2 \partial_\sigma \eta + \frac{\pi^2}{6}(\partial_\sigma \eta)^3\right] \tag{13}$$

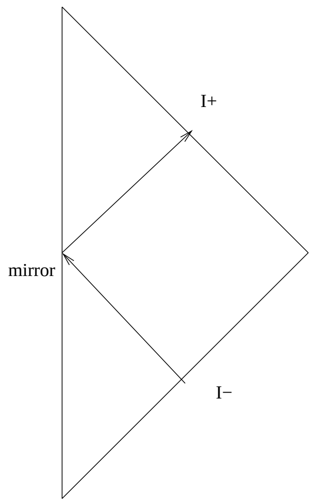


**Figure 1.** Penrose diagram of space-time produced by ground state solution showing an incoming ray getting reflected at the mirror.

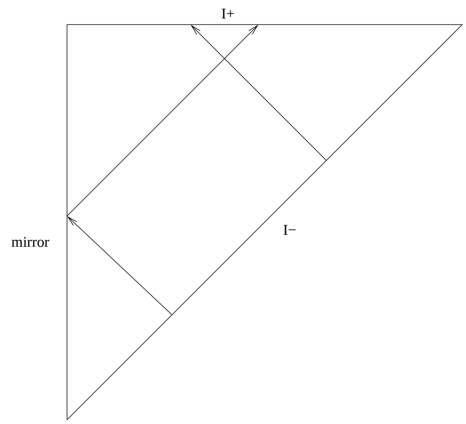


**Figure 2.** Penrose diagram for the closing hyperbola solution showing two classes of null rays.

so that the interactions are strong at the mirror $\sigma = 0$. In any case, the space-time generated is quite simple. The Penrose diagram is that of the two-dimensional Minkowski space with a mirror as shown in figure 1. The fluctuations are massless particles which come in from $\mathcal{I}^-_{\rm L,R}$, get reflected at the mirror and arrive at $\mathcal{I}^+_{\rm L,R}$. Interactions, which are strong at the mirror, lead to a non-trivial $S$-matrix.

Even though our theory looks like a usual quantum field theory in $1 + 1$ dimensions, it is actually a $1 + 1$ dimensional *closed* string theory in disguise. In $1 + 1$-dimensions there are no transverse oscillations of a string, so that the only dynamical degrees of freedom is its center of mass – which corresponds to a usual field theory. Indeed the two massless scalar fields displayed above are related to only two dynamical fields of Type 0B two-dimensional string theory. The relation is however *nonlocal at the string scale* and this has important consequences. This slight non-locality in fact encodes the gravitational attraction between objects in this model and is instrumental in showing that the $S$-matrix obtained in the matrix model is in exact agreement with the $S$-matrix obtained in a perturbative string computation.

Finally, the interpretation of the model in terms of a scalar field in 1+1 dimensions is a good interpretation only when the coupling is weak. Fortunately the couplings are weak at $\mathcal{I}^\pm$ – this allows a definition of asymptotic states and therefore a $S$-matrix. Near the mirror, the coupling is strong. The fact that the dispersion relation is relativistic at the free level has no significance in this region and it is not useful to interpret this model as a relativistic model of bosons. The fermions, however, are free everywhere – the understanding of this model in terms of fermions is exact and valid everywhere.

### 3.2 *Time-dependent solutions with singularities* [1]

The model (2) has an infinite number of global symmetries which are symmetries of the action, but do not generically commute with the Hamiltonian. The symmetry group is called $W_\infty$. If we act on the ground state with such a group element, e.g.

$$|\lambda\rangle = \exp[i\lambda Q]|\mu\rangle, \tag{14}$$

we obtain a new state of the theory with non-trivial time dependence which automatically satisfies the Schrödinger equation. However such a state would not be normalizable and would therefore not be included in the Hilbert space of the fluctuations around the ground state. Rather this would correspond to a new time-dependent background with a Hamiltonian

$$H' = \mathrm{e}^{-i\lambda Q} H \mathrm{e}^{i\lambda Q}. \tag{15}$$

At the semiclassical level some of these states are represented by classical solutions of the collective field theory (4), $\varphi_0(x,t)$. Our aim is to find out what kind of space-time is perceived by fluctuations around such a classical solution.

The strategy to do this is exactly the same as what we did for the ground state. We expand the field as in (6) and find the dynamics of $\eta(x,t)$. It turns out that at the free level, the fluctuations around any classical solution behave as massless scalars in $1+1$ dimensions with the space-time metric given by

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{(\mathrm{d}x + \frac{\partial_t\varphi_0}{\partial_x\varphi_0}\mathrm{d}t)^2}{(\pi\partial_x\varphi_0)^2}. \tag{16}$$

In $1+1$ dimensions, all metrics are conformally flat – so we can always determine the global structure of the space-time by making a coordinate transformation to Minkowskian coordinates $(\tau,\sigma)$. The latter will be called the closed string space-time.

The interesting fact is that there are several classical solutions of this type which lead to closed string space-times which appear to have *space-like boundaries*. A simple example is given by

$$\partial_x\varphi_0 = \frac{1}{\pi(1+\mathrm{e}^{2t})}\sqrt{x^2-(1+\mathrm{e}^{2t})}, \quad \partial_t\varphi_0 = -\frac{x\mathrm{e}^{2t}}{1+\mathrm{e}^{2t}}\ \partial_x\varphi_0. \tag{17}$$

Here we have rescaled $x$ and $t$ to set $2\mu = 1$ and have also shifted the origin of $t$ to absorb a parameter of the $W_\infty$ transformation. This solution starts off like the ground state at $t \to -\infty$, but the regions in $x$ space in which the fermions live is pushed out to infinity on both sides at late times. The fluctuations which correspond to small ripples on the Fermi sea, however, have a smooth time evolution. Any such fluctuation originating in the large $|x|$ region will get reflected and will be pushed by a moving mirror which will again end up in the large $|x|$ region.

The Minkowskian coordinates $(\tau,\sigma)$ in terms of which the metric is simply $\mathrm{d}s^2 = -\mathrm{d}\tau^2 + \mathrm{d}\sigma^2$ are given by the following transformations:

$$x = \cosh\sigma\sqrt{1+\mathrm{e}^{2t}}, \quad \mathrm{e}^\tau = \frac{\mathrm{e}^t}{\sqrt{1+\mathrm{e}^{2t}}}. \tag{18}$$

This immediately shows that as $-\infty < t < \infty$ the time $\tau$ has the range $-\infty < \tau < 0$. Since the dynamics of the matrix model ends at $t = \infty$ the resulting space-time appears to be geodesically incomplete with a space-like boundary at $\tau = 0$.

The Penrose diagram for this space-time is given in figure 2, which has a space-like $\mathcal{I}^+$ at $\tau = 0$, parametrized by $\sigma$. Note that the entire $\mathcal{I}^+$ is at $|x| = \infty$.

If such a space-time were encountered in the usual semiclassical gravity, one would have immediately completed this by attaching a region from $\tau = 0$ to $\tau = \infty$. However in this case this does not make sense. The fundamental dynamics of this model is given by matrix quantum mechanics which has a perfectly unitary time evolution in time $t$, and this time $t = \infty$ on $\mathcal{I}^+$. Extending the space-time beyond this would mean extending this fundamental time beyond $t = \infty$ and this does not make much sense. Recall that in this model space-time is a derived notion. What we are finding is that if we extrapolate this notion all the way to late times, we encounter a singularity which is space-like.

What is really happening in this model is that the interpretation of the model in terms of the usual relativistic two-dimensional space-time is breaking down at late times. This may be seen in several ways. Using the fact that the background is a $W_\infty$ transform of the ground state, one can exactly calculate correlation functions of fermion operators and hence the correlators of fluctuations around the ground state. The result is that near $\mathcal{I}^+$ these differ significantly from the semiclassical values, indicating $\mathcal{I}^+$ is strongly coupled. The fermion time evolution is perfectly smooth over the entire range.

There are several backgrounds of this type. They all have the common feature that while the 'open string time', i.e. the time in terms of which the quantum dynamics is defined at the fundamental level, runs over the full range, the 'closed string time' appears to end at some point. There are of course time reversed solutions where the closed string time appears to begin abruptly in a caricature of a Big Bang. And in all these instances, the space-time interpretation of the holographic model is breaking down at these space-like boundaries, even though the underlying theory makes perfect sense.

## 4. Ten-dimensional Big Bangs from matrix theory

There are similar examples in ten-dimensional string theory, which can be described in matrix theory and its descendants.

### 4.1 *Matrix string theory*

Let us briefly review the holographic description of Type IIA strings in terms of a 1 + 1-dimensional supersymmetric Yang–Mills theory (for a useful review, see [22]). This connection arises from duality transformations on the standard ten-dimensional flat background with string frame metric

$$\mathrm{d}s^2 = 2\mathrm{d}x^+\mathrm{d}x^- + \mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x}, \tag{19}$$

with the null coordinate $x^-$ compactified on a circle of radius $R$. The string coupling is $g_s$ and the string length is $l_s$. Consider the sector of the theory with a momentum

$$p_- = J/R \tag{20}$$

along $x^-$.

Now, ten-dimensional string theory has a rich set of symmetries called duality symmetries which generally relate one kind of string theory with another kind. In this case the above background may be shown to be dual to a Type IIB string theory with a string coupling $\tilde{g}_s$ and string length $\tilde{l}_s$ given by

$$\tilde{g}_s = \frac{R}{g_s l_s}, \quad \tilde{l}_s^2 = \frac{g_s l_s^3}{R}; \tag{21}$$

living on a compact space-like circle of radius $\tilde{R}$ given by

$$\tilde{R} = \frac{l_s^2}{R}; \tag{22}$$

and carrying $J$ units of D1-brane charge. The matrix theory conjecture in this case claims that a non-perturbative formulation of the theory is given by a $U(J)$ Yang–Mills theory with a dimensional coupling constant

$$g_{\mathrm{YM}} = \frac{R}{g_s l_s^2}. \tag{23}$$

The bosonic part of the action is given by

$$S = \int \mathrm{d}\tau \mathrm{d}\sigma \mathrm{tr} \left\{ \frac{1}{2} g_s^2 F_{\tau\sigma}^2 + \frac{1}{2}(D_\tau X^i)^2 - \frac{1}{2}(D_\sigma X^i)^2 + \frac{1}{4g_s^2}[X^i, X^j]^2 \right\}, \tag{24}$$

where $X^i, i = 1 \ldots 8$ are adjoint scalars and $F_{\tau\sigma}$ is the $U(J)$ gauge field strength. These fields live on a circle parametrized by $\sigma$ and $0 < \sigma < \tilde{R} = l_s^2/R$. However the space-time of this Yang–Mills theory is not dynamical.

In the regime of weak string coupling $g_s \ll 1$ the potential term in (24) suppresses $X^i$ which do not commute. Therefore in this limit the theory reduces to a theory of $8J$ scalar fields which may be chosen to be the diagonal components of the matrices $X^i$, which we denote by $X_a^i, a = 1 \ldots J$. Similarly, the gauge field strength is locally zero. However as we go around the $\sigma$ circle, the gauge symmetry allows non-trivial boundary conditions which are characterized by conjugacy classes of the gauge group. For example, we can have

$$\begin{aligned} X_1^i(\sigma + 2\pi\tilde{R}) &= X_2^i(\sigma) \\ X_2^i(\sigma + 2\pi\tilde{R}) &= X_3^i(\sigma) \\ &\cdots \\ X_J^i(\sigma + 2\pi\tilde{R}) &= X_1^i(\sigma). \end{aligned} \tag{25}$$

With these boundary conditions, the action (24) becomes the action of eight massless scalars living on a circle of radius $J\tilde{R}$. This is precisely the worldsheet description of a single Type IIA string in the light cone gauge. In a similar way one could have boundary conditions

$$\begin{aligned}
&X_1^i(\sigma + 2\pi\tilde{R}) = X_2^i(\sigma)\\
&X_2^i(\sigma + 2\pi\tilde{R}) = X_3^i(\sigma)\\
&\cdots\\
&X_K^i(\sigma + 2\pi\tilde{R}) = X_1^i(\sigma)\\
&X_{K+1}^i(\sigma + 2\pi\tilde{R}) = X_{K+2}^i(\sigma)\\
&X_{K+2}^i(\sigma + 2\pi\tilde{R}) = X_{K+3}^i(\sigma)\\
&\cdots\\
&X_J^i(\sigma + 2\pi\tilde{R}) = X_{K+1}^i(\sigma)
\end{aligned} \tag{26}$$

and one would have the worldsheet action for *two* strings. It is clear that for a given $J$ one would have various sectors of the theory which describes $J$ strings. Furthermore, the commutator interaction is capable of describing in a precise fashion the joining and splitting of these strings.

Therefore in the $g_{\rm s} \ll 1$ regime, this two-dimensional gauge theory describes a second quantized theory of closed strings in *ten space-time dimensions*, and therefore gravitational interactions in ten dimensions as well. It is important to realize that this happens only in this regime, which by virtue of (23) is the strongly coupled regime of the gauge theory. The fields $X^i$ metamorphize into transverse space-time coordinates. For generic $g_{\rm s}$ all the non-abelian excitations are important and there is no such ten-dimensional interpretation and therefore no clear interpretation in terms of a usual dynamical space-time.

### 4.2 *A matrix Big Bang*

It turns out that there is a remarkably simple modification of flat space which provides a useful model of a cosmological singularity [23]. This is a background which still has a flat string frame metric, but has in addition a dilaton which is linear in the null coordinate $x^+$,

$$\Phi = -Qx^+. \tag{27}$$

For $Q > 0$, the point $x^+ = -\infty$ is in fact a *null Big Bang singularity*. This is because the Einstein frame metric (i.e. in terms of which the low energy effective theory is the Einstein–Hilbert action) is geodesically incomplete, with geodesics reaching $x^+ = -\infty$ in a finite proper time. The effective string coupling $g_{\rm eff} = {\rm e}^{\Phi}$ is infinite at this point. In a similar way for $Q < 0$ we have a big crunch.

The logic which leads to a $1+1$-dimensional Yang–Mills theory in the $p_- = J/R$ sector now leads to a theory whose bosonic action is

$$S = \int \mathrm{d}\tau\, \mathrm{d}\sigma\, \mathrm{tr} \left\{ \frac{1}{2} g_{\mathrm{s}}^2 \mathrm{e}^{-2Q\tau} F_{\tau\sigma}^2 + \frac{1}{2}(D_\tau X^i)^2 - \frac{1}{2}(D_\sigma X^i)^2 + \frac{1}{4g_{\mathrm{s}}^2} \mathrm{e}^{2Q\tau} [X^i, X^j]^2 \right\}. \tag{28}$$

Thus the effect of the non-trivial dilaton is to make the coupling constant of the Yang–Mills theory time-dependent. It is useful to rewrite this action (28) as follows:

$$S = \int \mathrm{d}\tau\, \mathrm{d}\sigma\, \sqrt{h}\, \mathrm{tr} \left\{ \frac{1}{4} g_{\mathrm{s}}^2 h^{ab} h^{cd} F_{ac} F_{bd} + \frac{1}{2} h^{ab} (D_a X^i)(D_b X^i) + \frac{1}{4g_{\mathrm{s}}^2} [X^i, X^j]^2 \right\}, \tag{29}$$

where $h_{ab}$ denotes a two-dimensional metric

$$\mathrm{d}s_2^2 = h_{ab} \mathrm{d}\xi^a \mathrm{d}\xi^b = \mathrm{e}^{2Q\tau} [-\mathrm{d}\tau^2 + \mathrm{d}\sigma^2]. \tag{30}$$

Thus the theory may be viewed as one with *constant* couplings, but in a non-trivial space-time (30) – which is the Milne universe, or the future light cone of the origin. We would expect that in an appropriate limit the action (29) represents a string theory living in this Milne space-time with eight other transverse directions. It is clear from (30) that the space-time defined by $(\tau, \sigma)$ with $-\infty < \tau < \infty, 0 < \sigma < 2\pi\tilde{R}$ has a conical singularity at $\tau = -\infty$ and this singularity may be reached in a finite proper time. One way to see the geodesic incompleteness of the space-time is to go to the Minkowskian coordinates for the metric

$$T = \mathrm{e}^{Q\tau} \cosh\sigma, \quad X = \mathrm{e}^{Q\tau} \sinh\sigma,$$
$$\mathrm{d}s_2^2 = -\mathrm{d}T^2 + \mathrm{d}X^2. \tag{31}$$

The situation is quite similar to the two-dimensional backgrounds of the previous section. There is an *open string time* $\tau$ which runs over the full range. However the closed string time $T$ seems to begin at $T = 0$ at a null Big Bang.

From the action (28) we see that at $\tau \to \infty$, i.e. $T \to \infty$ the Yang–Mills coupling is strong. Therefore we expect that this regime would describe perturbative IIB strings in the manner discussed above. However, as $\tau \to -\infty$, i.e. at the 'Big Bang' the Yang–Mills coupling is *weak*. This means that there is nothing which suppresses $X^i$ which are non-commuting and all the $8J^2$ degrees of freedom are equally important. This means that there is no interpretation in terms of second quantized strings and no interpretation of the fields as coordinates in eight transverse space dimensions. The Yang–Mills theory, however, continues to make perfect sense. In fact the coupling of the theory goes to *zero* as we approach this 'singularity' and nothing is obviously wrong with a bunch of free fields !

What has happened is similar to the two-dimensional example. There is a certain regime of the parameters of the holographic theory where a space-time interpretation is valid. If we forcibly extrapolate this interpretation to early or late times, it appears that from the point of view of the closed string theory there is a singularity. However it is precisely in this region that it is illegal to ascribe a space-time interpretation: the holographic gauge theory is what it is and makes perfect sense.

### 4.3 *Zooming onto the Big Bang*

At the place where the closed string theory appears to perceive a Big Bang singularity the non-abelian nature of the theory becomes important. To get some idea of the nature of non-abelian excitations it is useful to consider strings moving on a time-dependent gravitational wave rather than flat space [24]. Fortunately, matrix theory may be formulated in a class of such gravitational waves whose string frame metric and dilaton are given by

$$\begin{aligned} \mathrm{d}s^2 &= 2\mathrm{d}x^+\mathrm{d}x^- - \left[\left(\frac{\mu}{3}\right)^2 \vec{x}^{\,2} + \left(\frac{\mu}{6}\right)^2 \vec{y}^{\,2}\right](\mathrm{d}x^+)^2 + \mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x}\,\mathrm{d}\vec{y}\cdot\mathrm{d}\vec{y}, \\ \Phi &= -Qx^+, \end{aligned} \tag{32}$$

where $\vec{x} = (x^1 \dots x^3), \vec{y} = (y^1 \dots y^5)$. There is, in addition, a background 5-form field strength also proportional to $\mu$. The resulting matrix theory is a deformation of the theory described in the previous subsection, additional terms involving $\mu$.

The introduction of a gravitational wave as above introduces a new length scale $1/\mu$ into the Yang–Mills theory. It turns out that the dimensionless ratio $\mu/G_{\mathrm{YM}}$ acts as a semiclassical parameter and when $\mu \gg G_{\mathrm{YM}}$ *classical* solutions of the Yang–Mills theory are relevant. Among such classical solutions are highly non-abelian configurations called *fuzzy spheres*

$$X^i = S(\tau)J^i, \quad i = 1, 2, 3, \tag{33}$$

where $S(\tau)$ is a real function of $\tau$ and $J^i$ are generators of a $N$-dimensional representation of $SU(2)$. These are called spheres since the Casimir condition implies that

$$\sum_{i=1}^{3}(X^i)^2 = \frac{N^2-1}{4} S^2(\tau) I_{N\times N} \tag{34}$$

which would represent a sphere if the $X^i$ were real numbers rather than matrices. They are called fuzzy because $X^i$ are not real numbers. Clearly, in the presence of a large number of fuzzy spheres, the matrices are far from being diagonal and the usual interpretation of the theory as a string theory in conventional ten-dimensional space-time is invalid. These fuzzy spheres are really discretized versions of D-branes.

The time dependence of the radius of the fuzzy sphere, $S(\tau)$ is determined by the dynamics of the YM theory. An examination of the action shows that at early times (near the Big Bang) such fuzzy spheres rather than strings proliferate. As time evolves, the size of these fuzzy spheres diminish and at late times they become zero size objects leaving only strings.

A Type IIB version of this model [25] exhibits another interesting aspect of the region near the singularity. It is well-known that quantum field theory in time-dependent backgrounds exhibit particle production. This results from the fact that typically the vacuum state at late times is not the vacuum at early times and vice versa. In these models, this same phenomenon leads to an interesting insight into the question of initial conditions. We saw that in this type of model one generically expects perturbative strings and hence conventional space-time at late times and non-abelian configurations at early times. The latter include D-branes which can be easily excited. The question we can now ask is the following: If we require that the state at late times contain only perturbative strings and nothing else, can we start with *any* conceivable initial state?

One might expect that the answer should be positive. After all at late times these D-branes and other non-abelian excitations are suppressed – we might expect that they all go away at sufficiently late times regardless of the initial state. Surprisingly the answer is negative. Because of particle production (or depletion) effects, the state which does not contain such D-branes at late times turns out to be a *squeezed state* of these D-branes at early times – with a thermal distribution of the number of D-branes with temperature $1/Q$. This means that if we require that at late time usual space-time emerges with no such remnant D-branes filling the vacuum – the initial state near the Big Bang must be close to this special squeezed state. It will be interesting to see whether this has some implication for the general question of initial conditions in cosmology.

## 5. Outlook

Raychaudhuri showed that a congruence of geodesics would tend to shrink and form singularities under generic conditions. These singularities clearly signal the breakdown of general relativity. It has been suspected that a proper theory of quantum gravity would 'resolve' such singularities, though it has never been clear what such a resolution might mean for space-like or null singularities.

The results from string theory are beginning to provide a concrete meaning to this by demonstrating that space-time is itself an approximate notion which emerges from more fundamental structures which form *holographic* descriptions of gravitational physics. In this article we have described how this happens in some versions of holography, viz. matrix models of two-dimensional strings and matrix theory of ten-dimensional strings. There have been some recent progress in understanding such cosmological singularities using the AdS/CFT correspondence as well [26,27]. These investigations have been in toy models of cosmology. However one would expect that we should be able to draw some general lessons which can be applied one day to realistic cosmology as well.

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