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Black hole dynamics in general relativity

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Abstract. Basic features of dynamical black holes in full, non-linear general relativity are summarized in a pedagogical fashion. Qualitative properties of the evolution of various horizons follow directly from the celebrated Raychaudhuri equation.

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1. Introduction

Ever since the mid-seventies when I first came across Professor Raychaudhuri's work, I have admired his remarkable creativity and his deep insights into the unique interplay between mathematics and physics that underlies general relativity. I met him only on a few occasions at conferences. Although these encounters were brief, like many others I was immediately struck by his simplicity, earnestness and above all his passion for science. Last year I also had the good fortune of seeing the documentary on his life made by IUCAA and Vignan Prasar. Through it I developed a better understanding of how he lived, where he taught and what inspired him. My admiration for this remarkable scientist has grown even further. There is a large body of physicists he molded both through his physics lectures and by the way he lived. It is easy to understand the unusually deep reverence they have always expressed. Personally, it is an honour and a pleasure to contribute to this volume.

Discussions of Professor Raychaudhuri's contributions to general relativity – especially of his celebrated equation – generally emphasize singularity theorems and cosmology. This is true, for example, of the excellent documentary I just mentioned. However, the Raychaudhuri equation has also had a deep impact on black hole physics. Perhaps the reason why this aspect is not as widely known is that much of the literature in this area has focused on properties of stationary black holes. But in Nature, black holes are rarely in equilibrium. They grow by swallowing stars and galactic debris as well as electromagnetic and gravitational radiation. These processes are generally modelled using perturbations of Kerr solutions. However, to understand how black holes form and grow in violent astrophysical processes including mergers, one must consider the dynamical regime of the full, non-linear general relativity. And it is here that one can see the Raychaudhuri equation in action, as it elegantly determines the qualitative features of the formation and growth of horizons in exact general relativity.

In the rest of this section, I will set the stage by briefly recalling the Raychaudhuri equation for null congruences. In §2, I will discuss event horizons and in §3, quasi-local horizons.

Fix a 4-dimensional space-time $(\mathcal{M}, g_{ab})$ and a closed, space-like 2-surface S in it. At each point of S there are exactly two future pointing null normals to it, generally denoted by ℓ^a and n^a. Consider the 3 surfaces $\mathcal{N}^{\pm}$ spanned by null geodesics tangential to ℓ^a and n^a. In a finite neighborhood of S, these null geodesics will not develop caustics. Let us restrict ourselves to this neighborhood so that $\mathcal{N}^{\pm}$ can be taken to be smooth 3-surfaces. By construction they are null. Physically, one can think of them as follows. Let us suppose S is illuminated just for an instant. Then $\mathcal{N}^{\pm}$ are spanned by the resulting light fronts. If (M, g_{ab}) were Minkowski space, for example, cross-sections of $\mathcal{N}^{+}$ would represent the outgoing, expanding light fronts and those of $\mathcal{N}^{-}$ the ingoing, contracting ones.

In what follows we return to general space-times. For definiteness, let us focus on ℓ^a and (with a slight abuse of terminology) denote the geodesic congruence obtained from it also by ℓ^a. If v denotes the affine parameter along the null geodesics tangential to ℓ^a, and if we set $v = 0$ on S, then 2-surfaces S_v defined by $v = \text{const}$ are diffeomorphic to S and represent the evolution of S along $\mathcal{N}^{+}$. We will refer to these S_v as *cross-sections* of $\mathcal{N}^{+}$. Next, let us denote by $\tilde{q}_a{}^b$ the projection operator into the tangent space of S_v. Thus $\tilde{q}_a{}^b \ell^a = 0$, $\tilde{q}_a{}^b n^a = 0$ and $\tilde{q}_a^b t^a = t^b$ for all vectors tangential to S_v. $\tilde{q}_{ab} = \tilde{q}_a{}^m g_{bm}$ is the intrinsic metric on S_v of signature $+,+$. While the discussion of this section can be readily extended to more general null congruences which may not emanate from a closed 2-surface S, this slightly restricted case makes the discussion more intuitive.

Since ℓ^a is a geodesic vector field, its acceleration $\ell^a \nabla_a \ell^b$ vanishes. Also, since ℓ^a is the null normal to a 3-surface, if the surface is specified by $f = \text{const}$, then $\ell_a \propto \nabla_a f$, whence $\nabla_{[a}\ell_{b]} = V_{[a}\ell_{b]}$ for some 1-form field V_a. Therefore

$$\omega_{ab} := \tilde{q}_a{}^m \tilde{q}_b{}^n \, \nabla_{[m}\ell_{n]} = 0. \tag{1}$$

The tensor field ω_{ab} is called the twist of ℓ^a. Geometrically, its vanishing is the necessary and sufficient condition that the null geodesic congruence ℓ^a is surface-forming. Physically, this can be interpreted as saying that the light rays represented by ℓ^a have no 'rotation'. Since $\omega_{ab} = 0$, the projection of $\nabla_a \ell_b$ into S_v is symmetric.

Our null congruence ℓ^a has no acceleration or twist. The remaining two fields constructed from its derivatives are called *expansion* and *shear*. The expansion Θ is defined as

$$\Theta = \tilde{q}^{ab} \, \nabla_a \ell_b \, . \tag{2}$$

As the name suggests it captures the rate of increase of the area of cross-sections S_v of $\mathcal{N}^{+}$. More precisely, denote by $\tilde{\epsilon}_{ab}$ the area 2-form on cross-sections S_v (defined by 2-metrics $\tilde{q}_{ab}$). Then $\mathcal{L}_{\ell} \tilde{\epsilon}_{ab} = \Theta \tilde{\epsilon}_{ab}$. Hence if at all we consider a small area element A of S_v any point p of $\mathcal{N}$ and follow it along the geodesic flow generated by ℓ^a, we have $\Theta = (1/A)(\mathrm{d}A/\mathrm{d}v)$.

While the expansion is obtained by taking the trace of the projection of $\nabla_a \ell_b$ into S, the shear is obtained by taking the trace-free part.

$$\sigma_{ab} = \tilde{q}_a{}^m \tilde{q}_b{}^n \nabla_m \ell_n - \frac{1}{2}\Theta\, \tilde{q}_{ab}. \tag{3}$$

Again as its name indicates σ_{ab} represents how the cross-sections S_v are sheared with respect to one another. The amount of shear on a cross-section S_v is a heuristic measure of the flux of gravitational radiation across it [1,2].

Since we know ℓ^a only on the 3-surface $\mathcal{N}^+$, its directional derivatives $t^a \nabla_a \ell_b$ are well-defined only when t^a is tangential to $\mathcal{N}^+$. All these are captured in the acceleration, the twist, expansion and shear. Finally, note that to calculate the twist ω_{ab}, the expansion Θ and the shear σ_{ab} of a cross-section S_v, one only needs to specify the vector field ℓ^a on that cross-section[1]. This fact will be important especially in §3. These three quantities are sometimes referred to as the *optical scalars* associated with the null congruence ℓ^a.

With these preliminaries out of the way, we can now introduce the Raychaudhuri equation for the null congruence introduced above. Having defined the optical scalars, a natural question is how they 'evolve' along the given null congruence and, in particular, which properties of space-time curvature feature in these evolution equations. For our purposes, it will suffice to restrict ourselves to the evolution of Θ. We have (see, e.g., [3,4]):

$$\frac{\mathrm{d}\Theta}{\mathrm{d}v} \equiv \ell^a \nabla_a \Theta = -\frac{1}{2}\Theta^2 - \sigma_{ab}\sigma^{ab} - R_{ab}\ell^a \ell^b\,, \tag{4}$$

where R_{ab} is the Ricci tensor of the space-time metric g_{ab}. This is the Raychaudhuri equation for our (twist-free) null congruence. Matter fields one normally uses in general relativity satisfy the null energy condition: $T_{ab}k^a k^b \geq 0$ for all null vector fields k^a. In this case, we automatically have $R_{ab}\ell^a \ell^b \geq 0$. Thus, remarkably, in physically interesting situations, the right side is negative definite. This property of null congruences has an extremely rich set of physical consequences, making Raychaudhuri equation a key element in the investigation of a variety of strong field phenomena in general relativity. In the next two sections we will see examples of this striking interplay between geometry and physics.

To conclude this section, I would like to note a simple – but surprisingly powerful – consequence of the Raychaudhuri equation. The idea is to exploit the fact that in physically interesting cases, we have

[1]For pedagogical simplicity, I introduced $\mathcal{N}^+$ using geodesics emanating from a cross-section S. If we are given only a null surface $\mathcal{N}^+$ and a specific choice of its null normal ℓ^a, the optical scalars can be defined at a point p of $\mathcal{N}^+$ by choosing just a 2-dimensional, space-like subspace of the tangent space of $\mathcal{N}^+$ at p; global cross-sections S_v are not needed.

$$\frac{\mathrm{d}\Theta}{\mathrm{d}v} + \frac{1}{2}\Theta^2 \le 0 \qquad \Leftrightarrow \qquad \frac{\mathrm{d}\Theta^{-1}}{\mathrm{d}v} \ge \frac{1}{2}. \tag{5}$$

Therefore, if the initial value $\Theta_{\rm o}$ is negative, i.e., the congruence ℓ^a is initially converging, we immediately have $\Theta(v) \le -2/[2|\Theta_{\rm o}^{-1}|-v]$. Thus, $\Theta(v)$ must diverge in an affine parameter $v \le 2/|\Theta_{\rm o}|$. This is a rather trivial mathematical consequence of (4), obtained just by solving an ordinary differential equation. However, it has a multitude of very interesting consequences.

2. Event horizons

To capture the intuitive notion that a black hole is a region from which signals cannot escape to the asymptotic part of space-time, one needs a precise definition of future infinity. The standard strategy is to use Penrose's conformal boundary $\mathcal{I}^+$ [5]. It is a future boundary: No point of the physical space-time lies to the future of any point of $\mathcal{I}^+$. It has topology $\mathbb{S}^{\not\vDash} \times \mathbb{R}$ and it is null (assuming that the cosmological constant is zero). In Minkowski space-time, one can think of $\mathcal{I}^+$ as the 'final resting place' of all future directed null geodesics. More precisely, the chronological past $I^-(\mathcal{I}^+)$ of $\mathcal{I}^+$ is the entire Minkowski space[2].

Given a general asymptotically flat space-time $(\mathcal{M}, g_{ab})$, one first finds the chronological past $I^-(\mathcal{I}^+)$ of $\mathcal{I}^+$. If it is not the entire space-time, then there is a region in $(\mathcal{M}, g_{ab})$ from which one cannot send causal signals to infinity. When this happens, one says that the space-time admits a black hole. More precisely, *black-hole region* $\mathcal{B}$ of $(\mathcal{M}, g_{ab})$ is defined as

$$\mathcal{B} = \mathcal{M} \setminus I^-(\mathcal{I}^+)\ , \tag{6}$$

where the right side is the set of points of $\mathcal{M}$ which are not in $I^-(\mathcal{I}^+)$. The boundary $\partial\mathcal{B}$ of the black hole region is called the *event horizon* and is denoted by $\mathcal{E}$ [6]. $I^-(\mathcal{I}^+)$ is often referred to as the asymptotic region and $\mathcal{E}$ is the boundary of this region within physical space-time[3]. I would like to emphasize that these notions use only asymptotic flatness and causal structure of the physical space-time. Since we have not introduced a Killing vector anywhere in the discussion, there is no restriction to stationarity. In particular, the event horizon $\mathcal{E}$ is not the set of points at which some Killing vector becomes null. Logically, it is distinct from the Killing horizon even when a Killing vector exists[4].

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[2] $I^-(\mathcal{I}^+)$ is the set of all points in the physical space-time from which there is a future directed time-like curve to a point on $\mathcal{I}^+$ in the conformally completed space-time. The term 'chronological' refers to the use of time-like curves. A curve which is everywhere time-like or null is called 'causal'.

[3] By a time reversal, i.e. by replacing $\mathcal{I}^+$ with $\mathcal{I}^-$ and I^- with I^+, one can define a white hole region $\mathcal{W}$. However, here I will consider only black holes.

[4] The only general relation between Killing vectors and event horizons is that if g_{ab} does admit a Killing vector, it must necessarily be tangential to $\mathcal{E}$. This follows from the fact that, since the event horizon is determined solely by the space-time metric, it is left invariant by every isometry of g_{ab}.

If a black hole region $\mathcal{B}$ exists, there are several *a priori* possibilities. A space-time may contain a single black hole while another may contain multiple black holes. At first glance one would also expect that one or more black holes may coalesce to form a single black hole in the distant future or a single black hole may bifurcate, giving rise to multiple black holes in the future. To obtain a precise formulation of these ideas and analyse if these possibilities are actually realised, one has to consider space-times in which there is a well-defined representation of 'instants of time'.

An appropriate notion is that of a *Cauchy surface* defined as follows: A 3-dimensional sub-manifold $\bar{M}$ of $\mathcal{M}$ is said to be achronal if no two points on it can be joined by a causal curve. A Cauchy surface $\bar{M}$ is an achronal sub-manifold of $(\mathcal{M}, g_{ab})$ such that every inextendible causal curve in $(\mathcal{M}, g_{ab})$ intersects $\bar{M}$ once and only once. A Cauchy surface represents 'an instant of time'. A space-time which admits a Cauchy surface is said to be globally hyperbolic. In such space-time, it is possible to predict the entire future and past using information specified only on the Cauchy surface $\bar{M}$. Finally, a globally hyperbolic space-time can be foliated by a family of Cauchy surfaces, each representing an instant of time.

Fix a globally hyperbolic space-time $(\mathcal{M}, g_{ab})$. Suppose it admits a black hole region $\mathcal{B}$. The intersection of $\mathcal{B}$ with a Cauchy surface $\bar{M}$ may have several disjoint components. Each then represents a separate black hole at that instant of time. With this general framework at hand, we can now ask for dynamics of black holes. How do they evolve? Can black holes merge? Can they bifurcate? If $\bar{M}'$ is a Cauchy surface to the future of $\bar{M}$, one can show that the number of disjoint components of $\bar{M}' \cap \mathcal{B}$ in the causal future of $\bar{M} \cap \mathcal{B}$ must be less than or equal to those of $\bar{M} \cap \mathcal{B}$ [3]. In this sense, then, black holes can merge but cannot bifurcate.

The next question one can ask is that of time evolution of the event horizon $\mathcal{E}$. Here, there is a striking result due to Hawking [3,6,7] which brought out an unforeseen relation between general relativity and thermodynamics. He showed that, under time evolution, the area of the event horizon cannot decrease. The horizon area is thus analogous to entropy. The Raychaudhuri equation discussed in §1 lies at the heart of this result and is thus an integral part of the foundation on which the rich literature on black hole thermodynamics rests.

Let $(\mathcal{M}, g_{ab})$ admit a black hole region B. As noted above, the event horizon $\mathcal{E}$ is a null 3-surface. Let us suppose that we can introduce a local coordinate v such that a small piece of $\mathcal{E}$ is given by $v = \text{const}$. Set $\ell_a = \nabla_a v$. Then ℓ_a is normal to $\mathcal{E}$ and, since $\mathcal{E}$ is null it is also tangential to $\mathcal{E}$. By definition, $\nabla_{[a}\ell_{b]} = 0$ and, on $\mathcal{E}$, $\ell^a \ell_a = 0$. Hence for any vector field t^a tangential to $\mathcal{E}$, we have $t^b \ell^a \nabla_a \ell_b = t^b \ell^a \nabla_b \ell_a = (1/2)\, t^b \nabla_b \ell \cdot \ell = 0$. Thus ℓ^a is a geodesic vector field. While this explicit and pedagogical argument assumes the existence of a function v with certain properties, this assumption is not necessary (and is not satisfied by the portion of $\mathcal{E}$ depicting a merger). Using general facts about causal structure one can show that $\mathcal{E}$ is a null 3-surface, ruled by future inextendible null geodesics whose expansion cannot become infinite at any point of $\mathcal{E}$ [3,4,6].

As in §1, let us denote the expansion of ℓ^a by Θ. As we saw, an immediate consequence of the Raychaudhuri equation (4) is that if Θ were to become negative at a point p of $\mathcal{N}$, it would become infinite within a finite affine parameter on the null geodesic through p. From the property of $\mathcal{E}$ just mentioned, this cannot happen if geodesics generated by ℓ^a are assumed to be complete [7]. Under this assumption then Θ must be non-negative everywhere on $\mathcal{N}$. Now suppose a cross-section S_2 of $\mathcal{E}$ is to the future of a cross-section S_1. Then, because of the relation between the expansion Θ of ℓ^a and area elements of cross-sections discussed in §1, we must have $a_{S_2} \geq a_{S_1}$. Thus, in any dynamical process the change Δa in the horizon area is always non-negative. This result is known as the *second law of black hole mechanics*. It is analogous to the second law of thermodynamics, the horizon area playing the role of entropy. Note however that the black hole is not treated as a closed system and, in the context of classical relativity considered here, area is defined without reference to any coarse graining.

This argument, originally given in [7], assumes that the geodesics ruling the event horizon are complete. This assumption can be replaced by another with direct physical interpretation [3,6]. Heuristically the new assumption is that there are no naked singularities in the asymptotic region $I^-(\mathcal{I}^+)$. More precisely, one assumes that there $\mathcal{M}$ admits a globally hyperbolic region containing $I^-(\mathcal{I}^+) \cup \mathcal{E}$. Now suppose that Θ becomes negative on some cross-section S of $\mathcal{E}$. Then, there exists a slight outward deformation S' of S which lies in the asymptotic region $I^-(\mathcal{I}^+)$ on which Θ is negative. Now one can again use the Raychaudhuri equation and properties of $I^-(\mathcal{I}^+)$ to arrive at a contradiction. Thus, Θ must be non-negative on $\mathcal{E}$. The argument of the last paragraph now implies the second law of black hole mechanics.

To summarize, the Raychaudhuri equation is a key ingredient in the derivation of the result that the area of the event horizon cannot decrease as it 'evolves to the future'.

3. Quasi-local horizons

3.1 *Limitations of the notion of an event horizon*

Event horizons and their properties have provided considerable insight into the dynamics of black holes. However, the notion has two severe limitations.

First, while the notion neatly encodes the idea that asymptotic observers cannot 'look into' a black hole, it is too global for many applications. For example, since it refers to null infinity, it cannot be used in spatially compact space-times. Clearly, one should be able to analyze black hole dynamics also in such space-times. Asymptotic flatness and the notion of $\mathcal{I}^+$ is used also in other contexts, in particular to discuss gravitational radiation in full, non-linear general relativity [5,8]. However, there $\mathcal{I}^+$ is used to facilitate the imposition of boundary condition and make notions such as '$1/r^n$-fall-off' precise. Concepts such as the 'Bondi-mass' and explicit expressions, e.g., of fluxes of gravitational energy across $\mathcal{I}^+$ are completely insensitive to geometry in the space-time interior. Situation with event horizons is quite different because they refer to the *full chronological past of* $\mathcal{I}^+$. As a consequence, by changing the geometry in a small – say Planck scale region – around the singularity, one can change the event horizon dramatically and even make it disappear [9]! And there is no reason to trust the space-time geometry provided by classical general relativity very close to the singularity. These considerations cast a long shadow on the physical meaning and relevance of event horizons.

Second, the notion is teleological; let us speak of a black hole *only after we have constructed the entire space-time.* Thus, for example, an event horizon may well be developing in the room you are now sitting *in anticipation* of a gravitational collapse that may occur in this region of our galaxy a million years from now. Clearly, when astrophysicists say that they have discovered a black hole in the center of our galaxy, they are referring to something much more concrete and quasi-local than an event horizon.

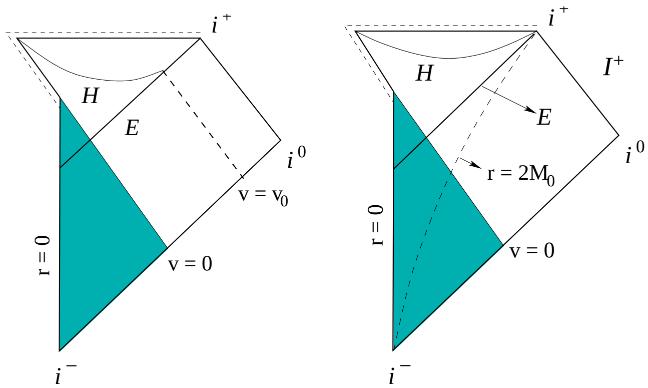
Because of these features, the notion of an event horizon is often not so useful in practice. A striking example is provided by numerical simulations of gravitational collapse leading to the formation of a black hole, or, of binary black holes which merge. Here the main task is to *construct* the space-time describing black hole dynamics. As one evolves initial data, one needs to know, at each time step, if a black hole has formed and, if so, where it is. Since space-time is the *end-product* of this procedure the event horizon cannot be located even in principle until after the simulation is complete. This teleological nature of the event horizon makes it totally unsuitable to detect black holes *during* these simulations. One needs another, quasi-local notion that can sense the presence of the black hole at each time step and home-in on its approximate location.

Finally, let us return to Hawking's area law discussed in the last section. Because of its similarity to the second law of thermodynamics, it has been extremely influential in fundamental physics. However, it is only a 'qualitative' result; it does not provide an explicit formula for the amount by which the area increases in any given physical process. One might hope that the change in area is related, in a direct manner, to the flux of matter fields and gravitational radiation falling into the black hole. Although this expectation seems natural on physical grounds, it is *impossible* to establish such a result for event horizons. For, the event horizons $\mathcal{E}$ can form and grow in a flat region of space-time in anticipation of a future collapse. *In this region, its growth occurs even though there is absolutely nothing falling across* $\mathcal{E}$. This is not just a qualitative idea but can be readily realized in general relativity. An explicit example is provided by the Vaidya solution where a spherically symmetric null fluid falls in from past null infinity $\mathcal{I}^-$ and collapses to form a black hole (see figure 1). In the distant past, before the in-fall has begun, space-time is flat and, if the in-fall lasts only for a finite time, in the distant future it is isometric to the Schwarzschild space-time. In between is the dynamical region during which the black hole forms and grows in mass. As figure 1 shows, the event horizon first forms and grows in the flat part of the space-time.

These considerations motivate the introduction and analysis of alternate, quasi-local notions to describe the non-linear dynamics of black holes. The rest of this section will be devoted to this task.

3.2 *Marginally trapped surfaces and tubes*

Since analytical solutions to Einstein's equations encoding dynamical black holes are only handful and represent idealised situations, bulk of the work in this area



(a) (b)

Figure 1. Black hole formation through the gravitational collapse of a null fluid. The figures show Penrose diagrams of Schwarzschild–Vaidya metrics for which there is no incoming radiation from $\mathcal{I}^-$ for retarded time $v \le 0$. Space-time is flat in the past of $v = 0$ (i.e., in the shaded region). In figure (**a**), the incoming flow stops at a retarded time $v = v_\mathrm{o}$. Space-time is isometric to the Schwarzschild space-time to the future of $v = v_\mathrm{o}$. The space-like dynamical horizon H and the event horizon E meet tangentially at $v = v_\mathrm{o}$. In figure (**b**) the incoming radiation continues till $v = \infty$ (but tapers off so that the total incoming energy is finite). Thus, the energy flux into the black hole mimics a more realistic collapse. The space-like dynamical horizon H, the null event horizon E and the time-like surface $r = 2M_\mathrm{o}$ (represented by the dashed line) all meet tangentially at i^+. In both figures, the event horizon originates in the shaded flat region while the dynamical horizon exists only in the curved region.

uses numerical techniques. Let me begin by sketching the standard procedure used in these simulations to get out of the quandary described in §3.1.

The first notion we need is that of marginally trapped 2-surfaces. In §1 we saw that if we instantaneously illuminate a compact, space-like 2-surface S in Minkowski space, there are two light fronts: the outgoing ones sweep out an expanding null surface $\mathcal{N}^+$ and the ingoing ones sweep out a contracting null surface $\mathcal{N}^-$. As before, let us use the convention that ℓ^a are the outgoing ones, future directed null normal to S and n^a the ingoing ones. Further, let us now normalize these vectors by demanding $\ell^a n_a = -2$. Since ℓ^a is normal to light fronts which are expanding, its expansion is positive ($\Theta_{(\ell)} > 0$) and since n^a is normal to contracting light fronts, its expansion is negative ($\Theta_{(n)} < 0$). Since light can escape from S to infinity, S is said to be *untrapped*. In Minkowski space, every 2-surface S is untrapped.

By contrast, black hole is a region from which light cannot escape. In this case, both light fronts would be ingoing, i.e. expansions of both null normals would be negative. This leads us to the first definition. A compact 2-surface S in a space-time $(\mathcal{M}, g_{ab})$ is said to be (outer) *trapped* if $\Theta_{(n)} < 0$ *and* $\Theta_{(\ell)} < 0$. The limiting case – which separates the trapped and untrapped 2-surfaces – occurs when the outgoing normal ℓ^a has zero expansion while the ingoing normal n^a has negative expansion, as usual. This brings us to the notion we set out to define. A compact

2-surface S is said to be (future, outer) *marginally trapped* if $\Theta_{(\ell)} = 0$ and $\Theta_{(n)} < 0$ on it[5].

Note that notions of trapped and marginally trapped surfaces are quasi-local. They do require us to distinguish between outgoing and incoming normals. But apart from this qualitative property, the notions refer only to geometry in the immediate neighborhood of S. They do not depend on asymptotic flatness, causal structure or, indeed, any property of the rest of the space-time. Thus, unlike the event horizon, these notions are neither global nor teleological. In particular you can be rest assured that there is no trapped or marginally trapped surface in the room you are now working in.

The idea in numerical simulations is to track the dynamics of black holes by following the evolution of marginally trapped surfaces. One generally uses the initial value formulation of Einstein's equations which can be summarized as follows. Let us consider just source-free general relativity since inclusion of sources does not add new conceptual elements. Fix a 3-manifold $\bar{M}$, topologically R^3. The initial data for general relativity consists of a pair $(\bar{q}_{ab}, \bar{K}_{ab})$ on $\bar{M}$ where $\bar{q}_{ab}$ is a metric of signature +,+,+ and $\bar{K}_{ab}$ is a symmetric tensor field. These fields have to satisfy constraints – the time-time and space-time components of Einstein's equations. In numerical simulations this is typically ensured using elliptic solvers (since the constraints are elliptic equations). The space–space part of Einstein's equations are then used to evolve these data numerically. The result is a 1-parameter family of pairs $(\bar{q}_{ab}(t), \bar{K}_{ab}(t))$. Next, consider the 4-manifold $\mathcal{M} := \bar{M} \times \mathbb{R}$ which, by construction is naturally foliated by copies of $\bar{M}$, denoted $\bar{M}_t$, each representing an 'instant of time'. The existence and uniqueness theorems for Einstein's equations imply that there is a 4-metric g_{ab} on $\mathcal{M}$ of signature −,+,+,+ such that each $\bar{M}_t$ is a Cauchy slice of $(\mathcal{M}, g_{ab})$ with $\bar{q}_{ab}(t)$ as its positive definite metric and $\bar{K}_{ab}(t)$ its extrinsic curvature.

To implement this procedure in the black hole context, one must first identify in the initial data one or several black holes. As I emphasized before, one cannot use the notion of the event horizon because its definition is teleological. *The idea is to use the quasi-local notion of marginally trapped surfaces.* But even this strategy seems difficult at first. For, as we just saw, the notion refers to the expansions $\Theta_{(\ell)}$ and $\Theta_{(n)}$ of null normals and these quantities seem to require the knowledge of the space-time metric which one does not yet have. However, it turns out possible to recast the definitions using just the Cauchy data. Let us first consider a space-time $(\mathcal{M}, g_{ab})$ with a Cauchy slice $\bar{M}$ admitting a marginally trapped surface S. Then, if $\bar{t}^a$ denotes the unit time-like normal to $\bar{M}$ and $\bar{r}^a$ the unit outward normal to S within $\bar{M}$, we can express the two null normals as $\ell^a = \bar{t}^a + \bar{r}^a$ and $n^a = \bar{t}^a - \bar{r}^a$. Then, using the definition of $\Theta_{(\ell)}$ and $\Theta_{(n)}$ and the expressions

$$\bar{K}_{ab} = \bar{q}_a{}^m \bar{q}_b{}^n \, \nabla_m \bar{t}_n \qquad \text{and} \qquad \tilde{K}_{ab} = \tilde{q}_a{}^m \tilde{q}_b{}^n \, \nabla_m \bar{r}_n \tag{7}$$

[5] I have included the parenthetical adjectives 'future' and 'outer' to facilitate comparisons with the more detailed discussions of [10–15]. They can be ignored for the rest of this article.

of the extrinsic curvature $\bar{K}_{ab}$ of $\bar{M}$ in $\mathcal{M}$ and of the extrinsic curvature $\tilde{K}_{ab}$ of S in $\bar{M}$ it follows immediately that

$$\Theta_{(\ell)} = \tilde{q}^{ab}(\bar{K}_{ab} + \tilde{K}_{ab}) = \bar{K} - \bar{r}^a\bar{r}^b\,\bar{K}_{ab} + \tilde{K}, \tag{8}$$

$$\Theta_{(n)} = \tilde{q}^{ab}(\bar{K}_{ab} - \tilde{K}_{ab}) = \bar{K} - \bar{r}^a\bar{r}^b\,\bar{K}_{ab} - \tilde{K}. \tag{9}$$

Therefore, given the Cauchy data $(\bar{M}, \bar{q}_{ab}, \bar{K}_{ab})$, it is possible to search for 2-surfaces S on which $\Theta_{(\ell)} = 0$ and $\Theta_{(n)} < 0$. In practice one restricts oneself to the physically interesting case of 2-surfaces which are topologically S^2. There may be one or more such surfaces on $\bar{M}$. Each is taken to represent a black hole at the 'initial instant of time' represented by $\bar{M}$. As explained above, evolution produces a 1-parameter family of Cauchy slices $\bar{M}_t$ and Cauchy data thereon. One then searches for marginally trapped surfaces on each $\bar{M}_t$. Efficient numerical programs – called apparent horizon finders – are available to find these surfaces (see [16] and references therein). The 3-dimensional world-tubes obtained by 'stacking' these marginally trapped 2-surfaces depict the dynamical evolutions of interest.

This brings us to the next definition. *A marginally trapped tube* M is a 3-dimensional sub-manifold (possibly with boundary) of $(\mathcal{M}, g_{ab})$ which admits a foliation by marginally trapped surfaces S. Note that this definition does not refer to a space-time foliation, i.e., does not require that M be generated by a Cauchy evolution. The marginally trapped tube is a 3-dimensional sub-manifold in its own right. This abstraction removes the excess baggage and makes the notion conceptually simpler to use [2,11,12,15] (see §3.3). However, the marginally trapped tubes that arise in numerical relativity do result from space-time foliations as explained above.

Are there any constraints on the marginally trapped tubes that a space-time can admit? These would provide insights into how black holes evolve in full general relativity. The first natural question is: What is the signature of the intrinsic metric on these tubes? Are they time-like, null or space-like? One's first impulse would be to say that they must be time-like because they are obtained by time evolution of space-like 2-surfaces S. However, this expectation is too naive. Take Minkowski space and consider the null cone C of a point p or the 'mass-shell' H based at p (i.e. spanned by all position vectors X relative to p satisfying $X \cdot X = -1$). If we slice Minkowski space by $t =$ constant planes, we obtain a foliation of C and H by 2-spheres, which can be thought of as related by time evolution. Yet, C is null and H is space-like. Similarly, marginally trapped tubes can have any signature (see figure 2).

In fact the signature is determined by a mixture of physics and geometry through the *Raychaudhuri equation*. Let M be a marginally trapped tube in $(\mathcal{M}, g_{ab})$ and denote by V^a a vector field which is tangential to M and everywhere orthogonal to its foliation by marginally trapped surfaces S and which furthermore preserves this foliation. Since V^a is orthogonal to the leaves S, there exists a function f such that $V^a = \ell^a - fn^a$. Since $\ell \cdot n = -2$ we have $V \cdot V = 4f$. Therefore M is respectively, space-like, null or time-like, depending on whether f is positive, zero or negative. Now, the definition of V^a immediately implies $\mathcal{L}_V\,\Theta_{(\ell)} = 0$, whence, $\mathcal{L}_\ell\,\Theta_{(\ell)} = f\mathcal{L}_n\,\Theta_{(\ell)}$. Therefore, the Raychaudhuri equation (4) for ℓ^a implies

$$f\,\mathcal{L}_n\,\Theta_{(\ell)} = -\tilde{q}^{ab}\tilde{q}^{mn}\sigma_{am}\sigma_{bn} - R_{ab}\ell^a\ell^b, \tag{10}$$

where, as before, σ_{ab} is the shear of ℓ^a (and ℓ^a is extended off S using the geodesic equation to evaluate $\mathcal{L}_n\Theta_\ell$). Because the right side is negative definite, this equation can be used to determine the sign of f and hence the signature of M [10].

The shear term is a measure of the energy carried by gravitational waves across M [1,2]. Similarly, through Einstein's equation, the second term on the right-hand side of (10) can be interpreted as a measure of the flux of matter energy across M [1,2]. (In both cases, 'energy' is defined with respect to ℓ^a.) Thus, f vanishes – i.e., M is null – only if there is no flux of matter or gravitational energy across M. Such a null marginally trapped tube is called *a non-expanding horizon* [12,17]. As the name suggests, in this case the area of any cross-section S of M is the same; the horizon does not expand or contract. The null normal ℓ^a to 2-spheres S is now also tangential to M. It is easy to show that the intrinsic metric on M (of signature 0,+,+) is Lie dragged by ℓ^a. Thus the intrinsic geometry of the marginally trapped tube is 'time independent', whence the tube represents a black hole *in equilibrium*. In every stationary black hole space-time the event horizon $\mathcal{E}$ is a non-expanding horizon. (Because of stationarity we are trivially spared of teleology.) However, there also exist more general examples. A class of Robinson–Trautman metrics provides an instructive illustration [18]. This space-time admits a non-expanding horizon but the space-time is dynamical and carries gravitational radiation in every of its neighborhoods, although none falls across the horizon! This example illustrates how surprising features can arise even in exact solutions. Non-expanding horizons also emerge as the future asymptotic limits in situations which are of direct physical interest. When black holes form through gravitational collapse, or pre-existing black holes merge, the marginally trapped tube asymptotes to a non-expanding horizon in the distant future [2,12]. In numerical simulations, this limit is reached rather rapidly [14] because the back-scattered radiation quickly becomes so small that it is indistinguishable from zero within numerical precision. Finally, the vanishing of right side of (10) is only a necessary condition for M to be a non-expanding horizon because so far we have allowed for the possibility that $\mathcal{L}_n\Theta_\ell$ itself may vanish. However, in numerical simulations one does not consider all mathematically viable possibilities but restricts oneself to physically interesting situations. In all simulations I am aware of the condition that has also proved to be sufficient: The marginally trapped tubes become null as soon as the right side of (10) vanishes.

If the right side of (10) is non-zero, on the other hand, M can be either time-like or space-like. Since the right side is negative definite, the sign of f is determined by whether $\mathcal{L}_n\Theta_{(\ell)} > 0$ or $\mathcal{L}_n\Theta_{(\ell)} < 0$. In the first case, an infinitesimal inward deformation along n^a of the marginally trapped surfaces S on M would produce an untrapped surface while in the second case it would produce a trapped surface. If M is time-like, it is called a *time-like membrane* (TLM). In this case M does not have the connotation of a horizon because light can travel from either of its sides to the other. If M is space-like, it is called a *dynamical horizon*. Both these possibilities are realized in numerical simulations [13,14].

The general situation is captured in a recent mathematical existence result [19]. Let $(\mathcal{M}, g_{ab})$ be a globally hyperbolic, smooth space-time on which the null energy condition holds. Let a Cauchy surface M admits a marginally trapped surface S

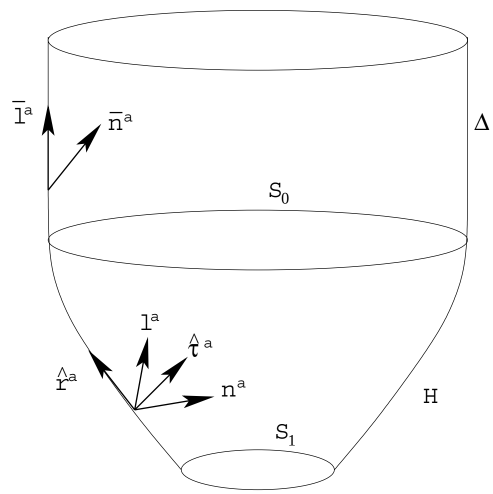


Figure 2. H is a dynamical horizon, foliated by marginally trapped surfaces S. $\hat{\tau}^a$ is the unit time-like normal to H and $\hat{r}^a$ the unit space-like normal within H to the foliations. Although H is space-like, motions along $\hat{r}^a$ can be regarded as time evolution with respect to observers at infinity. In this respect, one can think of H as a hyperboloid in Minkowski space and S as the intersection of the hyperboloid with space-like planes. In the asymptotic future H joins on to a non-expanding horizon Δ, mimicking the situation in numerical relativity.

at which the right side of (10) is non-zero. Suppose furthermore that S is stable in the sense that there exists an infinitesimal outward deformation which makes S untrapped. Then, S 'evolves' to a marginally trapped tube M which is space-like (at least) in a neighborhood of S. Note that this conclusion is arrived at assuming only 'stability' which, in practice, is substantially weaker than the condition $\mathcal{L}_n\Theta_{(\ell)} < 0$ referred to above. In particular, even when the right side of (10) is non-zero on a small portion of the 2-sphere S, M is guaranteed to be space-like 'all the way around' in a neighborhood of S (including the region where the right side may vanish).

In numerical simulations the following behavior has been observed [13,14]. Marginally trapped tubes appear to be smooth. In a gravitational collapse of a star, there are no marginally trapped surfaces in the distant past. Suppose the first marginally trapped surface appears on the Cauchy surface $M_{\rm o}$ corresponding to $t = t_{\rm o}$. Then in its immediate future, the marginally trapped tube has a cusp – or, an '$\bigcup$ shape' – each point of $\bigcup$ denoting a marginally trapped 2-sphere. As time evolves, the outer branch expands and the inner branch contracts. Both branches are space-like to begin with. The outer branch remains space-like – i.e., is a dynamical horizon – and asymptotes to a non-expanding horizon in the distant future. For black hole dynamics, this is the physically interesting component. The inner branch becomes time-like – i.e., a time-like membrane – after a while. While its time-evolution is interesting in its own right, as mentioned above, it is not of direct significance for black hole dynamics. In black hole mergers, in the asymptotic past when the black holes are very far apart there are two distinct marginally trapped tubes which are null. They become space-like as the black holes approach each other. On some Cauchy surface $M_{\rm o}$ a new marginally trapped surface forms enclosing the two distinct ones. The evolution of this common marginally trapped tube follows the same steps as the marginally trapped tube formed in a gravitational collapse, described above. In particular, the physically interesting outer branch of the tube is a dynamical horizon and it asymptotes a non-expanding horizon in the distant future.

3.3 *Dynamical horizons*

Discussion of §3.2 suggests that a quantitative handle on black hole dynamics can be obtained by examining the evolution of dynamical horizons in detail. This task is now being carried out in some detail in numerical simulations. In particular, programs are now available to monitor the change in the multipole moments [20] of the horizon geometry and analyze their approach to the asymptotic values corresponding to the Kerr horizon [14]. In this section I will conclude with an analytical result. We saw in §3.1 that while Hawking's theorem tells us that the area of the event horizon $\mathcal{E}$ cannot decrease, the teleological nature of $\mathcal{E}$ makes it impossible to relate the growth to the matter and gravitational waves falling into the black hole. Dynamical horizons on the other hand are well suited to obtain such a relation. Since we seek a *quantitative* relation, the Raychaudhuri equation will no longer suffice. Now we have to use Einstein's equations in detail.

Let us begin by fixing notation. Let $(\mathcal{M}, g_{ab})$ be a space-time satisfying the dominant energy condition. Thus, given any future directed causal vector V^a, the stress-energy T_{ab} is such that $-T^a{}_b V^b$ is also a future directed causal vector. Let $\mathcal{H}$ be a dynamical horizon, foliated by marginally trapped surfaces S. One can show that the foliation is unique [21]. Let $q_a{}^b$ and $\tilde{q}_a{}^b$ denote the projection operators on the tangent spaces to $\mathcal{H}$ and S respectively, $\hat{\tau}^a$ the unit future pointing normal to $\mathcal{H}$ and $\hat{r}^a$ the unit outward normal within $\mathcal{H}$ to the marginally trapped surfaces S. We first note an immediate consequence of the definition. Since $2\hat{r}^a = \ell^a - n^a$, $\Theta_{(\ell)} = 0$ and $\Theta_{(n)} < 0$ on each S, it follows that

$$\tilde{K} = \tilde{q}^{ab} D_a \hat{r}_b = \frac{1}{2}\, \tilde{q}^{ab} \nabla_a (\ell_b - n_b) > 0, \tag{11}$$

where D is the intrinsic derivative operator on $\mathcal{H}$, compatible with q_{ab}. Since the trace $\tilde{K}$ of the extrinsic curvature of S measures the rate of change of its area along $\hat{r}^a$, (11) implies that the area of marginally trapped surfaces increases monotonically. This is the dynamical horizon analog of Hawking's area law for event horizons. A more non-trivial task is to obtain an explicit expression for the change of area.

For this one has to use constraints – i.e., the time-time and space-time parts of Einstein's equations – on $\mathcal{H}$. Consider the portion $\Delta\mathcal{H}$ of $\mathcal{H}$ bounded by two marginally trapped surfaces S_1 and S_2. A series of rather straightforward manipulations of constraint equations leads to a surprising relation [2]:

$$\left(\frac{R_2}{2G} - \frac{R_1}{2G}\right) = \int_{\Delta H} T_{ab}\hat{\tau}^{\,a} N\ell^b \,\mathrm{d}^3V \;+\; \frac{1}{16\pi G} \times \int_{\Delta H} N \left\{\tilde{q}^{am}\tilde{q}^{bn}\,\sigma_{ab}\sigma_{mn} + 2\tilde{q}_{ab}\zeta^a\zeta^b\right\} \mathrm{d}^3V =: \mathcal{F}_{\mathrm{matt}} + \mathcal{F}_{\mathrm{gw}}. \tag{12}$$

Here $\zeta^a = \tilde{q}^{am}\hat{r}^b\nabla_b\ell_a$ is a vector field tangential to S; R is the 'area-radius', the area a_S of any marginally trapped surface being given by $a_S = 4\pi R^2$; and $N = |DR|$. Equation (12) provides an explicit, quantitative formula, relating the area change to processing happening at $\mathcal{H}$. The striking feature of (12) is that the integrand of each term on the right side is positive definite.

Let us now interpret the various terms appearing in this equation. The left side gives us the change in the horizon 'radius' caused by the dynamical process under which the horizon evolves from S_1 to S_2. The first integral on the right side of this equation is the flux $\mathcal{F}_{\mathrm{matt}}$ of matter energy (associated with the vector field $N\ell^a$) across $\Delta\mathcal{H}$. Since $N\ell^a$ is null and $\hat{r}^a$ time-like, by dominant energy condition this quantity is guaranteed to be non-negative. Since the second term is purely geometrical and emerged as the 'companion' of the matter term, it is natural to interpret it as the flux $\mathcal{F}_{\mathrm{gw}}$ of energy in the gravitational waves across $\Delta\mathcal{H}$. The presence of the shear term $|\sigma|^2$ in the flux formula (12) is expected from weak field considerations. The term $|\zeta|^2$, on the other hand, is two orders higher than the shear term in the weak field expansion [22]. Thus, it captures some genuinely non-linear, strong field physics. The interpretation of the last two, geometric terms as flux of energy carried by gravitational waves is supported by several independent considerations [2,12].

To summarize, it is remarkable that Einstein's equations provide a direct local relation between the growth of area of a dynamical horizon $\mathcal{H}$ and the flux of in-falling matter and gravitational waves. Furthermore, as a by-product a gauge invariant expression emerged for the energy-flux of gravitational waves across $\mathcal{H}$. As at null infinity, it is local to the cross-section and positive definite. This is surprising because in general relativity no such expressions exists for general 3-surfaces in the strong field regime. Indeed, had $\mathcal{H}$ been replaced by a 3-manifold which is foliated by either untrapped *or* trapped 2-surfaces, the procedure would fail to provide a positive definite expression. The fact that a satisfactory expression exists precisely when we try to calculate the energy-flux falling into a black hole is another illustration of the extraordinary interplay between geometry and physics contained in general relativity.

4. Discussion

Although a systematic mathematical theory of black holes was introduced over thirty years ago, in the ensuing decades most of the attention was focused on stationary black holes, specifically on properties of the Kerr family and perturbations thereof. In the fully dynamical regime, we only had qualitative results involving event horizons, most of them dating back to the seventies. While they spurred much activity in the area of black hole thermodynamics and associated fundamental physics, they have not been directly useful in unravelling the quantitative physics of black hole formation and growth. Furthermore, because of the global and teleological nature of event horizons these results have had limited practical applications, e.g., to numerical simulations.

Over the last three years this status-quo has begun to change. In particular, new properties of marginally trapped surfaces and tubes have been discovered, motivating the introduction of quasi-local horizons. These in turn are now providing detailed insights into the non-linear processes dictating black hole dynamics. As with event horizons, the Raychaudhuri equation plays a key role in determining the qualitative features of marginally trapped tubes. However, one can finally go beyond and use the detailed form of Einstein's equations to track, in quantitative detail, how black holes form and grow. This endeavour continues Professor Raychaudhuri's tradition of uncovering and exploiting new facets of the extraordinary interplay between geometry and physics that is hidden in general relativity.

Over the last three years, rapid progress could occur largely because of a synergistic interchange between mathematical and numerical relativity communities. If this interplay continues, the subject of dynamical black holes is likely to enrich both areas. By uncovering new structures and their properties, these efforts could further refine and clarify the intuitive scenarios that have emerged over the past three decades. More importantly, they could continue to open new avenues, leading to the discovery of unforeseen features and forcing us to revise older scenarios in important ways.

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