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On the Raychaudhuri equation

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Abstract. The Raychaudhuri equation is central to the understanding of gravitational attraction in astrophysics and cosmology, and in particular underlies the famous singularity theorems of general relativity theory. This paper reviews the derivation of the equation, and its significance in cosmology.

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1. Introduction

Amalkumar Raychaudhuri's remarkable paper [1] for the first time gave a general derivation of the fundamental equation of gravitational attraction for pressure-free matter, showing the repulsive nature of a positive cosmological constant, and underlying the basic singularity theorem (see below). He used special coordinates in the process, but wrote the result in a transparent covariant way. What was remarkable was how he derived this equation on his own, after reading writings of K Gödel on the ideas of shear and vorticity in cosmology (he defines the shear (eq. (8) in [1]) without fully explaining its meaning, apparently being unaware of Heckmann's writing on the topic, e.g. [2]). When combined with the other fundamental equation of gravity, the energy density conservation equation, it can be integrated to give the Friedmann equation (when the shear and vorticity vanish) and various generalisations of that equation (when the shear and vorticity have simple behaviour, e.g. in the case of Bianchi I models).

This equation was extended to arbitrary matter in an important paper by Ehlers [3], which included acceleration of the matter world lines and arbitrary stress tensors. This established the key result that the active gravitational mass density of a continuous medium is $\mu_{\rm grav} \equiv \mu + 3p/c^2$. Particular versions of this equation had been obtained earlier, at centers of symmetry by Tolman and Synge (see Raychaudhuri's footnote 12 of ref. [1]) and in the case of static space-time by Whittaker [4]. It was obtained for cosmology by Eddington [5] who used it to prove the instability of the Einstein static universe – a key result. It also underlies the estimates of ages in cosmological models obeying the energy conditions.

Its generalisation to the case of null geodesics (the null Raychaudhuri equation) plays a key role in geometrical optics in a curved space-time, as explored by Sachs, Ehlers, Penrose and others (see the summary in [6]). The combination of the time-like and null versions of the equation then played a key role in many singularity theorems – simple ones applicable in the case of Friedmann universes and inhomogeneous or anisotropic pressure-free irrotational models [3] and finally the famous Hawking–Penrose theorems showing the existence of singularities under rather generic conditions [6,7]. When completed with the full set of 1+3 covariant equations, it plays a central role in the investigations of the growth of inhomogeneities in perturbed cosmological models. These further developments will not be considered here. Rather, I focus on the direct application to time-like curves in cosmological and astrophysical contexts.

2. The equation

In fluid flows and cosmology, there is a preferred 4-velocity vector field $u^a : u^a u_a = -1$ that represents the average motion of matter [3,8]. Let τ be the proper time along these world lines: $u^a = \mathrm{d}x^a/\mathrm{d}\tau$. The time derivative of any tensor $T^{ab\cdot\cdot}_{\cdot\cdot cd}$ along the fluid flow lines is

$$\dot{T}^{ab\cdots}_{\cdots cd} = T^{ab\cdots}_{\cdots cd;e} u^e.$$

A particular application of time differentiation is the derivative of the 4-velocity itself in its own direction. This determines the acceleration vector

$$\dot{u}^a \;=\; u^a{}_{;b} u^b \;\Rightarrow\; \dot{u}^a u_a = 0, \tag{1}$$

which vanishes if and only if the flow lines are geodesics. It is convenient to define the representative length $\ell(x^i)$ such that comoving volumes scale like ℓ^3. The expansion $\Theta = u^a_{;a}$ gives the rate of change of volume d^3V:

$$\Theta = 3\dot{\ell}/\ell = (\mathrm{d}^3V)\dot{}/(\mathrm{d}^3V). \tag{2}$$

The shear and vorticity magnitudes are ω^2 and σ^2. Spatial gradients orthogonal to the preferred world lines are determined by the derivative operator $\nabla_a f := (g^b_a + u_a u^b) f_{,b}$.

The Einstein field equations (EFE) are

$$R_{ij} - \frac{1}{2} R g_{ij} + \Lambda g_{ij} = \kappa T_{ij}, \tag{3}$$

where Λ is the cosmological constant and κ is the gravitational constant. The Raychaudhuri equation, giving the evolution of Θ along the fluid flow lines [1,3,8], is obtained by contracting the equivalent form

$$R_{ij} = \kappa \left\{ T_{ij} - \frac{1}{2} T g_{ij} \right\} + \Lambda g_{ij}, \tag{4}$$

with $(u^a u^b)$ and using the Ricci identity for u^a. Thus one has

Raychaudhuri equation

$$\dot{\Theta} + \frac{1}{3}\Theta^2 + 2(\sigma^2 - \omega^2) - \dot{u}^a{}_{;a} + \frac{1}{2}\kappa(\mu + 3p/c^2) - \Lambda = 0. \tag{5}$$

This is the generic form of the Raychaudhuri equation (eq. (11) in [1] is for the case $p = 0$). It is the fundamental equation of gravitational attraction. To see its implications, we rewrite it in the form

$$3\frac{\ddot{\ell}}{\ell} = -2(\sigma^2 - \omega^2) + \dot{u}^a{}_{;a} - \frac{1}{2}\kappa(\mu + 3p/c^2) + \Lambda \tag{6}$$

which follows from the definition of the scale factor ℓ. This equation for the curvature $\ddot{\ell}$ of the curve $\ell(\tau)$ directly shows that shear, energy density and pressure tend to make matter collapse, as they tend to make the $\ell(\tau)$ curve bend down; while vorticity and a positive cosmological constant tend to make matter expand, as they tend to make the $\ell(\tau)$ curve bend up. The acceleration term is of indefinite sign. The equation shows that $\mu_{\rm grav} = \mu + 3p/c^2$ is the active gravitational mass density of a fluid. Hence any increase in the internal energy or the pressure increases its active gravitational mass density. The corresponding equation in Newtonian theory is the same except that $(\mu + 3p/c^2) \to \rho$ (the active gravitational mass density is just the mass density) [8].

3. Applications

3.1 *Static star models*

In the case of a static star, $\Theta = \omega = \sigma = 0$ and we can neglect the cosmological constant on this scale. So the equation reduces to $\dot{u}^a{}_{;a} = \frac{1}{2}\kappa(\mu + 3p/c^2)$, where the acceleration is determined from the pressure gradient by the momentum conservation equation. In the case of a perfect fluid,

$$(\mu + p/c^2)\dot{u}^a + \nabla^a p = 0.$$

So we obtain from (5)

$$((\nabla^a p)/(\mu + p/c^2))_{;a} = -\frac{1}{2}\kappa(\mu + 3p/c^2),$$

which is the basic balance equation between gravitational attraction and hydrostatic pressure for a static star. In the corresponding Newtonian equations, $\mu + 3p/c^2 \to \rho$ and $\mu + p/c^2 \to \rho$. It is these differences which cause gravitational collapse to be so much less severe in Newtonian theory than in general relativity.

3.2 *Static FLRW universe models*

In the case of a static Friedman–Lemaître–Robertsom–Walker (FLRW) universe, $\Theta = \omega = \sigma = 0 = \dot{u}^a$. So eq. (5) becomes

$$\frac{1}{2}\kappa(\mu + 3p/c^2) = \Lambda. \tag{7}$$

Thus *static universes with ordinary matter are only possible if* $\Lambda > 0$ [9] (cf. discussion after eq. (14) in [1]). Then, given the equation of state $p = p(\mu)$ and the cosmological constant, there is a unique radius ℓ_s for the static solution, at which the gravitational attraction caused by matter and the repulsion caused by the cosmological constant balance.

However, *this universe is unstable* [5] because if we change ℓ to a larger value, $\ell > \ell_s$, μ decreases but Λ stays constant. So $\ddot{\ell} > 0$ and the universe expands to infinity. Similarly, $\ell < \ell_s \Rightarrow \ddot{\ell} < 0$ and the universe collapses. This instability [10] leads us to believe that *the universe should either be expanding or contracting, but not static*. Indeed, the failure to perceive this in the 1920s can be regarded as one of the major lost opportunities in the history of cosmology (all the major figures in cosmology at that time believed that universe was static).

In the Newtonian case, the corresponding static model satisfies $\kappa\rho/2 = \Lambda$. The same qualitative results hold then as in general relativity theory (i.e. $\Lambda > 0$ and the model is unstable).

3.3 *Friedmann–Lemaître models*

In the FLRW case, the Raychaudhuri equation (6) becomes (cf. eq. (12) in [1]), for the case $p = 0$

$$3\ddot{\ell}/\ell = -\frac{1}{2}\kappa(\mu + 3p/c^2) + \Lambda. \tag{8}$$

Now the energy conservation equation for a perfect fluid is

$$\dot{\mu} + (\mu + p/c^2)\Theta = 0 \tag{9}$$

and this implies that $(\ell^2\mu)^{\cdot} = -\ell\dot{\ell}(\mu+3p/c^2)$. Thus, provided $\dot{\ell} \neq 0$, we can multiply (8) by $\ell\dot{\ell}$ and integrate to find

$$3(\dot{\ell})^2 - \kappa\mu\ell^2 - \Lambda\ell^2 = \text{const.} \tag{10}$$

This is just the *Friedmann equation* which governs the time evolution of FLRW universe models.

4. The basic singularity theorem

The fundamental singularity theorem follows immediately from the Raychaudhuri equation [1,3,8].

Theorem. *Irrotational geodesic singularities. If* $\Lambda \leq 0$, $\mu + 3p/c^2 \geq 0$ *and* $\mu + p/c^2 > 0$ *in a fluid flow for which* $\dot{u} = 0$, $\omega = 0$ *and* $H_0 > 0$ *at some time* s_0, *then a space-time singularity, where either* $\ell(\tau) \to 0$ *or* $\sigma \to \infty$, *occurs at a finite proper time* $\tau_0 \leq 1/H_0$ *before* s_0.

Proof. Consider the curve of ℓ against proper time τ. If $\ddot{\ell} = 0$, then $\ell \to 0$ a time $1/H_0$ ago. However, with the given conditions, $\ddot{\ell} < 0$. So, following the curve $\ell(\tau)$

back in the past, it must drop below this straight line and reach arbitrarily small positive values of ℓ at a time less than $1/H_0$ ago (unless some other space-time singularity intervenes before $\ell \to 0$, which can happen only if the shear diverges first).

In the exceptional case where the shear diverges first, a conformal space-time singularity will occur where $\ell \neq 0$. In the general case where $\ell \to 0$, the matter world lines converge together a finite time ago in the past and a space-time singularity will develop if $\mu + p/c^2 > 0$, for then as the universe contracts, the density and pressure increase indefinitely, implying the space-time curvature does so also. For ordinary matter this will additionally imply that $T \to \infty$, that is, the universe originates at a hot Big Bang. Furthermore, an age problem also becomes possible; for if we observe structures in the universe, such as stars, globular clusters, or galaxies, that are older than $1/H_0$, there is a contradiction with the assumptions of the theorem (for the universe must be older than its contents!). Furthermore, the presence of shear will shorten the lifetime of the universe (for a given value of the Hubble constant today) (cf. discussion after eq. (16) in [1]).

Similarly the argument implies that the universe must experience a very quick evolution through its hot early phase. For example at the time of decoupling the scale function ℓ_d is small: $\ell_d/\ell_0 \approx 1/1000$, which implies the Hubble parameter H_d at that time is $H_d > 1000H_0$ (the straight line estimate $\ddot{\ell} = 0 \Rightarrow H \propto 1/\ell \Rightarrow H_d \approx 1000H_0$; however the high densities will cause a considerable steepening of the $\ell(\tau)$ curve by those times, leading to the inequality). Similarly at the time of nucleosynthesis $\ell/\ell_0 \approx 10^{-8}$, showing that $H > 10^8 H_0$.

Application:

This result applies in particular to an expanding Friedmann–Lemaître universe, where $\ell \to 0$ and a hot Big Bang must occur (the shear is zero in this case, so a conformal singularity cannot occur). The proof makes it clear that *an increase in pressure does not resist the occurrence of the singularity*, but rather decreases the age of the universe and so makes the age problem worse (the pressure increases the active gravitational mass and there are no pressure gradients to resist the collapse).

This is the basic singularity theorem, on which further elaborations are built. How can one avoid the singularity? It is clear that shear anisotropy makes the situation worse. On the face of it, there are five possible routes to avoid the conclusion: a positive cosmological constant; acceleration; vorticity; an energy condition violation or alternative gravitational equations. We consider them in turn.

(a) *Cosmological constant* $\Lambda > 0$. In principle, this could dominate the matter and turn the universe around. However in practice this cannot happen because we have seen galaxies and quasars up to a red-shift of about 5, implying that the universe has expanded by at least a ratio of 5 to 6 to the present time. This means that if it bounced, then at the time of the turnaround, the density would have been greater than the present density by a factor of at least $5^3 = 125$. So the cosmological constant would have to be equivalent to a larger energy density, in order to dominate the Raychaudhuri equation then. We would certainly have detected so large a cosmological constant today, and have not done so, as can be deduced from present estimates of its magnitude. Further, if we accept that the microwave background radiation indicates that the universe has expanded by at least a factor of 1000, the argument is overwhelming; the cosmological constant would have to be equivalent to more than 10^9 times the present matter density to dominate the Raychaudhuri equation then! There is no way we could have avoided detecting this.

(b) *Pressure inhomogeneity* (acceleration) and

(c) *Rotational anisotropy* (the effect of 'centrifugal force'). Both of these involve abandoning the FLRW geometry. On the face of it, they could succeed. However, the powerful Hawking–Penrose singularity theorems [6,7] strongly restrict the allowable cases where they might in fact succeed, because of the microwave background radiation observations which show, for universes which are approximately Robertson–Walker, that the conditions of those more general theorems hold [6].

(d) *Violation of the energy condition*. The above result depended on the energy condition

$$\mu + 3p/c^2 \geq 0$$

which is obeyed by all normal matter. However, the false vacuum equation $\mu + p/c^2 = 0$ violates this, and can in principle cause a turnaround of the universe, avoiding an initial singularity. Nevertheless, we do not expect this equation to become relevant until temperatures of at least 10^{12} K. Thus even if violating the energy condition could enable us to avoid the initial singularity, the turnaround would only take place under extraordinarily extreme conditions when quantum effects are dominant. Hence we can rephrase the conclusion: a viable non-singular universe model cannot obey the laws of classical physics at all times in the past.

(e) *Other gravitational field equations*. Finally, we have of course assumed Einstein's field equations here. Alternative theories of gravity will certainly allow singularity violation, effectively by introducing negative energy terms into the Raychaudhuri equation. In particular, at very early times quantum gravitational effects will become important, almost certainly causing effective energy condition violations.

Thus the prediction of a singularity is a classical prediction. Physically, we may take it as the prediction that, as we follow the evolution back into the past, the universe cannot avoid entering the quantum-gravity domain. We do not yet have any reliable idea of what this implies.

In Newtonian theory, the discussion is as above except for one important point: then rotation *can* enable the universe to avoid the initial singularity (unlike in general relativity). This is shown by the existence of spatially homogeneous rotating and expanding but shear-free Newtonian universe models, in which the rotation spins up to enable the universe to avoid the initial singularity, whereas such universes cannot exist in relativity theory [11].

5. Evaluation today

We obtain very useful information by evaluating the Raychaudhuri equation at the present time. To do this, we define some useful parameters as follows. The *deceleration parameter* is

$$q_0 = -\left(\frac{\ddot{\ell}}{\ell}\right)_0 \frac{1}{H_0^2} \tag{11}$$

which is a dimensionless version of the second derivative $\ddot{\ell}$, with the sign chosen so that a positive value corresponds to deceleration. The total energy density is represented by the dimensionless *density parameter*

$$\Omega_0 = \frac{\kappa \mu_0}{3H_0^2} \ . \tag{12}$$

Similarly pressure and the cosmological constant are represented by

$$\Omega_{\mathrm{p}} = \frac{\kappa p_0}{3H_0^2 c^2}, \quad \Omega_\Lambda = \frac{\Lambda_0}{3H_0^2}.$$

It then follows directly from the Raychaudhuri equation that

$$2q_0 = \frac{4}{3}\left(\frac{\sigma_0^2}{H_0^2} - \frac{\omega_0^2}{H_0^2}\right) - \frac{2\dot{u}^a_{;a}}{3H_0^2} + \Omega_0 + 3\Omega_{\mathrm{p}} - \Omega_\Lambda. \tag{13}$$

If the rotation, shear, acceleration and pressure terms are small today compared with the others, as is highly plausible, then

$$2q_0 \simeq \Omega_0 + \Omega_\Lambda, \tag{14}$$

where the error is of the order of the magnitude of the terms neglected in passing from the previous equation. This becomes an exact equation in an FLRW universe with vanishing pressure. If $\Lambda = 0$, then this reduces to

$$2q_0 \simeq \Omega_0, \tag{15}$$

which again is exact in an FLRW universe with vanishing pressure, but this is no longer believed to be true. The Newtonian discussion is the same, except that there is no pressure contribution to (13).

Equation (14) is a direct relation between the deceleration and density parameters which is pivotal in observational cosmology. Dynamical estimates and lensing observations suggest $\Omega_{\mathrm{darkmatter}} \simeq 0.3$ with $\Omega_{\mathrm{baryons}} \simeq 0.04$. Supernovae observations of q_0, with this equation, then indicate that $\Omega_\Lambda \simeq 0.7$. This is the famed dark energy; the fact that its physical nature is unknown is a core problem for present day cosmology.

6. Conclusion

The Raychaudhuri equation is central to cosmology, as is made clear by all the applications discussed above. I have not discussed here its higher dimensional versions, which are of course just as important in higher dimensional gravity theories. Raychaudhuri's paper [1] contains other interesting results not discussed here (see §4), and was entitled "Relativistic cosmology, I". It has always been a matter of regret to me that Paper II was never written.

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