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Bifurcation methods of dynamical systems for handling nonlinear wave equations

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Abstract. By using the bifurcation theory and methods of dynamical systems to construct the exact travelling wave solutions for nonlinear wave equations, some new soliton solutions, kink (anti-kink) solutions and periodic solutions with double period are obtained.

Keywords. Bifurcation; periodic solution; soliton solution; kink solution.

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1. Introduction

This paper concerns the following well-known nonlinear soliton equations [1]:

(i) KdV equation: $u_t + auu_x + bu_{xxx} = 0.$ (1)

(ii) Boussinesq equation: $u_{tt} - au_{xx} + 3(u^2)_{xx} - bu_{xxxx} = 0.$ (2)

(iii) RLW equation: $u_t + au_x - 6uu_x - bu_{txx} = 0.$ (3)

(iv) Modified KdV equation: $u_t + au^2u_x + bu_{xxx} = 0.$ (4)

(v) Phi-four equation: $u_{tt} - u_{xx} - u + u^3 = 0.$ (5)

In [1], Wazwaz obtained several soliton solutions and periodic solutions of the above equations by using the sine–cosine method. Here we establish exact travelling wave solutions of the above models by using the methods of dynamical systems [2]. In different regions of the parametric space, we obtain not only all their bounded exact solutions which contain the results in ref. [1], but also the dynamical behavior of each solution.

To find the travelling wave solutions of the above equations, we make the transformation

$$u(x,t) = u(\xi), \quad \xi = x - ct, \tag{6}$$

where c is a wave speed.

(i) Substituting (6) into (1) yields

$$-cu_\xi + auu_\xi + bu_{\xi\xi\xi} = 0. \tag{7}$$

Integrating (7) once, we have

$$bu_{\xi\xi} + \frac{a}{2}u^2 - cu + d_1 = 0, \tag{8}$$

where d_1 is an integration constant. We assume that $b \neq 0$, otherwise we only get the trivial solutions of eq. (8). Thus eq. (8) is equivalent to the Hamiltonian system

$$\frac{\mathrm{d}u}{\mathrm{d}\xi} = y, \quad \frac{\mathrm{d}y}{\mathrm{d}\xi} = -\alpha u^2 - \beta u - \gamma, \tag{9}$$

with the Hamiltonian

$$H(u,y) = \frac{1}{2}y^2 + \frac{\alpha}{3}u^3 + \frac{\beta}{2}u^2 + \gamma u = h, \tag{10}$$

where $\alpha = \frac{a}{2b}$, $\beta = -\frac{c}{b}$, $\gamma = \frac{d_1}{b}$ and h is the Hamiltonian constant.

(ii) Using (6) we transform (4) into

$$-cu_\xi + au^2u_\xi + bu_{\xi\xi\xi} = 0. \tag{11}$$

Integrating (11) once and letting the constant of integration to be zero, we find that

$$bu_{\xi\xi} + \frac{a}{3}u^3 - cu = 0 \tag{12}$$

which is equivalent to the Hamiltonian system

$$\frac{\mathrm{d}u}{\mathrm{d}\xi} = y, \quad \frac{\mathrm{d}y}{\mathrm{d}\xi} = \mu u^3 + \nu u, \tag{13}$$

with the Hamiltonian

$$H^*(u,y) = \frac{1}{2}y^2 - \frac{\mu}{4}u^4 - \frac{\nu}{2}u^2 = h^*, \tag{14}$$

where $\mu = -\frac{a}{3b}$, $\nu = \frac{c}{b}$ and h^* is the Hamiltonian constant.

Since we can similarly drop the travelling wave equations of eqs (2), (3) and (5) which have the same forms as eqs (8) and (12), we omit the processes and only consider the exact solutions of eqs (1) and (4) (i.e., (9) and (13)).

From the theory of dynamical systems [3,4], the smooth travelling wave solutions of eq. (1) (or (4)) are given by smooth orbits of eq. (9) (or (13)): solitary wave solutions correspond to homoclinic orbits at a single equilibrium point; periodic waves come from periodic orbits; while heteroclinic orbits connecting two equilibrium points yield kink (or anti-kink) solutions. Thus in order to construct all solitary waves, kink (or anti-kink) waves and periodic waves of eq. (1) (or (4)), we need to find all periodic annuli, homoclinic and heteroclinic orbits of eq. (9) (or (13)) depending on the systemic parameters. The bifurcation methods of dynamical systems play an important role in our study.

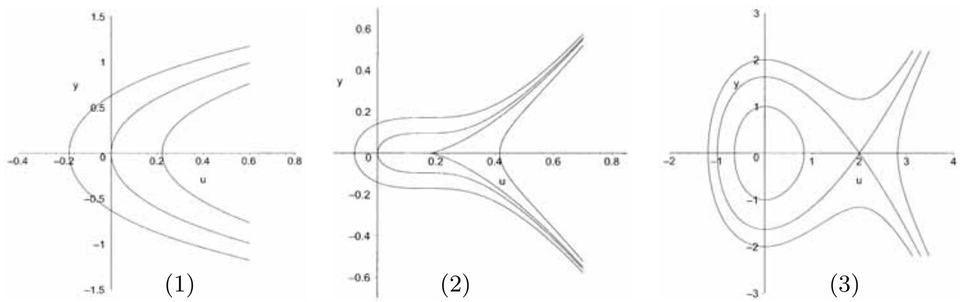


Figure 1. The phase portraits of eq. (9) when $\alpha < 0$: (**1**) $(\beta,\gamma) \in B_1$; (**2**) $(\beta,\gamma) \in L$; (**3**) $(\beta,\gamma) \in B_2$.

2. Exact bounded solutions of eq. (1)

In this section, we first consider the bifurcations of phase orbits of eq. (9) in its parameter space. The invariance of eq. (9) under the transformation $u \to -u,\ y \to -y,\ \alpha \to -\alpha$ enables us to consider the case $\alpha<0$ only. Obviously, the abscissas of equilibrium points of eq. (9) are the real roots of $f(u)=\alpha u^2+\beta u+\gamma$. For a fixed $\alpha<0$, there is a unique bifurcation curve L: $\gamma=\beta^2/(4\alpha)$ which divides the (β,γ)-parametric plane into two subregions:

$$B_1=\left\{(\beta,\gamma)\colon \gamma<\frac{\beta^2}{4\alpha},\quad \beta\in\mathbb{R}\right\};\quad B_2=\left\{(\beta,\gamma)\colon \gamma>\frac{\beta^2}{4\alpha},\quad \beta\in\mathbb{R}\right\}.$$

Equation (9) has no equilibrium if $(\beta,\gamma)\in B_1$ and has a unique equilibrium at $(-\frac{\beta}{2\alpha},0)$ if $(\beta,\gamma)\in L$; if $(\beta,\gamma)\in B_2$, eq. (9) has two equilibrium points at $(z_\pm,0)$ where $z_\pm=\frac{-\beta\pm\sqrt{\Delta}}{2\alpha}$, $\Delta=\beta^2-4\alpha\gamma$. The phase portraits of eq. (9) are shown in figure 1.

Then we consider the exact solutions of eq. (1) (i.e., (9)) defined by figure 1. Since we are considering the physical model where only bounded solutions are meaningful, we only study the case $(\beta,\gamma)\in B_2$ (i.e., $\Delta>0$). Denote $h_\pm = H(z_\pm,0)=\frac{\beta^3-6\alpha\beta\gamma\mp\sqrt{\Delta^3}}{12\alpha^2}$. By using the Hamiltonian (10) and the Jacobian elliptic functions [5], we have the following results.

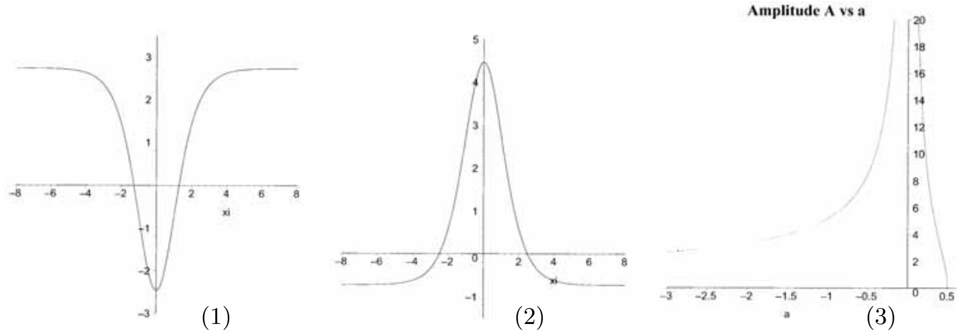
(1) For $h=h_-$ in (10), eq. (1) has a dark soliton solution (see figure 1(3))

$$u_1(\xi)=\frac{\beta+\sqrt{\Delta}-3\sqrt{\Delta}\,\mathrm{sech}^2(\frac{1}{2}\sqrt[4]{\Delta}\,\xi)}{2|\alpha|}. \tag{15}$$

(2) For $h\in(h_+,h_-)$ in (10), eq. (1) has a family of periodic solutions

$$u_2(\xi)=u_{11}+(u_{12}-u_{11})\,\mathrm{sn}^2\left(\frac{\sqrt{|\alpha|(u_{13}-u_{11})}}{\sqrt{6}}\,\xi,k_1\right) \tag{16}$$

with period $T=\frac{4\sqrt{6}\,K(k_1)}{\sqrt{|\alpha|(u_{13}-u_{11})}}$, where $K(k_1)$ is a complete elliptic integral with the



(1) (2) (3)

Figure 2. (**1**) Dark soliton of eq. (1) for a=1, b=−1, d_1=−1, c=1; (**2**) Bright soliton of eq. (1) for a=1, b=1, d_1=−1, c=1; (**3**) Plot of amplitude A vs a for d_1=1, c=1.

modulus k_1, $k_1 = \sqrt{\frac{u_{12}-u_{11}}{u_{13}-u_{11}}}$ and $u_{11}<u_{12}<u_{13}$ are the three real roots of the equation

$$-\frac{\alpha}{3}u^3 - \frac{\beta}{2}u^2 - \gamma u + h = 0. \tag{17}$$

Remark 2.1. Suppose that $\alpha>0$ and $\Delta>0$. Then eq. (1) has a bright soliton solution

$$u_3(\xi) = \frac{-\beta - \sqrt{\Delta} + 3\sqrt{\Delta}\,\mathrm{sech}^2(\frac{1}{2}\sqrt[4]{\Delta}\,\xi)}{2|\alpha|} \tag{18}$$

for $h=h_-$ and has a family of periodic solutions

$$u_4(\xi) = u_{23} - (u_{23} - u_{22})\,\mathrm{sn}^2\left(\sqrt{\alpha(u_{23}-u_{21})/6}\,\xi, k_2\right) \tag{19}$$

for $h\in(h_+, h_-)$, where $k_2=\sqrt{\frac{u_{23}-u_{22}}{u_{23}-u_{21}}}$ and $u_{21}<u_{22}<u_{23}$ are the three real roots of eq. (17).

Remark 2.2. The soliton solutions $u_1(\xi)$ and $u_3(\xi)$ have the same amplitude $A = \frac{3\sqrt{\Delta}}{2|\alpha|} = \frac{3\sqrt{c^2-2ad_1}}{|a|}$ which means that $c^2 > 2ad_1$ and both A and c are independent of b.

Figure 2 shows the difference between the bright and dark soliton solutions and the relation between the amplitude A and the systematic parameter a.

Remark 2.3.

(a) If $h\to h_-^-$, then $u_{11}\to\frac{2\sqrt{\Delta}-\beta}{2\alpha}$, $u_{12}\to z_-$, $u_{13}\to z_-$, $k_1\to 1$, $T\to\infty$, therefore $\mathrm{sn}(x,k_1)\to\tanh(x)$, $u_2(\xi)\to u_1(\xi)$. Similarly, $u_4(\xi)\to u_3(\xi)$ as $h\to h_-^-$.
(b) It can be shown from (15) and (18) that the balance between the weak nonlinear term uu_x and the dispersion term u_{xxx} gives rise to the soliton solutions $u_1(\xi)$ and $u_3(\xi)$.

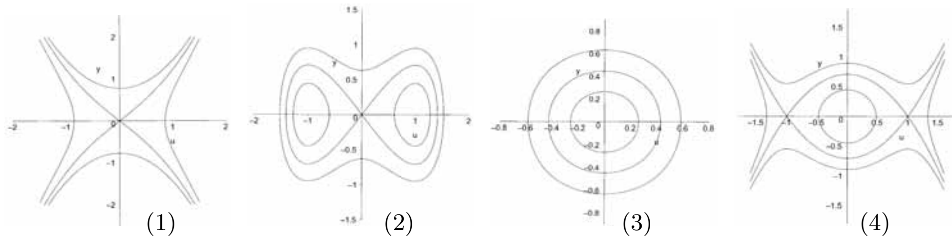


Figure 3. The phase portraits of eq. (13): (**1**) $(\mu,\nu) \in L_1^+ \cup C_1 \cup L_2^+$; (**2**) $(\mu,\nu) \in C_2$; (**3**) $(\mu,\nu) \in L_1^- \cup C_3 \cup L_2^-$; (**4**) $(\mu,\nu) \in C_4$.

3. Exact bounded solutions of eq. (4)

In this section, we first consider the bifurcations of phase orbits of eq. (13). There are two bifurcation curves $L_1^{\pm}$: $\nu = 0,\ \mu > 0(< 0)$ and $L_2^{\pm}$: $\mu = 0,\ \nu > 0(< 0)$ which divide the (μ,ν)-parametric plane into four subregions:

$$C_1 = \{(\mu,\nu)\colon \mu > 0, \nu > 0\};\quad C_2 = \{(\mu,\nu)\colon \mu < 0, \nu > 0\};$$
$$C_3 = \{(\mu,\nu)\colon \mu < 0, \nu < 0\};\quad C_4 = \{(\mu,\nu)\colon \mu > 0, \nu < 0\}.$$

Equation (13) has a unique equilibrium point at $O(0,0)$ for $(\mu,\nu) \in C_1 \cup C_3$ and has three equilibrium points at O and $(\pm\sqrt{-\nu/\mu}, 0)$ for $(\mu,\nu) \in C_2 \cup C_4$. The phase portraits of eq. (13) are shown in figure 3.

Then we construct the exact solutions of eq. (4) (i.e., (13)) defined by figure 3. Denote that $h_1^* = H^*(\pm\sqrt{-\nu/\mu}, 0) = \nu^2/(4\mu)$, $k_3 = \frac{1}{k_4} = \sqrt{\frac{2\Omega}{\nu+\Omega}}$, $k_5 = \frac{-\nu-\Omega}{2\sqrt{\mu h}}$ and $\Omega = \sqrt{\nu^2 - 4\mu h}$. Using the Hamiltonian (14) we have the following results:

(1) Suppose that $(\mu,\nu) \in C_2$.

(1.1) For $h^* = 0$ in (14), eq. (4) has a dark soliton and a bright soliton

$$u_1^*(\xi) = \pm\sqrt{-2\nu/\mu}\,\mathrm{sech}(\sqrt{\nu}\,\xi). \tag{20}$$

(1.2) For $h^* \in (h_1^*, 0)$ in (14), eq. (4) has two families of periodic solutions

$$u_2^*(\xi) = \pm\sqrt{\frac{-2\nu}{(2-k_3^2)\mu}}\,\mathrm{dn}\left(\sqrt{\frac{\nu}{2-k_3^2}}\,\xi, k_3\right). \tag{21}$$

(1.3) For $h^* \in (0,\infty)$ in (14), eq. (4) has a family of periodic solutions

$$u_3^*(\xi) = \sqrt{\frac{-2k_4^2\nu}{\mu(2k_4^2-1)}}\,\mathrm{cn}\left(\sqrt{\frac{\nu}{2k_4^2-1}}\,\xi, k_4\right). \tag{22}$$

Remark 3.1.

(a) $u_4^*(\xi) = -\,u_3^*(\xi)$ are a family of periodic solutions of eq. (4) which denote the same periodic orbits as the solutions $u_3^*(\xi)$ except for the phase difference $T_1^* = 2\sqrt{\frac{2k_4^2-1}{\nu}}\,K(k_4)$.

(b) The periodic solutions $u_i^*(\xi)$ $(i = 2,3,4)$ respectively converge to the soliton solutions $u_1^*(\xi)$ as $h^*\to 0$.

(2) Suppose that $(\mu,\nu)\in L_2^-$. For $h^*\in(0,\infty)$ in (14), eq. (4) has a family of periodic solutions

$$u_5^*(\xi) = \sqrt{-2h/\nu}\cos(\sqrt{-\nu}\,\xi). \tag{23}$$

(3) Suppose that $(\mu,\nu)\in C_3$. For $h^*\in(0,\infty)$ in (14), eq. (4) has a family of periodic solutions having the same parametric representation as (22).
(4) Suppose that $(\mu,\nu)\in L_1^-$. For $h^*\in(0,\infty)$ in (14), eq. (4) has a family of periodic solutions

$$u_6^*(\xi) = \sqrt[4]{-4h/\mu}\,\mathrm{cn}\left(\sqrt[4]{-\mu h}\,\xi, \sqrt{2}/2\right). \tag{24}$$

(5) Suppose that $(\mu,\nu)\in C_4$.
(5.1) For $h^*=h_1^*$ in (14), eq. (4) has a kink solution and an anti-kink solution

$$u_7^*(\xi) = \pm\sqrt{-\nu/\mu}\,\tanh(\sqrt{-\nu/2}\,\xi). \tag{25}$$

(5.2) For $h^*\in(0,h_1^*)$ in (14), eq. (4) has a family of periodic solutions

$$u_8^*(\xi) = \sqrt{\frac{-2k_5^2\nu}{(1+k_5^2)\mu}}\,\mathrm{sn}\left(\sqrt{\frac{-\nu}{1+k_5^2}}\,\xi, k_5\right). \tag{26}$$

Remark 3.2.

(a) $u_9^*(\xi) = -u_8^*(\xi)$ are a family of periodic solutions of eq. (4) which denote the same periodic orbits as the solutions $u_8^*(\xi)$ except that the phase difference $T_2^* = 2\sqrt{\frac{1+k_5^2}{-\nu}}\,K(k_5)$.
(b) As $h^*\to h_1^{*-}$, the periodic solutions $u_8^*(\xi)$ and $u_9^*(\xi)$ respectively converge to the kink (or anti-kink) solution $u_7^*(\xi)$.

Remark 3.3. It follows from (20) and (25) that the balance between the nonlinear term u^2u_x and the dispersion term u_{xxx} leads to the soliton and kink solutions of eq. (4).

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