PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 68, No. 5
May 2007
pp. 749–756

Synchronization and basin bifurcations in mutually coupled oscillators

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MS received 12 October 2006; revised 18 December 2006; accepted 28 February 2007

Abstract. Synchronization behaviour of two mutually coupled double-well Duffing oscillators exhibiting cross-well chaos is examined. Synchronization of the subsystems was observed for coupling strength $k > 0.4$. It is found that when the oscillators are operated in the regime for which two attractors coexist in phase space, basin bifurcation sequences occur leading to $n + 1, n \geq 2$ basins as the coupling is varied – a signature of Wada structure and final-state sensitivity. However, in the region of complete synchronization, the basins structure is identical with that of the single oscillators and retains its essential features including fractal basin boundaries.

Keywords. Synchronization; fractal basins; Duffing oscillator; cross-well chaos.

PACS Nos 05.45.-a; 05.45.Xt; 05.45.Pq

1. Introduction

Synchronization phenomena are of fundamental importance in many physical, biological, and technological systems. In particular, synchronization of chaotic dynamics has attracted much attention during the last decade since the seminal work of Pecora and Carroll [1]. The enormous research activity in this field has derived its motivation from its role in understanding the basic features of coupled nonlinear systems and in view of potential applications in communication systems, time series analysis, modelling brain and cardiac rhythm activity and earthquake dynamics [2–4].

Chaotic systems are characterized by a sensitive dependence on initial conditions, and indeed, synchronization between identical chaotic systems is an intriguing problem. For a system of two coupled chaotic oscillators: $(\mathrm{d}x/\mathrm{d}t) = f(x, y)$ and $(\mathrm{d}y/\mathrm{d}t) = g(x, y)$, where x and y are phase-space variables and f and g are the corresponding nonlinear functions, synchronization in direct sense implies that

$|x(t) - y(t)| \to 0$ as $t \to \infty$. When this occurs, the coupled systems are said to be completely synchronized [1,5,6]. Other forms of synchronization which have been observed includes phase synchronization [7,8], lag synchronization [7,9] and generalized synchronization [5,10].

Many researchers have demonstrated synchronization of coupled periodically driven non-autonomous chaotic systems theoretically [9,11–13], numerically [14–23], and experimentally [24,25]. Of particular interest is the popular Duffing oscillators [9,13,14,17,18,20] which is noted for its simplicity and rich complex dynamical behaviour and has for long served as a paradigm for nonlinear dynamics. Baker *et al* [14] reported the existence of intermittent synchronization in a unidirectional coupled Duffing oscillator. Intermittency was also reported in a nonlinearly coupled model [17]. Using dissipative coupling, Rajasekar and Murali [18] found six coexisting chaotic attractors with state variables of the subsystems perfectly synchronized for some and asynchronized for the others. Very recently, we studied phase synchronization (PS) in coupled hyperchaotic Duffing oscillators and complete synchronization in coupled double-well Duffing oscillators [20]. In ref. [20], we employed velocity coupling in the unidirectional configuration to investigate PS and CS respectively. In addition, Pisarchik and Jaimes-Reategui studied intermittent lag synchronization in a driven nonlinearly coupled Duffing oscillator [9]. In the present paper, we employ linear feed-back, and examine the synchronization behaviour of two mutually coupled double-well Duffing oscillators that exhibit cross-well chaos [26]. The transition to the synchronous state is studied by examining the structural changes associated with the basins of attraction of two coexisting resonance attractors in the phase space.

2. The model

Let us consider two identically coupled periodically forced double-well Duffing oscillators (DDOs) described by the following second-order nonautonomous differential equations:

$$\ddot{x}_1 + b\dot{x}_1 + \frac{\mathrm{d}V(x_1)}{\mathrm{d}x_1} = g\cos(\omega, t) + k(x_2 - x_1), \tag{1}$$

$$\ddot{x}_2 + b\dot{x}_2 + \frac{\mathrm{d}V(x_2)}{\mathrm{d}x_2} = g\cos(\omega, t) + k(x_1 - x_2), \tag{2}$$

where the state variables, $x_{1,2}$ stand for the displacement from the equilibrium position, dot denotes differentiation with respect to time (t); b and g are the damping parameter and the forcing strength respectively. ω is the angular frequency of the driving force. k is the coupling parameter that controls the strength of the feed-back. The coupling considered here can be interpreted as a perturbation of each oscillator through a feed-back signal proportional to the difference of their amplitudes [27]. $V(x)$ is the double-well potential approximated by a finite Taylor series [26]:

$$V(x_{1,2}) = -\frac{x_{1,2}^2}{4} + \frac{x_{1,2}^4}{8}. \tag{3}$$

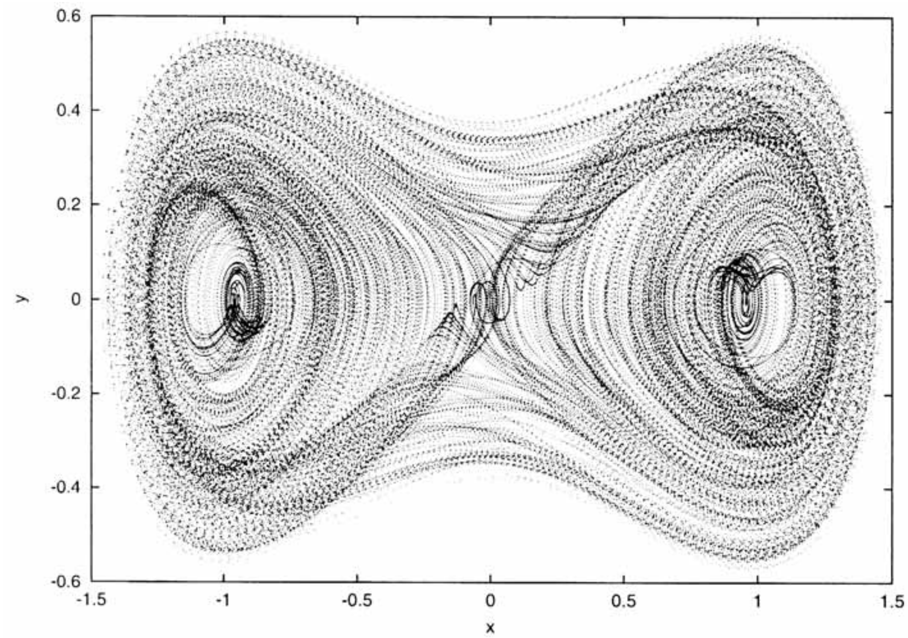

Figure 1. Cross-well chaotic attractor in the x, y ($y = \dot{x}$) plane for $b = 0.1, g = 0.095$ and $\omega = 0.79$.

With $k = 0$ and $V(x_{1,2})$ given by eq. (3), the uncoupled systems (1) and (2) exhibit cross-well motion when the particle is under the influence of a strong periodic forcing.

3. Results and discussion

3.1 *Synchronization*

All the results to be presented here were computed with the following parameter settings: $g = 0.095, \omega = 0.79$ and $b = 0.1$. This places the coupled oscillators in a stable chaotic state as shown in figure 1. Numerical solutions were obtained using a fourth-order Runge–Kutta routine with time-step of $0.01(2\pi/\omega)$ as well as the software *Dynamics* [28].

When the coupling is switched on, we find that the two systems begin to re-adjust and build up correlation; and when k reaches a certain threshold (say $k_{\text{th}} = 0.4$) the oscillators become synchronized after some initial transient. Complete synchronization of the subsystems was found for $k > k_{\text{th}} = 0.4$. As an illustration, we plot in figure 2 the error dynamics $E_x = x_2 - x_1$, $E_y = \dot{x}_2 - \dot{x}_1$; and E given by

$$E = \sqrt{E_x^2 + E_y^2}, \tag{4}$$

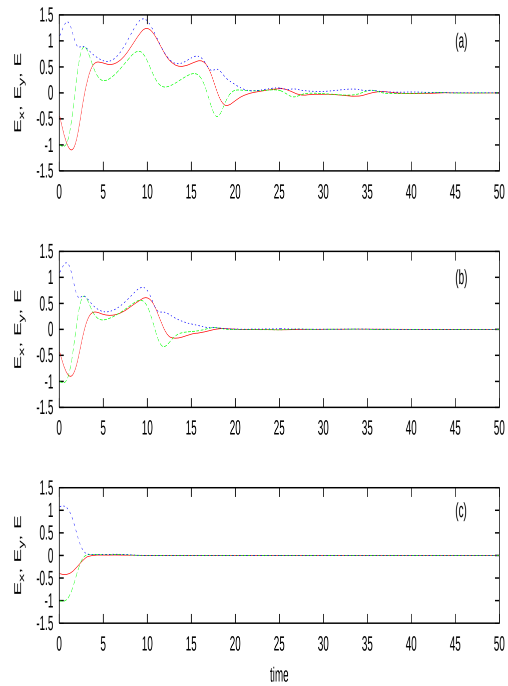

Figure 2. Error dynamics E_x (red), E_y (green) and E (blue) for increasing coupling strengths for (**a**) $k = 0.256$, (**b**) $k = 0.4$ and (**c**) $k = 1.2$. The coupling is switched on at $t = 0$.

which is an intuitive measure of the quality of synchronization [14]. For $k = 0$, E_x, E_y and E vary chaotically. When the coupling is gradually switched on, the error states E_x, E_y and E progressively converge to zero as $t \to \infty$ as shown in figures 2a–c, where the coupling has been switched on at $t = 0$. We find that the transient time before complete synchronization occur gradually diminishes with increasing coupling strength and becomes negligible in the synchronized state. At $k_{\text{th}} = 0.4$ (see figure 2b), the correlation between the oscillators is stronger and finally for larger coupling strength (see figure 2c), where the transient is negligible, the correlation is strongest. This shows that the two oscillators asymptotically approach identical trajectories, i.e. they are completely synchronized.

To examine the transition to the stable synchronized state shown in figure 2, we plot in figure 3, the average error dynamics (E) given by eq. (4) together with the Lyapunov exponent spectrum (λ) as functions of the coupling strength k. Notably, E approaches zero asymptotically as k increases and is positive for $k < k_{\text{th}}$; while for $k \geq k_{\text{th}}$, E is zero, implying stable synchronized state (see figure 3a). In figure 3b, we see that λ exhibits spikes as k increases prior to synchronization. This suggests sudden changes in the system which could be associated with changes in the bifurcation structures as will be discussed later. It is clear that stable synchronization is readily achieved as k exceeds k_{th}, with λ being negative in the region of synchronization.

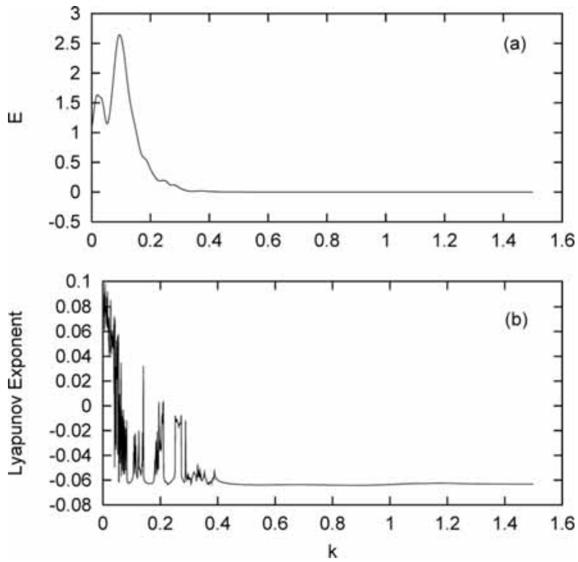

Figure 3. Average error dynamics E (**a**) and Lyapunov exponent spectrum (λ) (**b**) as functions of coupling strength k.

3.2 *Basin bifurcations*

In ref. [26], it was reported that the one-dimensional (1D) model of the DDO has two coexisting resonant attractors to the right of the cross-well chaotic zone ($g = 0.095, \omega = 0.88$) and are the only resonant oscillations in a broad range of the system parameter plane. In connection with these attractors, we note that when the coupled system achieves synchronism, the coupling term vanishes ($x_1 = x_2$). By implication, two coexisting resonant attractors should also be found in the synchronized state of the coupled oscillators respectively with two basins of attraction as in the case of the single oscillator.

By using Poincaré section representation, we visualize the basins of attraction of the coexisting resonant attractors using the software *Dynamics* [28]. We computed the basins for several coupling strengths ($0 \leq k \leq 2.0$). We show typical basins for $k = 0, 0.25, 0.8$ and 1.5 in figure 4. $k = 0$ (figure 4a) corresponds to the single (1D) oscillator. The attractors are symmetric with respect to each other and to the $\dot{x}$ coordinate. When the coupling is switched on, additional attractors besides the resonant attractors of the 1D model were found. That is, there is a probability of finding $n+1, n \geq 2$ attractors, coexisting in phase space for the range of values studied. The additional attractors (hereafter called 'coupling-induced attractors') arising from coupling effect modify the basin structure significantly. To illustrate this, we display the basins of attraction of four coexisting attractors for $k = 0.25$ and of three coexisting attractors for $k = 0.8$ in figures 4b and 4c respectively.

One significant feature of these basins is the complexities associated with the basin boundaries. The existence of three or more basins prior to the synchronous state also suggests the possibility of the Wada property. If the basin boundaries

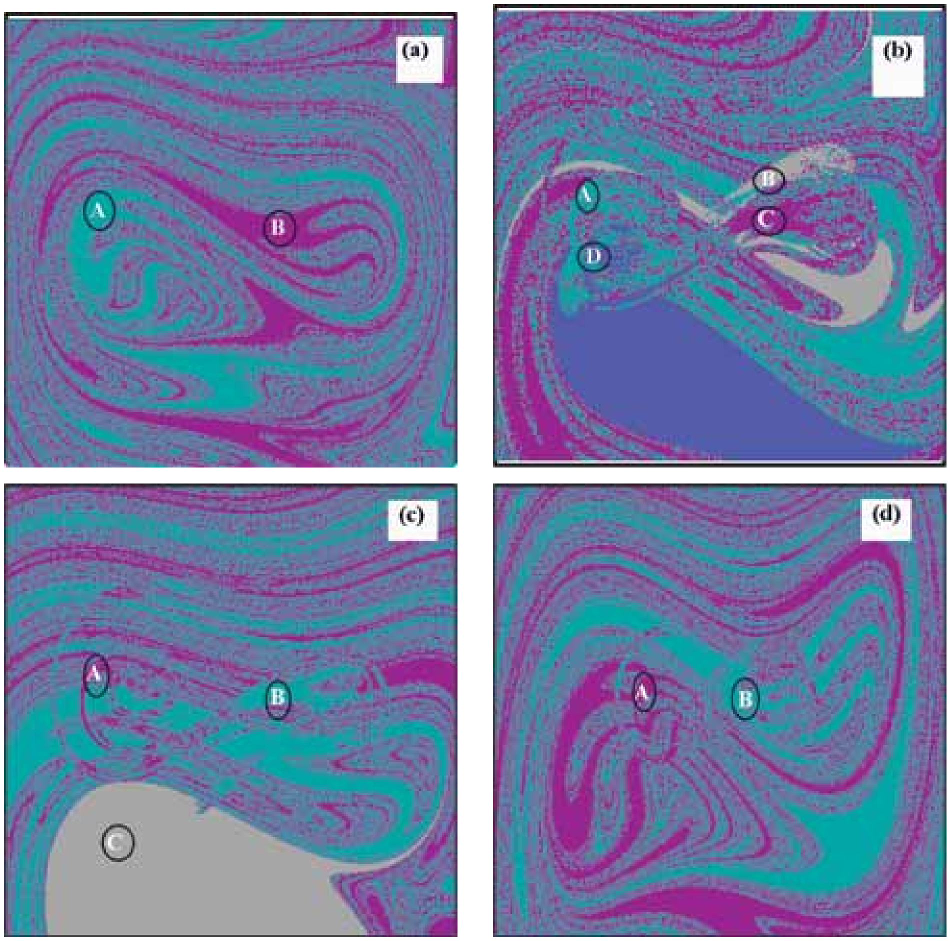

Figure 4. The basin structures for the coupled double-well Duffing oscillators for different values of the coupling strength. The parameters are: $g = 0.095, b = 0.1$ and $\omega = 0.88$. (**a**) $k = 0$, (**b**) $k = 0.25$, (**c**) $k = 0.8$, (**d**) $k = 1.5$. The x and y variables are plotted in the oscillator 1 coordinates in the range $-2.0 \leq x,\, y \leq 2.0$.

satisfy the necessary and sufficient conditions for Wada basins, the consequence is final-state sensitivity or unpredictability [29–32]. A basin boundary satisfies the Wada property if every open neighbourhood of any point on the boundary has a nonempty intersection with at least three different basins. The Wada property was first introduced in the physics literature by Kennedy and Yorke [29]. Since then the Wada basin boundary has been reported for both Hamiltonian and dissipative systems [33–38]; including experimental works in chaotic scattering [39]. Recently, Aguirre and Sanjuan [40,41] showed the existence of Wada basins in Duffing oscillators. To the best of our knowledge, there has not been any report on coupling-induced Wada basins in coupled chaotic systems. Detailed studies and proof of the Wada property for the basins obtained here and in some other systems are the issues addressed in a future report which is in progress.

As the coupling was further increased passing $k = 0.8$, we find that a transition from multiple basins to two-basins structure occurred around $k = 0.95$. Here, the coupling-induced attractors disappear completely leaving the symmetric resonant attractors as the only attractors in the subspace; and the basin structure corresponds to those of the uncoupled DDOs. The sudden changes observed in the basin structures as the coupling parameter was varied is referred to as basin bifurcations [42]. Several basin bifurcations were observed around the k values for which transient behaviour prior to complete synchronization was achieved.

It turns out that one can associate a synchronous state in this system with the features indicated below: (i) The same number of attractors in the single oscillator and the synchronized state of the coupled oscillators. (ii) The symmetry of the attractors with respect to each other and to the $\dot{x}$ coordinate in both cases. (iii) The fractal basin boundaries in both cases. We conjecture that this scenario can be observed for systems operated in the regime for which two or more attractors coexist in phase space.

4. Conclusions

We have examined in this paper the synchronization of cross-well chaos in mutually coupled double-well Duffing oscillators. Synchronization of the subsystems was observed for coupling strength $k > 0.4$. We have also found that coupling induces additional attractors in the subspace besides the two resonant attractors to the right of the cross-well chaotic zone already found in the single Duffing oscillator. Prior to the synchronous state, sudden changes occur in the basin structures as the coupling strength is varied. These changes are simply basin bifurcations leading to $n+1, n \geq 2$ basins – a signature of Wada structure and final-state sensitivity. However, in the region of complete synchronization, the only attractors in the subspace are the two resonant attractors corresponding to the single oscillators – the basins of which are identical with that of the single oscillator and retains its essential features including fractal basin boundaries. Alhough detailed theoretical study of this observation is currently being addressed and will be reported in a future paper, we believe that basin bifurcations as reported here should be a common phenomenon in coupled oscillators. Thus, our findings should complement existing results as far as coupled oscillators are concerned.

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