PRAMANA
— journal of
physics

Vol. 68, No. 4
April 2007
pp. 689–699

Damage spreading on networks: Clustering effects

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MS received 6 August 2006; revised 26 October 2006; accepted 4 December 2006

Abstract. The damage spreading of the Ising model on three kinds of networks is studied with Glauber dynamics. One of the networks is generated by evolving the hexagonal lattice with the star-triangle transformation. Another kind of network is constructed by connecting the midpoints of the edges of the topological hexagonal lattice. With the evolution of these structures, damage spreading transition temperature increases and a general explanation for this phenomenon is presented from the view of the network. The relationship between the transition temperature and the network measure-clustering coefficient is set up and it is shown that the increase of damage spreading transition temperature is the result of more and more clustering of the network. We construct the third kind of network-random graphs with Poisson degree distributions by changing the average degree of the network. We show that the increase in the average degree is equivalent to the clustering of nodes and this leads to the increase in damage spreading transition temperature.

Keywords. Ising model; damage spreading; Glauber dynamics; network.

PACS Nos 05.50.+q; 75.10.Hk; 75.40.Mg ; 75.10.Nr

1. Introduction

Damage spreading (DS) technique [1] has been proved to be a useful technique in the study of dynamical properties of statistical models, especially magnetic models, like Ising, Clock, Potts, spin glass, etc. With the DS technique, one can learn how a perturbation spreads throughout a cooperative system composed of interacting subunits [2,3]. The most important advantage of this technique is that it is less sensitive to statistical fluctuations compared to conventional Monte Carlo methods [2]. Till now, many elements which characterize the DS process have been considered in literatures [4], including the interactions (ferromagnetic, antiferromagnetic, spin glass, etc.), the Monte Carlo rules (heat bath, Glauber, Kawasaki, etc.). As for the lattice geometry, various kinds of networks have been discussed [4–8], including the simple regular lattices, such as two-dimensional square and 3-12 lattice; the two-dimensional random structures, such as the soap froth and Voronoi structures;

and the so-called complex networks, which have attracted much attention in recent years. The importance of DS has also been stressed in its application in economic [9] and social phenomena [10,11].

On the other hand, though much work has been done on DS, better understanding of the DS behaviors need to be presented. For example, DS on a set of hierarchical trivalent cellular structures was studied in refs [4,5]. It was found that, by increasing the number of the smallest polygons, the DS transition becomes more and more difficult. Similar property was also found in another set of two-dimensional structures [6]. Now a problem arises: whether simple or complex, regular or random, all these structures are networks. How can we give a general explanation for these phenomena from the view of the network? During the last ten years, the study on complex networks has made great advance [12–15]. Many measures are put forward to describe the properties of complex networks. Among theses measures, degree distribution and cluster coefficient are the most important ones [15]. How can we relate the topological properties or the structure-dependent properties of the DS with the two network measures? These are the main considerations of this paper. For the lattices mentioned above, we will construct the relationship between the DS transition temperature and the cluster coefficient. We will demonstrate that the increase in DS transition temperature in the process of network evolution studied in refs [4,6] is the result of more and more clustering of the network. We will also study the DS on the random graphs with Poisson degree distributions by changing the average degree of the network. We will show that the increase in the average degree $\langle k \rangle$ equivalents to the clustering of the nodes leads to the increase of DS transition temperature T_{d}.

2. Theory

We deal with the DS problem on networks based on Ising model with Glauber dynamics and by using Monte Carlo method [4–7]. We consider the Hamiltonian of nearest-neighbor Ising model to be of the form

$$H = -\sum_{\langle i,j \rangle} J_{ij} s_i s_j, \tag{1}$$

where $J_{ij} > 0$ is the ferromagnetic exchange interaction coefficient between the nearest-neighbor sites i and j, i.e., the interaction between the nearest-neighbor polygons (nodes).

First, let the system A evolves for a long time to reach equilibrium, then a replica B of the system is made. At $t = 0$, the spin in the center cell of the lattice B is flipped (damaged) and fixed all the time. Note that many kinds of perturbation sources are introduced in literatures, and one of them is the case of constant perturbation source, which is modeled by fixing the initially flipped spin all the time. In this paper we only discuss this special case and we choose the perturbation source at the center of the network. The Hamming distance (or damage) in phase space for the trivalent structures is calculated by

$$D(t) = \frac{1}{N}\sum_{i=1}^{N}(1 - \delta_{s_i^A(t),s_i^B(t)}), \tag{2}$$

where $\{s_i^A(t)\}$ and $\{s_i^B(t)\}$ are the two spin configurations of the system which is subjected to the same thermal noise and the same set of random numbers, N is the number of total spins on the lattice studied. In order to make a configuration $s = \{s_i(t)\}$ evolve in time, we use Glauber dynamics, the transition probability of flipping spin i is

$$w_i(s) = \min\left[1, \exp\left(-\frac{\Delta E_i}{k_{\mathrm{B}}T}\right)\right], \tag{3}$$

where ΔE_i is the change in energy when spin i is flipped.

3. A set of two-dimensional trivalent hierarchical lattices

A set of hierarchical trivalent cellular structures are introduced in ref. [4], based on the star-triangle transformation operating on the vertices of the hexagons, including the pure hexagonal lattice, the 3-12 lattice, the 3-6-24 lattice and the 3-6-12-48 lattice. The star-triangle transformation can be described as follows. On each vertex of the simplest trivalent regular lattice – the pure hexagonal lattice (figure 1a), replace the star (Y) by a triangle ($\triangledown$), so that we have a triangle on each vertex of the original hexagon. The result is a crystal made with 12-gons and triangles. In doing the star-triangle transformation, the side length of the 12-gon is so chosen that the 12-gon is a regular polygon. As Liebmann [16] did for the 4-8 lattice, we call this crystal the 3-12 lattice (figure 1b). By repeating the application of star-triangle transformation on the trivalent vertex, we get the 3-6-24 lattice made with 24-gon, 6-gon and triangles (after the second-order star-triangle transformation) and the 3-6-12-48 lattice made with 48-gon, 12-gon, 6-gon and triangles (after the third-order star-triangle transformation), etc. (Formation and evolution of patterns on the Euclidean plane have been reviewed by some authors, and readers can find them in literatures, for example, in ref. [17].)

We put the spins in the centers of the polygons. Connecting the centers of the polygons forms the network. Obvious frozen-chaotic phase transition occurs in these structures and if $T_{\mathrm{d}}(6)$, T_{d}(3-12), T_{d}(3-6-24) and T_{d}(3-6-12-48) stand for the critical temperatures corresponding to the pure hexagonal, the 3-12, the 3-6-24, and the 3-6-12-48 lattices, we have $T_{\mathrm{d}}(6) < T_{\mathrm{d}}$(3-12) $< T_{\mathrm{d}}$(3-6-24) $< T_{\mathrm{d}}$(3-6-12-48). Thus, the more complicated the lattice is, the higher the critical temperature for damage spreading. This trend can be understood by considering the role of the polygons with the smallest side number [4].

In order to get better understanding of the structures, we introduce two lattice parameters: $N_{\mathrm{p}}(i)$ and N_{t}. For a given lattice size N (the number of the main polygons along one direction), let $N_{\mathrm{p}}(i)$ be the number of i-gons (cells with edge number of i) and N_{t} the total number of polygons (cells). Thus, $N_{\mathrm{p}}(6) = N_{\mathrm{t}} = N^2$ for the pure hexagonal lattice; $N_{\mathrm{p}}(3) = 2N^2$, $N_{\mathrm{p}}(12) = N^2$ and N_{t}(3-12) $= 3N^2$ for the 3-12 lattice; $N_{\mathrm{p}}(3) = 6N^2$, $N_{\mathrm{p}}(6) = 2N^2$, $N_{\mathrm{p}}(24) = N^2$ and N_{t}(3-6-24) $= 9N^2$

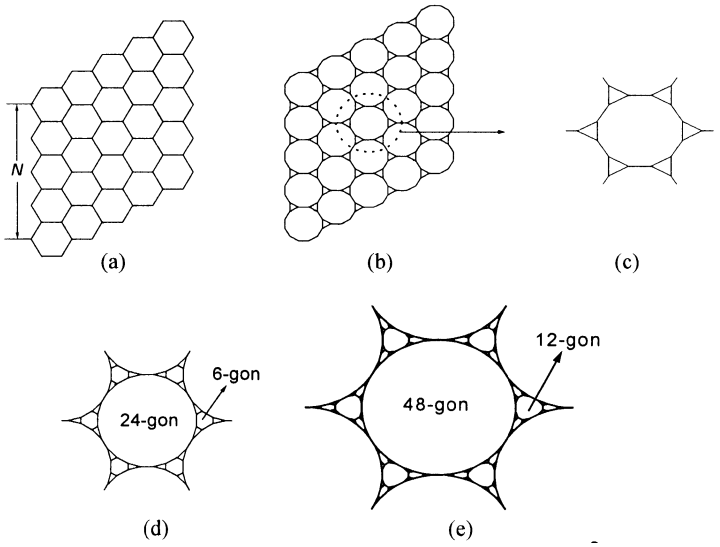

Figure 1. Hierarchical trivalent structures generated by the star-triangle transformation. (**a**) The pure hexagonal lattice, (**b**) the 3-12 lattice generated by the first-order star-triangle transformation, (**c**) the primary cell of the 3-12 lattice, (**d**) the primary cell of the 3-6-24 lattice and (**e**) the primary cell of the 3-6-12-48 lattice.

for the 3-6-24 lattice; $N_{\rm p}(3) = 18N^2$, $N_{\rm p}(6) = 6N^2$, $N_{\rm p}(12) = 2N^2$, $N_{\rm p}(48) = N^2$ and $N_{\rm t}(3\text{-}6\text{-}12\text{-}48) = 27N^2$ for the 3-6-12-48 lattice.

Next, we define

$$p(i) = \frac{N_{\rm p}(i)}{N_{\rm t}}, \tag{4}$$

which is the fraction of i-gons in the lattices or the probability distribution of edge number in the cells. We calculate the fractions of i-gons in the lattices as follows: $p(6) = 1$ for the pure hexagonal lattice; $p(3) = 2/3$ and $p(12) = 1/3$ for the 3-12 lattice; $p(3) = 2/3$, $p(6) = 2/9$ and $p(24) = 1/9$ for the 3-6-24 lattice; $p(3) = 2/3$, $p(6) = 2/9$, $p(12) = 2/27$ and $p(48) = 1/27$ for the 3-6-12-48 lattice. Note that the fractions of triangles in these structures are the same.

Since we put the spins in the centers of the polygons, if we consider the place of spins as the nodes of the network, this network is a lattice dual to the original one. For example, connecting the nodes locating on the centers of the pure hexagonal lattice forms the triangle lattice, which is dual to the pure hexagonal lattice. So, the side number of the polygons corresponds to the node degree. The 3-12 lattice is constructed by triangles and the 12-gons, so it's dual lattice is constructed by connecting nodes with degree 3 and degree 12 (see figure 2).

The averaged degrees of the above lattices can be calculated as follows:

$$\langle k \rangle = \sum_i p(i)k(i), \tag{5}$$

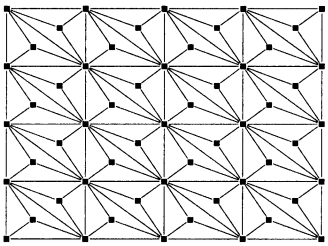

Figure 2. The dual lattice of the 3-12 lattice.

in which $k(i)$ is the degree of node i. For the pure hexagonal lattice, $\langle k\rangle = k(6) \times p(6) = 6 \times 1 = 6$; for the 3-12 lattice $\langle k\rangle = k(3) \times p(3) + k(12) \times p(12) = 3 \times 2/3 + 12 \times 1/3 = 6$; for the 3-6-24 lattice and the 3-6-12-48 lattice the same results $\langle k\rangle = 6$ are obtained. So the damage properties cannot be explained by degree difference among the lattices and so, we have to turn to another quantity: the cluster coefficient. The key topological difference among the above structures may be their clustering properties, which may greatly affect the evolution of damage spreading in the above networks. According to Newman [15], two measures can be defined to describe the clustering of a network (called the clustering coefficients):

$$C_1 = \frac{3 \times \text{Number of triangles}}{\text{Number of connected triples}} \tag{6}$$

and

$$C_2 = \frac{1}{n}\sum_{i=1}^{n} c_i, \tag{7}$$

where n is the number of nodes and c_i is the ratio of the number of triangles connected to vertex i over the number of triples centered on vertex i. Often C_2 is referred to as the average 'network density' [15] and it is the average of the ratio, while C_1 is the ratio of the average. In the following discussion on the relation between clustering effect and damage spreading, we focus on C_2 since the dynamics of damage spreading is affected mainly by the local clustering, and the calculation of the average of ratio is relatively easy to be performed on each given vertex i. For the networks connected by nodes with only limited kinds of degrees, the network density C_2 can be calculated in a straightforward way, using the following formula:

$$C_2 = \sum_i p(i)\frac{i}{i(i-1)/2} = \sum_i p(i)\frac{2}{i-1}. \tag{8}$$

Thus, $C_2(6) = \sum_{i=6} p(i)[2/(i-1)] = 0.4$; C_2(3-12) $= \sum_{i=3,12} p(i)[2/(i-1)] = 0.727$; and in the same way we get C_2(3-6-24) $= 0.765$; C_2(3-6-12-48) $= 0.7706$. The transition temperatures are shown in figure 3 as a function of C_2 for the pure hexagonal lattice, the 3-12, the 3-6-24 and the 3-6-12-48 lattices and we can see that network clustering causes difficulty of DS.

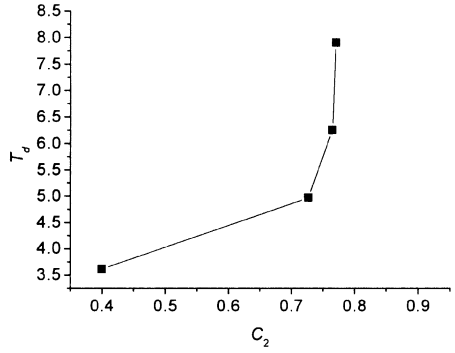

Figure 3. Transition temperatures as a function of C_2 for the pure hexagonal lattice, the 3-12, the 3-6-24 and the 3-6-12-48 lattices.

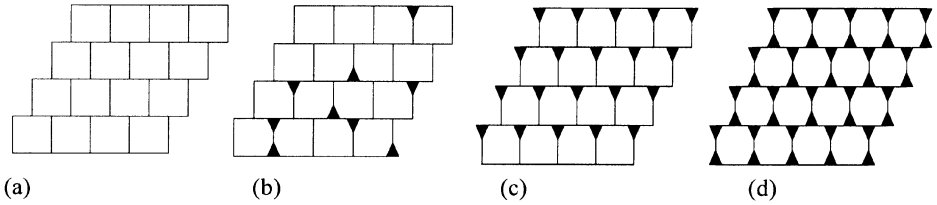

Figure 4. (**a**) The topological hexagonal lattice, (**b**) the star-triangle transformation ($N_i = 10$) made at the vertices randomly, (**c**) the 3-9 lattice and (**d**) the 3-12 lattice.

In the above examples, network clustering is caused by increasing the number of the smallest polygons. This process can be simulated by inserting triangles continuously onto the topological hexagonal lattice [5]. Since only the topology of the structures is emphasized, we consider the simplest star-triangle transformation in a hexagonal lattice. We do the transformation on the vertices of the topological hexagonal lattice with the dimension of $N \times N$ (figures 4a and 4b). The transformation is made at the vertices at random. But if we do the transformation regularly at each of the other row or at each row, we get the 3-9 lattice (figure 4c) and the 3-12 lattice (figure 4d). Notice that we generate N_i random points for inserting the triangles. But the real number of triangles may be less than N_i since the randomly generated points may be coincided. For example, we want to generate 10 triangles but we may get only 9 at last (see figure 4b). On the other hand, the real number of the triangles varies among configurations and our results are averaged for 100 configurations.

We denote the inserting triangle number as N_i and we can show that as N_i increases both T_{d} and C_2 increase (figure 5). So we know that the DS transition temperature is related to the clustering of the networks. In order to see if this conclusion is of generality, next we will turn to other networks.

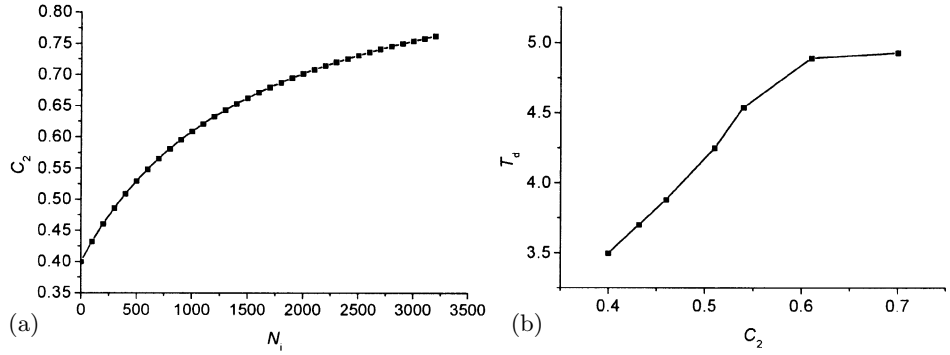

Figure 5. (**a**) The clustering coefficients of the network in figure 4b vs. the number of inserted triangles ($N = 40$); (**b**) The DS transition temperatures of the network in figure 4b as a function of C_2.

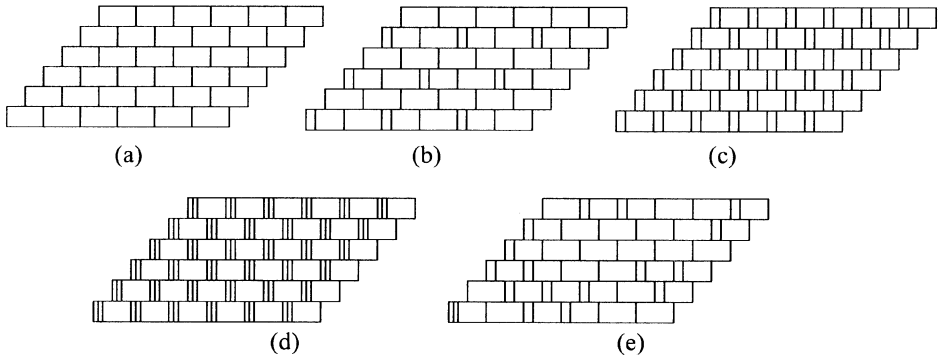

Figure 6. A set of hierarchical trivalent structures. (**a**) The topological hexagonal lattice, (**b**) the 4-6-7 lattice, (**c**) the 4-8 lattice, (**d**) the 4-10 lattice and (**e**) the random lattice with $N_i = 18$.

4. Examples of other two-dimensional regular lattice

Guo *et al* [6] studied the DS on another set of hierarchical trivalent structures. By connecting the midpoints of the edges of the topological hexagonal lattice, we insert the rectangles into the topological hexagonal lattice (figure 6a) and form the 4-6-7 lattice (figure 6b, inserting at each of the other row), the 4-8 lattice (figure 6c, inserting at each row) and the 4-10 lattice (figure 6d, inserting twice at each row). Some workers have paid their attention to the 4-8 lattice [16,18] and this lattice is often referred to as the $(4,8^2)$ Archimedean lattice [19]. But our 4-8 lattice (figure 6c) is generated from the topological hexagonal lattice (figure 6a), so it is slightly different from that studied in refs [16,18].

As in the above section, we put the spins in the centers of the polygons, so our networks are in fact the dual ones of those of figure 6. We plot some of the networks in figures 7 and 8.

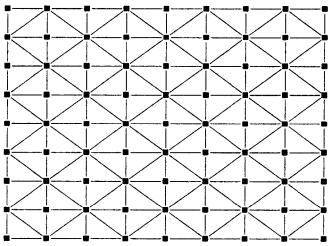

Figure 7. The dual lattice of the 4-8 lattice.

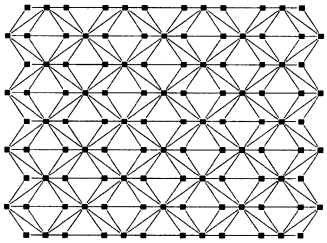

Figure 8. The dual lattice of the 4-10 lattice.

For the Glauber dynamics, we have $T_{\rm d}(6) < T_{\rm d}(4\text{-}6\text{-}7) < T_{\rm d}(4\text{-}8) < T_{\rm d}(4\text{-}10)$ [6]. Here $T_{\rm d}(6)$, $T_{\rm d}(4\text{-}6\text{-}7)$, $T_{\rm d}(4\text{-}8)$ and $T_{\rm d}(4\text{-}10)$ stand for the DS transition temperatures corresponding to the topological hexagonal lattice, the 4-6-7, the 4-8, and the 4-10 lattices. This phenomenon is similar to the DS on the set of 3-12,3-6-24,3-6-12-48 lattices studied in ref. [4]. We can understand this trend by considering the effects of the polygons with the smallest side number, the rectangles here, as pointed out in ref. [6].

For a given N, the number of sites for the topological hexagonal, the 4-6-7, the 4-8, the 4-10 lattices is N^2, $5N^2/4$, $2N^2$ and $3N^2$, respectively, while the number of the rectangles in the corresponding 4-6-7, 4-8 and 4-10 lattices is $N^2/4$, N^2 and $2N^2$, respectively. The increase of the lattice sites is equivalent to the increase of spin density. But we can also show that this result is due to the clustering of the networks.

Table 1 presents the main parameters related to the topological structures, in which $\langle k\rangle$ and C_2 are obtained from eqs (5) and (6). The relation between the transition temperature and the clustering coefficient C_2 is plotted in figure 9, which also tells us that the increase in $T_{\rm d}$ is due to the network clustering.

Table 1. Structural parameters of the topological hexagonal, the 4-6-7, 4-8 and 4-10 lattices.

Lattice	Polygon type	Polygon number	$p(i)$	$\langle k\rangle$	C_2
6	6	N^2	1	6	0.4
	4	$N^2/4$	1/5		
4-6-7	6	$2N^2/4$	2/5	6	0.424
	7	$2N^2/4$	2/5		
4-8	4	N^2	1/2	6	0.47
	8	N^2	1/2		
4-10	4	$2N^2$	2/3	6	0.5
	10	N^2	1/3		

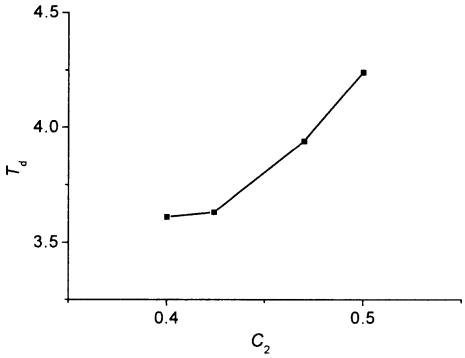

Figure 9. Transition temperatures as a function of C_2 for the topological hexagonal lattice, the 4-6-7, the 4-8 and the 4-10 lattices.

Of course, we can check our idea by inserting rectangles continuously onto the topological hexagonal lattice, as done in ref. [6], and we can get similar conclusions [6].

Studies have shown that the critical temperatures of the Voronoi and soap froth structures are nearly equal to that of the pure hexagonal lattices [4]. In order to understand that, we calculate the network densities for the Voronoi and soap froth structures, which are respectively C_2=0.425 for Voronoi and C_2=0.427 for soap froth structures. We notice that these values are similar to the pure hexagonal lattice, implying that the clustering degree among Voronoi, soap froth and the pure hexagonal lattice is similar, so that DS behaves in the same way on these structures.

5. DS on random graphs with Poisson distributions

In the above calculations, the networks have constant average degree distribution $\langle k\rangle = 6$. It seems that when $\langle k\rangle$ is fixed, the phase transition is determined mainly by the clustering. Now, the problem is, if we change (increase) $\langle k\rangle$, does it result in the increase in phase transition critical temperature? The answer is yes. In order to show this, we investigate the DS problem on a random graph with Poisson degree distribution.

Our network model consists of N nodes parametrized by a Poisson degree distribution $P(k) = \mathrm{e}^{-\langle k\rangle}(\langle k\rangle^k/k!)$, with degree lying between $d_{\min}$ and $d_{\max}$. Our model parameters are $N = 5000$, $d_{\min} = 1$ and $d_{\max} = 100$.

Averaged damage spreading as a function of temperature T is shown in figure 10 for the random graph with Poisson distribution. As expected, $T_{\mathrm{d}}(\langle k\rangle)$ increases with $\langle k\rangle$. From figure 11, we know that increasing $\langle k\rangle$ is equivalent to increasing the cluster coefficient C_2. This shows the same effect of network clustering as in the above two sections. Figure 11 also shows us a very interesting phenomenon that the relationship between C_2 and $\langle k\rangle$ is linear.

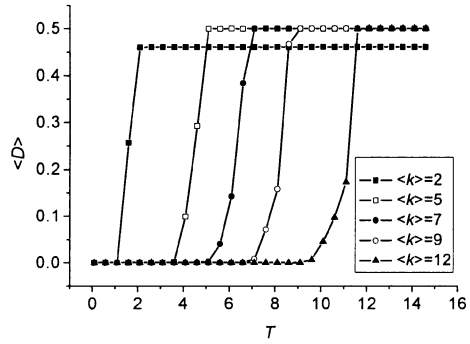

Figure 10. Averaged damage spreading as a function of temperature for the random graph with Poisson degree distribution. T is in units of J/k_{B}.

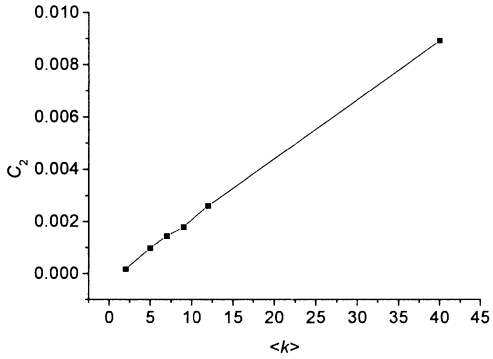

Figure 11. Clustering coefficient vs. $\langle k \rangle$ for the random graph with Poisson degree distribution.

6. Summary

The damage spreading of the Ising model on three kinds of networks is studied with Glauber dynamics. In §3, a set of hierarchical trivalent cellular structures are generated by evolving the hexagonal lattice with the star-triangle transformation. In §4, another set of hierarchical trivalent structures are constructed by connecting the midpoints of the edges of the topological hexagonal lattice. With the evolution of these structures, DS transition temperature T_{d} increases and a general explanation for this phenomenon is presented from the view of the network. The relationship between the transition temperature and the network measure-clustering coefficient is set up and it is shown that the increase of DS transition temperature is the result of more and more clustering of the network. In §5, we construct the third set of network structures-random graphs with Poisson degree distributions by changing the average degree of the network. We show that the increase in the average degree $\langle k \rangle$ is equivalent to the clustering of the nodes, leading to the increase of DS transition temperature T_{d}.

Acknowledgments

Stimulating discussions with K Y Szeto are gratefully acknowledged. This work was supported by the Inner Mongolia Natural Science Foundation (Grant No. 200408020106) and the Educational Committee of Inner Mongolia Autonomous Region (No. NJ04033).

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