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Painlevé test for integrability for a combination of Yang's self-dual equations for $SU(2)$ gauge fields and Charap's equations for chiral invariant model of pion dynamics and a comparative discussion among the three

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Abstract. Painlevé test for integrability for the combined equations generated from Yang's self-dual equations for $SU(2)$ gauge fields and Charap's equations for chiral invariant model of pion dynamics faces some peculiar situations that allow none of the stages (leading order analysis, resonance calculation and checking of the existence of the requisite number of arbitrary functions) to be conclusive. It is also revealed from a comparative study with the previous results that the existence of abnormal behaviour at any of the stated stages may have a correlation with the existence of chaotic property or some other properties that do not correspond to solitonic behaviour.

Keywords. Painlevé analysis; integrability; $SU(2)$ gauge field; chaotic property; solitonic behaviour.

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1. Introduction

This communication reports Painlevé test for integrability as proposed by Weiss *et al* [1] for a set of equations which is a combination of Yang's self-dual equation for $SU(2)$ gauge fields [2] and Charap's equations for chiral invariant model of pion dynamics [3]. This combination was proposed by the present authors [4].

In this report the Painlevé test has been executed for a particular set of values of the coupling constants that combine the Yang's equations [2] and Charap's equations [3].

The combined equation as proposed by Chakraborty and Chanda [4] is given as

$$\begin{aligned}
&\phi_{11} + \phi_{22} + \phi_{33} + \varepsilon\phi_{44} \\
&\quad = k'[(1/\phi)(\phi_1^2 + \phi_2^2 + \phi_3^2 + \varepsilon\phi_4^2) \\
&\quad - (1/\phi)(\psi_1^2 + \psi_2^2 + \psi_3^2 + \varepsilon\psi_4^2) - (1/\phi)(\chi_1^2 + \chi_2^2 + \chi_3^2 + \varepsilon\chi_4^2) \\
&\quad - (2/\phi)(\psi_1\chi_2 - \psi_2\chi_1 + \psi_4\chi_3 - \psi_3\chi_4)] \\
&\quad + k''\{2\phi[\exp(-\beta)](\phi_1^2 + \phi_2^2 + \phi_3^2 + \varepsilon\phi^2) \\
&\quad + 2\psi[\exp(-\beta)](\phi_1\psi_1 + \phi_2\psi_2 + \phi_3\psi_3 + \varepsilon\phi_4\psi_4) \\
&\quad + 2\chi[\exp(-\beta)](\phi_1\chi_1 + \phi_2\chi_2 + \phi_3\chi_3 + \varepsilon\phi_4\chi_4)\} \qquad (1.2a)
\end{aligned}$$

$$\begin{aligned}
&\psi_{11} + \psi_{22} + \psi_{33} + \varepsilon\psi_{44} \\
&\quad = k'[(2/\phi)(\phi_1\psi_1 + \phi_2\psi_2 + \phi_3\psi_3 + \varepsilon\phi_4\psi_4) \\
&\quad + (2/\phi)(\phi_1\chi_2 - \phi_2\chi_1 + \phi_4\chi_3 - \phi_3\chi_4)] \\
&\quad + k''\{2\psi[\exp(-\beta)](\psi_1^2 + \psi_2^2 + \psi_3^2 + \varepsilon\psi_4^2) \\
&\quad + 2\phi[\exp(-\beta)](\phi_1\psi_1 + \phi_2\psi_2 + \phi_3\psi_3 + \varepsilon\phi_4\psi_4) \\
&\quad + 2\chi[\exp(-\beta)](\psi_1\chi_1 + \psi_2\chi_2 + \psi_3\chi_3 + \varepsilon\psi_4\chi_4)\} \qquad (1.2b)
\end{aligned}$$

$$\begin{aligned}
&\chi_{11} + \chi_{22} + \chi_{33} + \varepsilon\chi_{44} \\
&\quad = k'[(2/\phi)(\phi_1\chi_1 + \phi_2\chi_2 + \phi_3\chi_3 + \varepsilon\phi_4\chi_4) \\
&\quad + (2/\phi)(\phi_2\psi_1 - \phi_1\psi_2 + \phi_3\psi_4 - \phi_4\psi_3) \\
&\quad + k''\{2\chi[\exp(-\beta)](\chi_1^2 + \chi_2^2 + \chi_3^2 + \varepsilon\chi_4^2) \\
&\quad + 2\phi[\exp(-\beta)](\phi_1\chi_1 + \phi_2\chi_2 + \phi_3\chi_3 + \varepsilon\phi_4\chi_4) \\
&\quad + 2\psi[\exp(-\beta)](\psi_1\chi_1 + \psi_2\chi_2 + \psi_3\chi_3 + \varepsilon\psi_4\chi_4)\} \qquad (1.2c)
\end{aligned}$$

where $\beta = \ln(f_\pi^2 + \phi^2 + \psi^2 + \chi^2), f_\pi$ is a constant and k' and k'' are arbitrary coupling constants and $\phi_1 \equiv (\partial\phi/\partial x^1)$, $\phi_{11} \equiv [\partial^2\phi/\partial(x^1)^2]$, $\psi_1 \equiv (\partial\psi/\partial x^1)$, $\psi_{11} \equiv [\partial^2\psi/\partial(x^1)^2]$, $\chi_1 \equiv (\partial\chi/\partial x^1)$, $\chi_{11} \equiv [\partial^2\chi/\partial(x^1)^2]$ etc.

Equations (1.2) reduce to Yang's self-dual equations for $SU(2)$ gauge fields when $\varepsilon = 1, k' = 1, k'' = 0$ and to Charap's equations when $\varepsilon = -1, k' = 0, k'' = 1$.

For convenience, Chakraborty and Chanda [4] called eqs (1.2) with $\varepsilon = 1, k' = 1, k'' = 0$ as the 'Yang's equations' and eqs (1.2) with $\varepsilon = -1$, $k' = 0$, $k'' = 1$ as the 'Charap's equations'. Equations (1.2) without any restriction on ε, k' and k'' are called as the 'combined Yang–Charap (Y–C) equations'. For $\varepsilon = 1, k' \neq 0, k'' \neq 0$ eqs (1.2) are 'extended Yang's equations'. And for $\varepsilon = -1, k' \neq 0, k'' \neq 0$ eqs (1.2) are 'extended Charap's equations'.

In the same publication [4], the present authors gave some exact solutions of this combined equation (for $k' = 1$ and $k'' = 1$) and rediscovered the same type of exact solutions for Yang's equations [2] and Charap's equations [3]. In another publication [5] (again for $k' = 1$ and $k'' = 1$) they reported some graphical representations of all those exact solutions. Therefrom different physical characteristics are observable. They are as follows:

(i) Spreading wave with solitary profile. The profile tends to vanish as time tends to infinity (in the case of Yang's equations [2] and combined (extended Yang) equations [4]).

(ii) Solitary wave with oscillatory profile (Charap's equations [3]).

(iii) Localized wave with solitary profile which becomes plane wave periodically and abruptly and wave packet/s which are oscillatory in nature and become zero periodically and abruptly (combined (extended Charap) equation [4]).

(iv) All the solutions are basically localised in character.

In a recent publication [6] the present authors revisited the Painlevé test for the integrability as proposed by Weiss *et al* [1] of the Yang's equations for $SU(2)$ gauge fields. Jimbo *et al* [7] analyzed the complex form of the equations with a rather restricted form of singularity manifold. They did not discuss about exact solutions in that context. But, the present authors analyzed the same equations starting from the real form of the same equations keeping the singularity manifold completely general in nature. It has been found that the equations, in real form, pass the Painlevé test for integrability. The truncation procedure of the same analysis leads to non-trivial exact solutions obtained previously and auto-Backlund transformation [8] between two pairs of those solutions.

In another recent publication [9], the present authors demonstrate that the field equations for Charap's chiral invariant model of the pion dynamics pass the Painlevé test for complete integrability as proposed by Weiss *et al* [1]. The truncation procedure of the same analysis leads to auto-Backlund transformation [8] between two pairs of solutions. With the help of this transformation non-trivial exact solutions obtained previously have been rediscovered.

In this report we apply the same procedure as in [6] and [9]. While doing the Painlevé test for integrability for the combined equation it appears that some new mathematical behaviours, not known in the literature, are seen. It is revealed from a comparative study with the previous results that the existence of abnormal behaviour at any stages of Painlevé test for integrability as proposed by Weiss *et al* [1] may have a correlation with the existence of chaotic property or some other properties that do not correspond to solitonic behaviour.

2. Painlevé test for integrability of the combined equations

The results presented here are valid for $k' = 1, k'' = 1$ only for eq. (1.2). One has to investigate separately to ascertain whether similar behaviour will be seen for other values of k' and k''. Again the results reported here remain independent of ε. Or in other words, the observations of this analysis are the same for extended Yang's equations and extended Charap's equations.

For eqs (1.2) with $k' = 1, k'' = 1$,we define the singularity manifold given by

$$u = u(x^1, x^2, x^3, x^4) = 0 \tag{2.1}$$

and set

$$\phi = u^{\alpha} \sum_{j=0}^{\infty} \phi_j u^j, \quad \psi = u^{\beta} \sum_{j=0}^{\infty} \psi_j u^j, \quad \chi = u^{\gamma} \sum_{j=0}^{\infty} \chi_j u^j \tag{2.2a, b, c}$$

where $\phi(x^1, x^2, x^3, x^4)$, $\psi(x^1, x^2, x^3, x^4)$, $\chi(x^1, x^2, x^3, x^4)$ are a set of solutions of (1.2) with $k' = 1$ and $k'' = 1$; ϕ_j, ψ_j, χ_j are all analytic functions of (x^1, x^2, x^3, x^4) in the neighborhood of the manifold (2.1); $\phi_0 \neq 0, \psi_0 \neq 0, \chi_0 \neq 0$.

The test may be divided into three main steps after the substitution of (2.2) in the differential equations concerned, i.e. (1.2) with $k' = 1$ and $k'' = 1$.

(i) Make the leading order analysis (where one gets all possible α, β, γ, ϕ_0, ψ_0 and χ_0 in (2.2)).

(ii) Define the recursion relations for ϕ_j, ψ_j, χ_j for leading orders obtained in step I and determine the resonance positions (those values of j for which the system matrix, defined in eqs (2.8) and (2.9) vanishes). $j = -1$ is always a resonance point [10]. It indicates that the singularity manifold defined by (2.1) is arbitrary.

(iii) Check whether the expansions allow requisite number of arbitrary functions at the resonance positions.

2.1 *Leading order analysis*

We assume

$$\phi \sim \phi_0 u^{\alpha}, \quad \psi \sim \psi_0 u^{\beta}, \quad \chi \sim \chi_0 u^{\gamma}. \tag{2.3a, b, c}$$

We substitute (2.3a,b,c) in (1.2a,b,c) respectively and equate the coefficients of the negative powers of u (considering that all α, β and γ are negative). This leads to $\alpha = \beta = \gamma$.

From the leading order terms of eqs (1.2a,b,c) one gets

$$(\phi_0^2 + \psi_0^2 + \chi_0^2)[-\alpha(2\alpha + 1)\phi_0^2 + \alpha^2\psi_0^2 + \alpha^2\chi_0^2] = 0 \tag{2.4a}$$

$$\phi_0\psi_0(\phi_0^2 + \psi_0^2 + \chi_0^2)[\alpha(\alpha - 1) - 4\alpha^2] = 0 \tag{2.4b}$$

$$\phi_0\chi_0(\phi_0^2 + \psi_0^2 + \chi_0^2)[\alpha(\alpha - 1) - 4\alpha^2] = 0. \tag{2.4c}$$

From (2.4) there appear two possibilities. The first possibility appears to be

$$(\phi_0^2 + \psi_0^2 + \chi_0^2) = 0 \tag{2.5}$$

and the second possibility appears to be the combination of three equations given by

$$[-\alpha(2\alpha + 1)\phi_0^2 + \alpha^2\psi_0^2 + \alpha^2\chi_0^2] = 0 \tag{2.6a}$$

$$[\alpha(\alpha - 1) - 4\alpha^2] = 0 \tag{2.6b}$$

$$[\alpha(\alpha - 1) - 4\alpha^2] = 0. \tag{2.6c}$$

It is interesting to note that eqs (2.6b,c) lead to $\alpha = -1/3$ and if one puts this value (i.e. $\alpha = -1/3$) in eq. (2.6a), one again arrives at $(\phi_0^2 + \psi_0^2 + \chi_0^2) = 0$, which is the same as (2.5)

So finally we get,

$$\phi = u^\alpha \sum_{j=0}^{\infty} \phi_j u^j, \quad \psi = u^\alpha \sum_{j=0}^{\infty} \psi_j u^j, \quad \chi = u^\alpha \sum_{j=0}^{\infty} \chi_j u^j \tag{2.7a, b, c}$$

with

$$(\phi_0^2 + \psi_0^2 + \chi_0^2) = 0, \quad \alpha = \text{ arbitrary}. \tag{2.7d, e}$$

2.2 *Resonance positions*

If we directly substitute (2.7a,b,c) in eqs (1.2) and then equate the coefficients of powers of u in the various terms and thereby we observe the behaviour of the expansion coefficients, then the recursion relation for ϕ_j, ψ_j and χ_j are

$$[T] \begin{bmatrix} \phi_m \\ \psi_m \\ \chi_m \end{bmatrix} = [\text{other terms with } \phi_j,\ \psi_j,\ \chi_j \text{ and their derivatives where } j < m], \tag{2.8}$$

where $[T]$ is the system matrix and it is written as

$$[T] = \begin{bmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{bmatrix}, \tag{2.9}$$

where

$$\begin{aligned} A_1 = {} & \{(m+\alpha)(m+\alpha-1) - 6\alpha(m+\alpha) + 3\alpha(\alpha-1) - 6\alpha^2\}\phi_0^3 \\ & + \{(m+\alpha)(m+\alpha-1) - 4\alpha(m+\alpha) + \alpha(\alpha-1)\}\phi_0\psi_0^2 \\ & + \{(m+\alpha)(m+\alpha-1) - 4\alpha(m+\alpha) + \alpha(\alpha-1)\}\phi_0\chi_0^2, \\ B_1 = {} & \{2\alpha(m+\alpha) + 2\alpha^2\}\psi_0^3 + \{2\alpha(\alpha-1) - 4\alpha^2\}\phi_0^2\psi_0 \\ & + \{2\alpha(m+\alpha) + 2\alpha^2\}\phi_0\chi_0^2, \\ C_1 = {} & \{2\alpha(m+\alpha) + 2\alpha^2\}\chi_0^3 + \{2\alpha(\alpha-1) - 4\alpha^2\}\phi_0^2\chi_0 \\ & + \{2\alpha(m+\alpha) + 2\alpha^2\}\psi_0^2\chi_0, \end{aligned}$$

$$\begin{aligned} A_2 = {} & \{\alpha(\alpha-1) - 2\alpha(m+\alpha) - 2\alpha^2\}\psi_0^3 \\ & + \{3\alpha(\alpha-1) - 4\alpha(m+\alpha) - 8\alpha^2\}\phi_0^2\psi_0 \\ & + \{\alpha(\alpha-1) - 2\alpha(m+\alpha) - 2\alpha^2\}\psi_0\chi_0^2, \end{aligned}$$

$$
\begin{aligned}
B_2 &= \{(m+\alpha)(m+\alpha-1) - 4\alpha(m+\alpha)\}\phi_0^3 \\
&\quad + \{(m+\alpha)(m+\alpha-1) - 6\alpha(m+\alpha) + 2\alpha(\alpha-1) - 6\alpha^2\}\phi_0\psi_0^2 \\
&\quad + \{(m+\alpha)(m+\alpha-1) - 4\alpha(m+\alpha)\}\phi_0\chi_0^2, \\
C_2 &= \{2\alpha(\alpha-1) - 2\alpha(m+\alpha) - 6\alpha^2\}\phi_0\psi_0\chi_0, \\
A_3 &= \{\alpha(\alpha-1) - 2\alpha(m+\alpha) - 2\alpha^2\}\chi_0^3 \\
&\quad + \{3\alpha(\alpha-1) - 4\alpha(m+\alpha) - 8\alpha^2\}\phi_0^2\chi_0 \\
&\quad + \{\alpha(\alpha-1) - 2\alpha(m+\alpha) - 2\alpha^2\}\psi_0^2\chi_0, \\
B_3 &= \{2\alpha(\alpha-1) - 2\alpha(m+\alpha) - 6\alpha^2\}\phi_0\psi_0\chi_0, \\
C_3 &= \{(m+\alpha)(m+\alpha-1) - 4\alpha(m+\alpha)\}\phi_0^3 \\
&\quad + \{(m+\alpha)(m+\alpha-1) - 4\alpha(m+\alpha)\}\phi_0\psi_0^2 \\
&\quad + \{(m+\alpha)(m+\alpha-1) - 6\alpha(m+\alpha) + 2\alpha(\alpha-1) - 6\alpha^2\}\phi_0\chi_0^2.
\end{aligned}
$$

Applying (2.7d), i.e. $(\phi_0^2 + \psi_0^2 + \chi_0^2) = 0$, the system matrix $[T]$ reduces to a simpler form given by

$$
[T] = \begin{bmatrix} G\phi_0^3 & G\phi_0^2\psi_0 & G\phi_0^2\chi_0 \\ G\phi_0^2\psi_0 & G\phi_0\psi_0^2 & G\phi_0\psi_0\chi_0 \\ G\phi_0^2\chi_0 & G\phi_0\psi_0\chi_0 & G\phi_0\chi_0^2 \end{bmatrix}, \tag{2.10}
$$

where $G = -[6\alpha^2 + 2\alpha(m+1)]^3$. Obviously, the determinants for the matrix become zero irrespective of the values of α and m.

Because the determinant takes the form

$$
|T| = -[6\alpha^2 + 2\alpha(m+1)]^3\phi_0\psi_0\chi_0 \begin{vmatrix} \phi_0^2 & \phi_0\psi_0 & \phi_0\chi_0 \\ \phi_0^2 & \phi_0\psi_0 & \phi_0\chi_0 \\ \phi_0^2 & \phi_0\psi_0 & \phi_0\chi_0 \end{vmatrix},
$$

we get no information about the resonance position. Only, we can say that so long as the singularity manifold defined by $u = 0$ is arbitrary we have $m = -1$ to be a resonance position [10]. Again it appears by virtue of (2.7d) that any two coefficients of ϕ_0, ψ_0, χ_0 can be kept arbitrary. This is represented as there is a possibility of having resonance at $m = (0, 0)$. However, this has to be confirmed in the next section.

2.3 *Further investigation for the resonance position*

For m=1: Along with $(\phi_0^2 + \psi_0^2 + \chi_0^2) = 0$ in (2.9) we arrive at a situation for the three equations of (1.1a,b,c) with $k' = 1$ and $k'' = 1$ given by

$$
A\phi_1 + B\psi_1 + C\chi_1 = f'/\phi_0, \tag{2.11a}
$$
$$
A\phi_1 + B\psi_1 + C\chi_1 = f''/\psi_0, \tag{2.11b}
$$
$$
A\phi_1 + B\psi_1 + C\chi_1 = f'''/\chi_0, \tag{2.11c}
$$

where f', f'' and f''' are functions of ϕ_0, ψ_0 and χ_0 and their derivatives. And

$$A = [-(6\alpha^2 + 4\alpha)^3 \phi_0]\phi_0$$
$$B = [-(6\alpha^2 + 4\alpha)^3 \phi_0]\psi_0$$
$$C = [-(6\alpha^2 + 4\alpha)^3 \phi_0]\chi_0.$$

On simple manipulation we get from (2.11)

$$\frac{f'}{\phi_0} = \frac{f''}{\psi_0} = \frac{f'''}{\chi_0} = D \text{ (say)}, \tag{2.12}$$

where

$$D = [1/(u_1^2 + u_2^2 + u_3^2 + u_4^2)][(\phi_{01}u_1 + \phi_{02}u_2 + \phi_{03}u_3 + \phi_{04}u_4)(2\phi_0^2)$$
$$+(\psi_{01}u_1 + \psi_{02}u_2 + \psi_{03}u_3 + \psi_{04}u_4)(2\alpha\phi_0\psi_0)$$
$$+(\chi_{01}u_1 + \chi_{02}u_2 + \chi_{03}u_3 + \chi_{04}u_4)(2\alpha\phi_0\chi_0)].$$

Thus, (2.11a,b,c) become identical and we have the possibility of getting resonance at $m = (1, 1)$. This means that there is a possibility of two of the three coefficients ϕ_1, ψ_1 and χ_1 being arbitrary. At this stage we can say that the resonance at $m = (0, 0)$ is confirmed, which means that two of the three coefficients ϕ_0, ψ_0 and χ_0 are proved to be arbitrary.

For m=2: Along with $(\phi_0^2 + \psi_0^2 + \chi_0^2) = 0$ in (2.9) we arrive at a situation for the three equations of (1.2a,b,c) with $k' = 1$ and $k'' = 1$ given by

$$A'\phi_2 + B'\psi_2 + C'\chi_2 = g'/\phi_0, \tag{2.13a}$$
$$A'\phi_2 + B'\psi_2 + C'\chi_2 = g''/\psi_0, \tag{2.13b}$$
$$A'\phi_2 + B'\psi_2 + C'\chi_2 = g'''/\chi_0, \tag{2.13c}$$

where g', g'' and g''' are functions of ϕ_1, ψ_1, χ_1, ϕ_0, ψ_0 and χ_0 and their derivatives, and

$$A' = [-(6\alpha^2 + 6\alpha)\phi_0]\phi_0$$
$$B' = [-(6\alpha^2 + 6\alpha)\phi_0]\psi_0$$
$$C' = [-(6\alpha^2 + 6\alpha)\phi_0]\chi_0.$$

Thus,

$$g'/\phi_0 = [1/(u_1^2 + u_2^2 + u_3^2 + u_4^2)][(\phi_0 p + \psi_0 q + \chi_0 r)$$
$$\times(-2\alpha\phi_0^2)(u_{11} + u_{22} + u_{33} + u_{44})$$
$$+2\alpha\phi_0^3(u_1p_1 + u_2p_2 + u_3p_3 + u_4p_4)$$
$$+2\alpha\phi_0^2\psi_0(u_1q_1 + u_2q_2 + u_3q_3 + u_4q_4)$$
$$+2\alpha\phi_0^2\chi_0(u_1r_1 + u_2r_2 + u_3r_3 + u_4r_4)$$
$$+\{(2\alpha - 1)\phi_0^2 p + (4\alpha + 2)\phi_0(\phi_0 p + \psi_0 q + \chi_0 r)\}$$
$$\times(\phi_{01}u_1 + \phi_{02}u_2 + \phi_{03}u_3 + \phi_{04}u_4)$$
$$+\{2\phi_0\psi_0 p - 4\alpha\psi_0(\phi_0 p + \psi_0 q + \chi_0 r)\}$$
$$\times(\psi_{01}u_1 + \psi_{02}u_2 + \psi_{03}u_3 + \psi_{04}u_4)$$

$$+\{2\alpha\phi_0(2\chi_0 p+\phi_0 r)+2\phi_0\chi_0 p-4\alpha\chi_0(\phi_0 p+\psi_0 q+\chi_0 r)\}$$
$$\times(\chi_{01}u_1+\chi_{02}u_2+\chi_{03}u_3+\chi_{04}u_4)$$
$$+2\phi_0^2(\phi_{01}^2+\phi_{02}^2+\phi_{03}^2+\phi_{04}^2)$$
$$+2\phi_0\psi_0(\phi_{01}\psi_{01}+\phi_{02}\psi_{02}+\phi_{03}\psi_{03}+\phi_{04}\psi_{04})$$
$$+2\phi_0\chi_0(\phi_{01}\chi_{01}+\phi_{02}\chi_{02}+\phi_{03}\chi_{03}+\phi_{04}\chi_{04})],$$

$$g''/\psi_0=[1/(u_1^2+u_2^2+u_3^2+u_4^2)][(\phi_0 p+\psi_0 q+\chi_0 r)$$
$$\times(-2\alpha\phi_0\psi_0)(u_{11}+u_{22}+u_{33}+u_{44})$$
$$+2\alpha\phi_0^2\psi_0(u_1p_1+u_2p_2+u_3p_3+u_4p_4)$$
$$+2\alpha\phi_0\psi_0^2(u_1q_1+u_2q_2+u_3q_3+u_4q_4)$$
$$+2\alpha\phi_0\psi_0\chi_0(u_1r_1+u_2r_2+u_3r_3+u_4r_4)$$
$$+\{4\alpha\psi_0(\phi_0 p+\psi_0 q+\chi_0 r)+4\alpha\phi_0\psi_0 p$$
$$+2(\alpha+1)\phi_0^2 q\}(\phi_{01}u_1+\phi_{02}u_2+\phi_{03}u_3+\phi_{04}u_4)$$
$$+\{(2\alpha+2)\phi_0^2 p-2\alpha\chi_0^2 p+4\alpha\phi_0\psi_0 q$$
$$+(4\alpha+2)\phi_0\chi_0 r)\}(\psi_{01}u_1+\psi_{02}u_2+\psi_{03}u_3+\psi_{04}u_4)$$
$$+\{2\alpha\psi_0\chi_0 p+2\alpha\phi_0\chi_0 q+2(\alpha+1)\phi_0\psi_0 r\}$$
$$\times(\chi_{01}u_1+\chi_{02}u_2+\chi_{03}u_3+\chi_{04}u_4)$$
$$+2\phi_0^2(\phi_{01}\psi_{01}+\phi_{02}\psi_{02}+\phi_{03}\psi_{03}+\phi_{04}\psi_{04})$$
$$+2\phi_0\psi_0(\psi_{01}^2+\psi_{02}^2+\psi_{03}^2+\psi_{04}^2)$$
$$+2\phi_0\chi_0(\phi_{01}\chi_{01}+\phi_{02}\chi_{02}+\phi_{03}\chi_{03}+\phi_{04}\chi_{04})],$$

$$g'''=[1/(u_1^2+u_2^2+u_3^2+u_4^2)][(\phi_0 p+\psi_0 q+\chi_0 r)$$
$$\times(-2\alpha\phi_0\chi_0)(u_{11}+u_{22}+u_{33}+u_{44})$$
$$+2\alpha\phi_0^2\chi_0(u_1p_1+u_2p_2+u_3p_3+u_4p_4)$$
$$+2\alpha\phi_0\psi_0\chi_0(u_1q_1+u_2q_2+u_3q_3+u_4q_4)$$
$$+2\alpha\phi_0\chi_0^2(u_1r_1+u_2r_2+u_3r_3+u_4r_4)$$
$$+\{4\alpha\chi_0(\phi_0 p+\psi_0 q+\chi_0 r)+4\alpha\phi_0\chi_0 p$$
$$+2(\alpha+1)^2\phi_0^2 r\}(\phi_{01}u_1+\phi_{02}u_2+\phi_{03}u_3+\phi_{04}u_4)$$
$$+\{2\alpha\psi_0\chi_0 p+2\alpha\phi_0\chi_0 q+2(\alpha+1)\phi_0\psi_0 r\}$$
$$\times(\psi_{01}u_1+\psi_{02}u_2+\psi_{03}u_3+\psi_{04}u_4)$$
$$+\{2\alpha p(2\phi_0^2+\chi_0^2)+2\phi_0^2 p+4\alpha\phi_0\chi_0 r+(2\alpha+4)\phi_0\psi_0 q)\}$$
$$\times(\chi_{01}u_1+\chi_{02}u_2+\chi_{03}u_3+\chi_{04}u_4)$$
$$+2\phi_0^2(\phi_{01}\chi_{01}+\phi_{02}\chi_{02}+\phi_{03}\chi_{03}+\phi_{04}\chi_{04})$$
$$+2\phi_0\psi_0(\psi_{01}\chi_{01}+\psi_{02}\chi_{02}+\psi_{03}\chi_{03}+\psi_{04}\chi_{04})$$
$$+2\phi_0\chi_0(\chi_{01}^2+\chi_{02}^2+\chi_{03}^2+\chi_{04}^2)],$$

where (i) the coefficients ϕ_1, ψ_1, χ_1 in (2.2a,b,c) are rewritten as p, q, r respectively in order to differentiate them from $(\partial\phi/\partial x^1)$, $(\partial\psi/\partial x^1)$, $(\partial\chi/\partial x^1)$ etc. and (ii) $p_1 \equiv (\partial p/\partial x^1), q_1 \equiv (\partial q/\partial x^1), r_1 \equiv (\partial r/\partial x^1)$ etc.

So, here we see that,

$$\frac{g'}{\phi_0} \neq \frac{g''}{\psi_0} \neq \frac{g'''}{\chi_0}.$$

However, the left-hand sides of (2.13) are identical and hence one should forcefully have

$$\frac{g'}{\phi_0} = \frac{g''}{\psi_0} = \frac{g'''}{\chi_0} \tag{2.14}$$

which generates two equations in $\phi_1, \psi_1, \chi_1, \phi_0, \psi_0, \chi_0$ and their derivatives.

So finally one has three distinct equations in ϕ_1, ψ_1 and χ_1 – one is given by (2.13a) and the other two are given by (2.14). Thus, none of the three coefficients ϕ_1, ψ_1, χ_1 can be kept arbitrary. Thus, there cannot be any resonance at $m = 1$.

Thus even at this stage we do not have any conclusive inference regarding leading order, resonance position and existence of requisite number of arbitrary functions. In this way one has to proceed indefinitely. For the same reason it has not been possible to truncate the series and to obtain auto-Backlund transformation which was done in our previous publications [5] and [7] related to Yang's equations for $SU(2)$ gauge fields and Charap's equations for chiral invariant model of pion dynamics.

3. Summary

We arrive at peculiar situations regarding the Painlevé test (as proposed by Weiss *et al*) for the combined equations given by (1.2a,b,c), of course, with $k' = 1, k'' = 1$. They do not allow any of the stages (leading order analysis, resonance calculation and checking the existence of requisite number of arbitrary functions) to be conclusive.

Here we represent observations of the Painlevé test and graphical representations of exact solutions for the Yang's equations, Charap's equations and their combined form (1.2a,b,c) in a tabular form (see table 1).

From the above we can add to the observation of several other authors (see for example [11]) that there is

(i) a correlation between the existence of Painlevé property and the absence of chaotic behaviour, and

(ii) a correlation between the absence of Painlevé property and the presence of chaotic behaviour.

Normally the existence of Painlevé property is defined by the presence of a requisite number of arbitrary functions in the Laurent-like expansion (1.1) at the resonance points. Our observation is that this does not inform one about the existence of physical behaviour free from chaos or specifically the existence of solitonic behaviour. In this context we like to introduce a conjecture that is given as follows: The basic requirement for the existence of the regular behaviour as stated above is the existence of a requisite number of arbitrary functions at the resonance points in the Laurent-like expansion used for the Painlevé test. In addition to that, the

Table 1.

	Characteristics			
	Leading order analysis	Resonance calculation	Existence of number of arbitrary functions	Graphical representation of exact solution
Yang's equation (eq. (1.2) with $\varepsilon = 1, k' = 1, k'' = 0$)	One arrives at $\phi_0^2 + \psi_0^2 + \chi_0^2 = 0$ (*)	Clear result	Clear existence	Initial solitary profile which tends to vanish as time tends to infinity and spreading wave packet
Extended Yang's equation (eq. (1.2) with $\varepsilon = 1, k' = 1, k'' = 1$)	One arrives at $\phi_0^2 + \psi_0^2 + \chi_0^2 = 0$ (*)	Not conclusive	Not conclusive	Initial solitary profile which tends to vanish as time tends to infinity and spreading wave packet
Charap's equation (eq. (1.2) with $\varepsilon = 1, k' = 0, k'' = 1$)	Clear result	Clear result	Clear existence	Existence of solitonic solution with oscillatory profile
Extended Charap's equation (eq. (1.2) with $\varepsilon = 1, k' = 1, k'' = 1$)	One arrives at $\phi_0^2 + \psi_0^2 + \chi_0^2 = 0$ (*)	Not conclusive	Not conclusive	Solitary profile (for and packet (ϕ) for ψ and χ) that pass through stages of plane wave and zero value abruptly

*This is valid only when ϕ_0, ψ_0, χ_0 are complex quantities. One cannot execute the truncation procedure with this result.

Laurent-like expansion must be well-behaved at all the stages – namely, leading order analysis, resonance calculation and checking of the existence of the requisite number of arbitrary functions.

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