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Linear delta expansion technique for the solution of anharmonic oscillations

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Abstract. The linear delta expansion technique has been developed for solving the differential equation of motion for symmetric and asymmetric anharmonic oscillators. We have also demonstrated the sophistication and simplicity of this new perturbation technique.

Keywords. Linear delta expansion technique; case study.

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We have presented the linear delta expansion (LDE) technique [1] for the resolution of oscillatory non-linear problems. This is a powerful technique which has been originally introduced to deal with problems of strong coupling in quantum field theory. This method has also been applied to a wide class of problems [2–9]. In the original formulation of LDE technique, the Lagrangian density $\mathcal{L}_\delta$ which is not exactly solvable, is interpolated with a solvable Lagrangian $\mathcal{L}_0$. The full Lagrangian was written as $\mathcal{L}_\delta = \mathcal{L}_0 + \delta\mathcal{L}$. For $\delta = 0$, one obtains solvable Lagrangian $\mathcal{L}_0$. Here $\delta\mathcal{L}$ is treated as a perturbation and δ is used to keep track of the perturbative order.

In the theory of harmonics, there is an important phenomenon which should be pointed out because of its practical importance in demonstrating non-linear effects. The thermal expansion of a crystalline solid can be understood from the non-linear force acting between the atoms. The restoring force between a pair of atoms is asymmetric.

For large amplitude of vibration, the restoring force often contains additional terms involving higher power of displacement x. In such case, the restoring force is expressed as

$$F(x) = -(sx + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots), \tag{1}$$

where s, a_2, a_3, a_4, a_5 etc. are constants. For this case, the oscillator becomes non-linear and the vibration contains a fundamental frequency and higher harmonics. The oscillations of a non-linear oscillator are called anharmonic oscillations. The differential equation of motion for this type of oscillation is

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}+\omega^2x=-\frac{1}{m}(a_2x^2+a_3x^3+a_4x^4+a_5x^5+\cdots), \tag{2}$$

which is obtained by splitting the given force (1) into two parts. The main part corresponds to the simple harmonic motion, for which the solution is known analytically and the other term will be treated as a perturbation. Here, the oscillating frequency $\omega=\sqrt{s/m}$, where m is the mass of the oscillating particle. This type of non-linear equation represents both symmetric and asymmetric vibrations depending on the powers of x in (2). Therefore, for a system to execute asymmetric vibration of large amplitude, the correction terms containing x^2, x^4, x^6 etc. and for symmetric vibration of large amplitude x^3, x^5, x^7 etc. should be considered.

Equation (2) can be rewritten as a non-linear ordinary differential equation of motion as

$$(\mathrm{d}^2x(t)/\mathrm{d}t^2)+\omega^2x(t)=f(x(t)). \tag{3}$$

This equation describes a conservative system, oscillating with an unknown period T. Here, the non-linear term

$$f(x(t))=-\frac{1}{m}(a_2x^2+a_3x^3+a_4x^4+a_5x^5+\cdots)=-\frac{1}{m}\Sigma_i a_{i+1}x^{i+1}, \tag{4}$$

is treated as a perturbation. In LDE technique, we can write (3) as

$$(\mathrm{d}^2x(t)/\mathrm{d}t^2)+\omega^2x(t)=\delta(f(x(t))). \tag{5}$$

Clearly, for $\delta=1$, eq. (5) takes the form of (3). In LDE formalism, we have expanded the functions x and ω related to the perturbation in the power of δ as $x(t)=\Sigma_{j=0}^{\infty}\delta^j x_j(t)$ and $\omega=\Sigma_{j=0}^{\infty}\delta^j\omega_j$. These expansions are basically of Lindstedt–Poincare (LP) type [10]. Substituting the above expansions of $x(t)$ and ω into (5) we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\sum_{j=0}^{\infty}\delta^j x_j(t)\right)+\left(\sum_{j=0}^{\infty}\delta^j\omega_j\right)^2\left(\sum_{j=0}^{\infty}\delta^j x_j(t)\right)=-\delta\left(\frac{1}{m}\sum_{i=1}a_{i+1}\left(\sum_{j=0}^{\infty}\delta^j x_j(t)\right)^{i+1}\right). \tag{6}$$

This equation of motion has a more general form than the available perturbation theories for these types of problems. In eq. (6), the unperturbed equations are linear and perturbed equations contain nonlinear terms. When expanded in a perturbation series, one obtains linear non-homogeneous equations to be solved in order. Actually, the system is not exactly coupled and can be solved in hierarchy starting from the unperturbed equation. The non-homogeneous term is a known function but different at each order of approximation. In this article, we will show through the applications that the feature of the present method leads to a simple framework in obtaining the corrections to the displacement x and oscillating frequency ω without using complicated and tedious mathematical procedures. This equation also produces a hierarchy of inhomogeneous (and linear) equations whose solutions provide approximate forms of the truly periodic motion at various orders. As in LP method, the secular terms, produced by driving forces with the fundamental frequencies, are eliminated by choosing the n-order correction of ω such that these resonant forms are cancelled out. In this manner, the n-order solution is expressed as a series of the kind $x_n(t) = \Sigma_{j=0}^{\infty} A_{n,j} \cos(j\omega_0 t)$, where ω_0 is the 0-order frequency.

Case I: Let us start with an asymmetric restoring force of quadratic form

$$F(x) = -sx - a_2 x^2. \tag{7}$$

Applying LDE technique, from (6) we obtained the equation of motion for this type of force as

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\sum_{j=0}^{\infty}\delta^j x_j(t)\right) + \left(\sum_{j=0}^{\infty}\delta^j \omega_j\right)^2 \left(\sum_{j=0}^{\infty}\delta^j x_j(t)\right) = -\delta\left(\alpha\left(\sum_{j=0}^{\infty}\delta^j x_j(t)\right)^2\right), \tag{8}$$

where $\alpha = a_2/m$ is known as non-linear parameter. Now, we want to solve this equation in the light of LDE technique to test its effectiveness. Equating terms with the same power of δ on both sides of (8) up to $O(\delta^2)$, we obtain one linear homogeneous (harmonic) and two inhomogeneous coupled equations as

$$\frac{\mathrm{d}^2 x_0}{\mathrm{d}t^2} + \omega_0^2 x_0 = 0, \tag{9}$$

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} + \omega_0^2 x_1 = -\alpha x_0^2 - 2\omega_0\omega_1 x_0 \quad \text{and} \tag{10}$$

$$\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} + \omega_0^2 x_2 = -2\alpha x_0 x_1 - 2\omega_0\omega_2 x_0 - 2\omega_0\omega_1 x_1 - \omega_1^2 x_0. \tag{11}$$

The solution of linear homogeneous eq. (9) is well-known in the literature and is $x_0 = a \cos \omega_0 t$, where a is the amplitude of vibrations.

For the first-order correction of the displacement, substituting the value of x_0 in (10), we get

$$\frac{\mathrm{d}^2 x_1}{\mathrm{d}t^2} + \omega_0^2 x_1 = -\frac{\alpha a^2}{2} - \frac{\alpha a^2}{2} \cos 2\omega_0 t - 2\omega_0\omega_1 a \cos \omega_0 t. \tag{12}$$

We consider the solution of (12) as

$$x_{n=1} = \sum_{j=0}^{2} A_{1,j} \cos j\omega_0 t. \tag{13}$$

Substituting the value of $x_{n=1}$ in (12) and equating the coefficients without cosine function and $\cos \omega_0 t$ and $\cos 2\omega_0 t$ on both sides, we obtain the solution with the help of boundary condition at $t = 0$, $x_1 = 0$ as

$$x_{n=1} = -\frac{\alpha a^2}{2\omega_0^2} + \frac{\alpha a^2}{3\omega_0^2} \cos \omega_0 t + \frac{\alpha a^2}{6\omega_0^2} \cos 2\omega_0 t. \tag{14}$$

Hence, the total displacement up to the first-order correction is

$$x(t) = a \cos \omega_0 t - \frac{\alpha a^2}{2\omega_0^2} + \frac{\alpha a^2}{3\omega_0^2} \cos \omega_0 t + \frac{\alpha a^2}{6\omega_0^2} \cos 2\omega_0 t, \tag{15}$$

which is the standard result [11]. This type of oscillation contains a fundamental frequency ω_0 and a second harmonic frequency $2\omega_0$. The frequency of oscillation $\omega = \sqrt{s/m}$.

Putting the value of x_0 and $x_{n=1}$ in (11), we get the differential equation for the second-order correction of the displacement as

$$\frac{\mathrm{d}^2 x_2}{\mathrm{d}t^2} + \omega_0^2 x_2 = -\frac{\alpha^2 a^3}{3\omega_0^2} + \frac{5}{6}\frac{\alpha^2 a^3}{\omega_0^2}\cos\omega_0 t - \frac{\alpha^2 a^3}{3\omega_0^2}\cos 2\omega_0 t - \frac{\alpha^2 a^3}{6\omega_0^2}\cos 3\omega_0 t - 2\omega_0\omega_2 a\cos\omega_0 t. \tag{16}$$

Let us consider the solution of (16) as

$$x_{n=2} = \sum_{j=0}^{3} B_{2,j}\cos j\omega_0 t. \tag{17}$$

With the same mathematical procedure as above, we obtain the second-order correction of frequency $\omega_2 = 5\alpha^2 a^3/12\omega_0^4$ and with the help of the boundary condition at $t = 0, x_2(t) = 0$, we obtain the second-order correction of the displacement as

$$x_{n=2}(t) = -\frac{\alpha^2 a^3}{3\omega_0^4} - \frac{29\alpha^2 a^3}{144\omega_0^4}\cos\omega_0 t + \frac{\alpha^2 a^3}{9\omega_0^4}\cos 2\omega_0 t + \frac{\alpha^2 a^3}{48\omega_0^4}\cos 3\omega_0 t. \tag{18}$$

It is observed that the total displacement with n-order correction contains power series of the parameter α. These results converge only for $|\alpha| < 1$.

Case II: We consider another asymmetric restoring force of the form

$$F(x) = -sx - a_4 x^4. \tag{19}$$

Applying LDE technique, we rewrite eq. (6) for the above asymmetric restoring force as

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\sum_{j=0}^{\infty}\delta^j x_j\right) + \left(\sum_{j=0}^{\infty}\delta^j \omega_j\right)^2\left(\sum_{j=0}^{\infty}\delta^j x_j\right) = -\delta\left(\beta\left(\sum_{j=0}^{\infty}\delta^j x_j\right)^4\right), \tag{20}$$

where the non-linear parameter $\beta = a_4/m$. With the same mathematical procedure, we get the first-order correction of oscillating frequency $\omega_1 = 0$ and also with the help of boundary condition at $t = 0$, $x_{n=1} = 0$, we obtain the first-order correction of the displacement as

$$x_{n=1} = -\frac{3\beta a^4}{8\omega_0^2} + \frac{\beta a^4}{5\omega_0^2}\cos\omega_0 t + \frac{\beta a^4}{6\omega_0^2}\cos 2\omega_0 t + \frac{\beta a^4}{120\omega_0^2}\cos 4\omega_0 t. \tag{21}$$

The second-order correction of the displacement is carried out in the same manner as before and we get

$$x_{n=2}(t) = -\frac{3\beta^2 a^7}{10\omega_0^4} + \frac{1435\beta^2 a^7}{8400\omega_0^4}\cos\omega_0 t + \frac{2\beta^2 a^7}{15\omega_0^4}\cos 2\omega_0 t - \frac{9\beta^2 a^7}{640\omega_0^4}\cos 3\omega_0 t + \frac{\beta^2 a^7}{150\omega_0^4}\cos 4\omega_0 t + \frac{\beta^2 a^7}{320\omega_0^4}\cos 5\omega_0 t + \frac{\beta^2 a^7}{8400\omega_0^4}\cos 7\omega_0 t, \tag{22}$$

and the second-order correction of oscillating frequency $\omega_2 = 189\beta^2 a^6/480\omega_0^3$. Here it is also observed that the total displacement with n-order correction contains power series of the parameter β. This result is expected to converge only for $|\beta| < 1$.

Case III: We consider the symmetric restoring force of the form

$$F(x) = -sx - a_2 x^3. \tag{23}$$

Applying the LDE technique, from (6) we get the differential equation for the above symmetric restoring force as

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\sum_{j=0}^{\infty}\delta^j x_j\right) + \left(\sum_{j=0}^{\infty}\delta^j \omega_j\right)^2 \left(\sum_{j=0}^{\infty}\delta^j x_j\right) = -\delta\left(\gamma\left(\sum_{j=0}^{\infty}\delta^j x_j\right)^3\right), \tag{24}$$

where the non-linear parameter $\gamma = a_3/m$. The non-linear parameter γ can be both positive and negative. If γ is positive, $F(x)$ increases over the linear value sx for large displacements so that the spring is hard. When γ is negative, $F(x)$ decreases below the linear value sx, then the spring is soft.

With the help of the same mathematical procedure and with the boundary condition at $t = 0$, $x_{n=1} = 0$, we obtain the displacement for the first-order correction as

$$x_1 = \frac{\gamma a^3}{32\omega_0^2}(\cos 3\omega_0 t - \cos \omega_0 t) \tag{25}$$

and the first-order correction of frequency of oscillation $\omega_1 = -(3\gamma a^2/8\omega_0)$.

With the same technique and with the boundary condition at $t = 0, x_{n=2} = 0$, we obtain the second-order correction of the displacement as

$$x_{n=2} = \frac{\gamma^2 a^5}{1028\omega^4}(\cos 5\omega_0 t - \cos \omega_0 t) \tag{26}$$

and the second-order correction of frequency $\omega_2 = -(9\gamma^2 a^4/32\omega_0^3)$. It is observed that total displacement with n-order correction contains power series of γ. Such expression is converged for unitary harmonic frequency only for $|\gamma| < 1$ which is indeed a limitation given that the motion is known to be bounded and periodic for $\gamma > -1$.

Case IV: Let us consider a symmetric restoring force of the form

$$F(x) = -sx - a_5 x^5. \tag{27}$$

Applying the LDE technique, from (6), we get the differential equation for the above symmetric restoring force as

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\sum_{j=0}^{\infty}\delta^j x_j\right) + \left(\sum_{j=0}^{\infty}\delta^j \omega_j\right)^2 \left(\sum_{j=0}^{\infty}\delta^j x_j\right) = -\delta\left(\epsilon\left(\sum_{j=0}^{\infty}\delta^j x_j\right)^5\right), \tag{28}$$

where the non-linear parameter $\epsilon = a_5/m$. The non-linear parameter ϵ can be both positive and negative. With the boundary condition at $t = 0$, $x_{n=1} = 0$, the first-order correction of the displacement is carried out in the same way as before and we get

$$x_{n=1} = -\frac{\epsilon a^5}{24\omega_0^2}\cos\omega_0 t + \frac{5\epsilon a^5}{128\omega_0^2}\cos 3\omega_0 t + \frac{\epsilon a^5}{384\omega_0^2}\cos 5\omega_0 t \tag{29}$$

which converge only for $|\epsilon| < 1$ and the first-order correction of oscillating frequency $\omega_1 = -(5\epsilon a^5/16\omega_0)$. The second-order correction is carried out in the same way and it has been found that the solution converges faster. Hence, the contribution of the second-order correction is very small. Therefore, for this case the first-order correction is sufficient for the total displacement and oscillating frequency.

Now, we conclude that in this article, we develop a more economical scheme [10] which yields simple perturbation theory formulae from which one can obtain all perturbative corrections to both displacement and frequency of symmetric and asymmetric anharmonic oscillators. Here, we attain added realism and sophistication for this LDE technique by dealing with the differential equation of motion for anharmonic oscillators. We feel that this perturbation technique is not only simple, but also may be the best to study important applications for a wide class of realistic non-exactly solvable problems.

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