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On non-extensive nature of thermal conductivity

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Abstract. In this paper we study non-extensive nature of thermal conductivity. It is observed that there is similarity between non-extensive entropic index and fractal dimension obtained for the silica aerogel thermal conductivity data at low temperature.

Keywords. Non-extensive; thermal conductivity; specific heat; silica aerogels.

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1. Motivation

Non-extensive statistics is being increasingly used to explain anomalous behaviour observed in the properties of various physical systems. Tsallis statistics has been used to study physical systems/phenomena which include turbulence in plasma [1], cosmic ray background radiation [2], self-gravitating systems [3], econo-physics [4], electron–positron annihilation [5], classical and quantum chaos [6], linear response theory [7], Levy-type anomalous super-diffusion [8], thermalization of electron–phonon systems [9], low-dimensional dissipative systems [10] etc. It has been shown that non-extensive features get manifested in those systems which have long range forces, long memory effects or in those systems which evolve in (non-Euclidean like space–time) fractal space–time ([11] and reference therein). Apart from other applications, it has been suggested [11] that non-extensive statistics can be applied to complex systems like glassy materials and fractal/multi-fractal or unconventional structures also. Anomalous low-temperature silica aerogel thermal conductivity has motivated us to use non-extensive statistics. Another important motivation is to identify physical nature of entropic parameter.

2. Thermal conductivity

The steady state flow of heat gives the flux J [12] of the thermal conductivity as

$$J = -K\frac{\mathrm{d}T}{\mathrm{d}x}, \tag{1}$$

where K is the thermal conductivity coefficient and $\mathrm{d}T/\mathrm{d}x$ is the temperature gradient. The thermal conductivity coefficient [12] is given as

$$K \sim C_{\mathrm{p}} V_{\mathrm{a}} l_{\mathrm{m}}, \tag{2}$$

where C_{p} is the specific heat of phonon, V_{a} is the average phonon velocity and l_{m} is the mean path of phonon between collisions. At high temperatures $l_{\mathrm{m}} \sim (1/T)$ and at low temperatures (1–10 K) it is independent of temperature. In general, V_{a} is independent of temperature. Specific heat C_{p} is modelled by Debye model which results in T^3 dependence at low temperatures. Thus, at low temperatures, thermal conductivity is mainly influenced by specific heat and follows T^3 dependence. Specific heat depends on density of states $g(\omega)$.

$$C_{\mathrm{p}} = \frac{k_{\mathrm{B}}^2}{\hbar} \int_0^{\infty} g\left(\frac{k_{\mathrm{B}} T}{\hbar} x\right) \frac{x^2 e^x}{(e^x - 1)^2} \mathrm{d}x, \tag{3}$$

where $x = (\hbar\omega/k_{\mathrm{B}}T)$. The above equation has been obtained using Boltzmann–Gibbs statistical mechanics. In general, most of the experimental data of various systems follow eq. (3). However, there are some anomalous experimental results which do not follow eq. (3). For example, specific heat of various glassy systems at low temperature shows quasi-linear behaviour, i.e. approximate linear dependence on temperature T [13,14]. Silica aerogel specific heat also displays quasi-linear dependence on T instead of T^3 dependence. Thus, it is expected that thermal conductivity of glasses at very low temperature should display a quasi-linear dependence on temperature T. Earlier it has been reported that thermal conductivity of silica aerogels [15] indeed displays quasi-linear dependence in the 1–10 K temperature range.

3. Non-extensive approach

Non-extensive entropy [11] is defined as

$$S_q = \frac{1 - \sum_i P_i^q}{q - 1}, \tag{4}$$

where q is the non-extensive entropic index and P_i are the probabilities of the microscopic states with $\sum P_i = 1$.

To calculate non-extensive thermal conductivity we need to derive non-extensive specific heat (NSH). In order to derive NSH we need non-extensive quantum distribution function of bosons. It is very difficult to compute exact analytical expression of non-extensive distribution function. However, there are many studies which provide an approximate form of the non-extensive distribution functions [2,16]. In the present paper we use dilute gas approximation (DGA) for boson distribution function. The similarity between DGA and Tsallis *et al* [2] results is discussed in ref. [16]. For DGA case, the average occupation number is given as [16]

$$\langle n_q \rangle = \frac{1}{(1 + (q-1)\beta(E_i - \mu)^{1/(q-1)} - 1)}. \tag{5}$$

For boson gas at the temperature β [16], we have

$$\langle n_q \rangle = \frac{\hbar\omega_i}{(1+(q-1)\frac{\hbar\omega_i}{kT})^{1/(q-1)} - 1} \tag{6}$$

and total vibrational energy is

$$U_q = \int \hbar\omega \langle n_q \rangle g(\omega) \tag{7}$$

from which specific heat can be obtained as

$$C_{\mathrm{p}} = \frac{\partial U_q}{\partial T} \tag{8}$$

where $g(\omega)$ represents density of states. For the case of Debye approximation $g(\omega) \propto (\omega)^2$. Equation (8) represents specific heat using Tsallis statistics where n_q is given by eq. (6). We have solved eq. (8) numerically using eq. (6). It has been possible to explain some of the anomalous specific heat properties of glasses [17] observed in 1–10 K range of temperature. Our numerical results were analytically confirmed by [18] and found to be correct in the very low-temperature range.

Non-extensive thermal conductivity can be written as

$$K_q \sim C_{\mathrm{p}}^q V_{\mathrm{a}} l_{\mathrm{m}}, \tag{9}$$

where C_{p}^q is given by eq. (8).

4. Results and discussion

Low-temperature specific heat studies in silica aerogels and other disordered systems do not follow Debye model and concepts of fractals have been used to explain such results [15]. For a fractal approach, density of states $g(\omega) \propto \omega^{\tilde{d}}$ where $\tilde{d}$ is the spectral dimension. Experimental data of SiO_2-aerogels for specific heats follows $C_{\mathrm{p}} \sim T^d$, where d is the fractal dimension [15]. For SiO_2-aerogels experimental thermal conductivity [15] data in the temperature range 1–10 K are given by

$$K(T) \sim T^{1.2}. \tag{10}$$

We have solved eq. (8) numerically and calculated non-extensive thermal conductivity as given by eq. (9). In figure 1, we have plotted non-extensive thermal conductivity given by eq. (9). The curve ‘a’ corresponds to $q = 2.5$ and curve ‘c’ corresponds to $q = 3.0$. Curve ‘b’ has been obtained using eq. (10). It is clear from figure 1 that curve ‘a’ and curve ‘b’ match in the temperature range 1–10 K and for high temperatures (>40 K) curve ‘b’ moves towards to curve ‘c’. Our numerical results may not hold good in high-temperature range. Thus non-extensive thermal conductivity in the range 1–10 K can be approximated by

$$K_q(T) \sim T^{\alpha}. \tag{11}$$

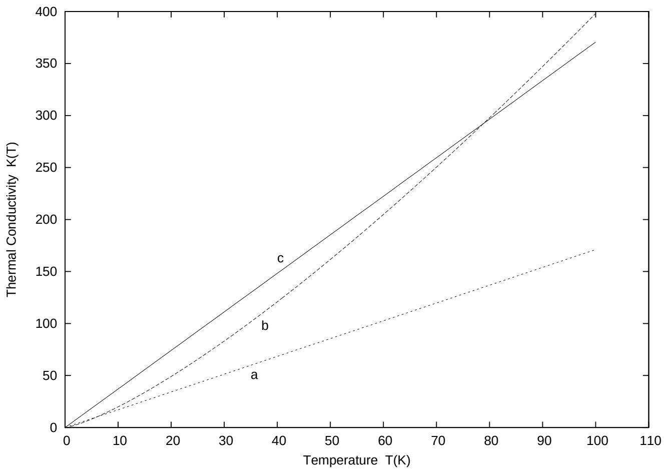

Figure 1. Thermal conductivity.

In Tsallis formalism, entropic parameter q measures the degree of non-extensivity. At theoretical level, similarities between non-extensivity and deformation algebra [19,20] have been discussed. Non-extensivity and deformation [21] have been, even, used together to explain COBE data. At experimental level also, there have been many studies [22] to identify entropic parameter q, directly or indirectly to a physical parameter. In the present studies, parameter q behaves as an upscaled or modified fractal dimension. This similarity between q and d is expected because non-extensive statistics corresponds to non-Euclidean space–time which is fractal in nature. In earlier studies also the relationship between q and d has been indirectly established. For example, the indirect similarity between fractal and non-extensive results was reported for Lamb Mössbauer factor and specific heat [17,23].

5. Conclusions

In this paper we have shown the similarity between fractal thermal conductivity and non-extensive thermal conductivity in the temperature range 1–10 K. For silica aerogels thermal conductivity is given by eq. (10) which has been obtained by considering fractal nature. The similarity between eq. (10) and non-extensive thermal conductivity (eq. (9)) in the temperature range 1–10 K points towards a relationship between fractal dimension and non-extensive entropic parameter q.

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