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Energy–momentum localization in Marder space–time

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Abstract. Considering the Einstein, Møller, Bergmann–Thomson, Landau–Lifshitz (LL), Papapetrou, Qadir–Sharif and Weinberg's definitions in general relativity, we find the momentum four-vector of the closed Universe based on Marder space–time. The momentum four-vector (due to matter plus field) is found to be zero. These results support the viewpoints of Banerjee–Sen, Xulu and Aydoğdu–Saltı. Another point is that our study agrees with the previous works of Cooperstock–Israelit, Rosen, Johri *et al.*

Keywords. Energy–momentum distribution; Marder Universe; Bergmann–Thomson; Einstein; Møller prescription.

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1. Introduction

In both the general theory of relativity and the teleparallel gravity a large number of physicists have devoted considerable attentions to the problem of computing the conserved quantities, such as energy and/or momentum, associated with various space–times. Following Einstein's original pseudo-tensor for energy–momentum, many expressions have been introduced in the literature. Einstein's energy–momentum complex, used for calculating the energy in general relativistic system, was followed by many prescriptions, e.g. Møller [1], Papapetrou [2], Landau–Lifshitz [3], Tolman [4], Bergmann–Thomson [5], Qadir–Sharif [6], Weinberg [7] and the teleparallel gravity versions of Einstein, Bergmann–Thomson, Landau–Lifshitz [8] and Møller's [9] energy–momentum complexes. The energy and momentum prescriptions give meaningful results when the line element is transformed to the Cartesian coordinates. However, for the Møller's prescriptions it is not necessary to use Cartesian coordinates. Virbhadra and his collaborators have considered many space–times and have shown that several energy–momentum complexes give the same and acceptable results [10]. Aguirregabiria *et al* [11] showed that several energy–momentum complexes give the same result for any Kerr-Shild class metric. Recently, Chang *et al* [12] showed that every energy–momentum complex can be associated with a particular Hamilton boundary term, and therefore the energy–momentum complexes may also be considered as quasi-local.

Tryon [13] suggested that in our Universe, all conserved quantities have to vanish. Tryon's Big Bang model predicted a homogeneous, isotropic and closed Universe inclusive of matter and anti-matter equally. He also argued that any closed Universe has zero energy. The notion of the energy and momentum distributions of the closed Universe was opened by an interesting work of Rosen [14] and Cooperstock [15]. Rosen considered a closed homogeneous isotropic Universe described by a Friedmann–Robertson–Walker (FRW) line element. He used the Einstein energy–momentum complex and found that the total energy is vanishing. This interesting result attracted the attention of some general relativists, for example: Johri *et al* [16], Banerjee–Sen [17] and Xulu [18]. Johri *et al* using the Landau–Lifshitz's energy–momentum complex, found that the total energy of an FRW spatially closed Universe is zero at all times. Banerjee–Sen who investigated the problem of the energy of the Bianchi Type-I space–times using the Einstein complex, and Xulu who investigated the same problem using the Landau–Lifshitz, Papapetrou and Weinberg's energy–momentum complexes, obtained that the total energy is zero. Aydoğdu and Saltı [19] have investigated energy of the Universe in Bianchi Type-I models in Møller's tetrad theory of gravity and Saltı [20] investigated energy–momentum distribution in closed Universe and they found zero energy and/or momentum density at all times. This result agrees with the studies of Johri *et al.* The basic purpose of this paper is to obtain the total energy and momentum of Marder metric using the energy–momentum expression of Einstein, Bergmann–Thomson, Møller, Landau–Lifshitz, Papapetrou, Quadir–Sharif and Weinberg in general relativity. The layout of the paper is as follows. In §2, we introduce the Marder space–time. Then in §3, we give the momentum 4-vector definitions of the Bergmann–Thomson, Einstein, Møller, Landau–Lifshitz, Papapetrou, Quadir–Sharif and Weinberg in general theory of relativity and related calculations. Finally, we summarize and discuss our results. Throughout this paper, the Latin indices $(i, j, ...)$ represent the vector number and the Greek $(\mu, \nu, ...)$ represent the vector components; all indices run 0 to 3. We use geometrized units where $G = 1$ and $c = 1$.

2. The Marder-type space–time

Cosmological models which are anisotropic and homogeneous have a significant role in describing the Universe in the early stages of its evolution [21]. In this paper we consider Marder's cylindrically symmetric metric in the form [22]

$$\mathrm{d}s^2 = A^2(\mathrm{d}x^2 - \mathrm{d}t^2) + B^2\mathrm{d}y^2 + C^2\mathrm{d}z^2, \tag{1}$$

where, A, B, C are functions of t only. This metric represents the anisotropic homogeneous universe. The model becomes conformal to flat space–time in particular cases. Cylindrically symmetric space–time plays an important role in the study of the Universe on a scale in which anisotropy and inhomogeneity are not ignored [23].

Cylindrically symmetric cosmological models have made significant contributions in understanding some essential features of the Universe such as the formation of galaxies during the early stages of their evolution [24]. The cylindrically symmetric space–times representing material distribution were obtained by Marder [22]. The metric can be transformed to the Bianchi Type-I form by the coordinate transformation $t \to \int A(t)\mathrm{d}t$. The Bianchi cosmologies play an important role in theoretical cosmology and have been much studied since the 1960s. A Bianchi cosmology represents a spatially homogeneous Universe, since by definition the space–time admits a three-parameter group of isometries whose orbits are space-like hyper-surfaces. These models can be used to analyze aspects of the physical Universe which pertain to or which may be affected by anisotropy in the rate of expansion, for example, the cosmic microwave background radiation, nucleosynthesis in the early Universe, and the question of the isotropization of the Universe itself [25]. Spatially homogeneous cosmologies also play important roles in attempts to understand the structure and properties of the space of all cosmological solutions of Einstein field equations.

3. Energy–momentum in general relativity

In this section, we introduce Bergmann–Thomson, Einstein, Møller, Landau–Lifshitz, Papapetrou, Qadir–Sharif and Weinberg energy–momentum definitions, respectively.

3.1 *Bergmann–Thomson's energy–momentum formulation*

The energy–momentum prescription of Bergmann–Thomson is given by

$$\Xi^{\mu\nu} = \frac{1}{16\pi}\Pi^{\mu\nu\alpha}_{,\alpha}, \tag{2}$$

where

$$\Pi^{\mu\nu\alpha} = g^{\mu\beta}V^{\nu\alpha}_{\beta} \tag{3}$$

with

$$V^{\nu\alpha}_{\beta} = -V^{\alpha\nu}_{\beta} = \frac{g_{\beta\xi}}{\sqrt{-g}}[-g(g^{\nu\xi}g^{\alpha\rho} - g^{\alpha\xi}g^{\nu\rho})]_{,\rho} \tag{4}$$

Ξ^0_0 is the energy density, Ξ^0_μ are the momentum density components, and Ξ^μ_0 are the components of the energy current density. The Bergmann–Thomson energy–momentum definition satisfies the following local conservation laws:

$$\frac{\partial \Xi^{\mu\nu}}{\partial x^\nu} = 0 \tag{5}$$

in any coordinate system. The energy and momentum components are given by

$$P^\mu = \int\int\int \Xi^{\mu 0}\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{6}$$

Further, Gauss's theorem furnishes

$$P^{\mu} = \frac{1}{16\pi} \int\int \Pi^{\mu 0\alpha} \kappa_{\alpha} \, \mathrm{d}S. \tag{7}$$

κ_{α} stands for the 3-components of unit vector over an infinitesimal surface element dS. The quantities P^i for i=1,2,3 are the momentum components, while P^0 is the energy. Using the metric in eq. (1), we found the required components of $\Pi^{\mu\nu\alpha}$ to be zero. From this point of view, by using eq. (7), we obtain

$$P_{\mu} = 0. \tag{8}$$

3.2 *Einstein's energy–momentum formulation*

The energy–momentum complex as defined by Einstein is given by

$$\Theta_{\mu}^{\nu} = \frac{1}{16\pi} H_{\mu,\alpha}^{\nu\alpha}, \tag{9}$$

where

$$H_{\mu}^{\nu\alpha} = \frac{g_{\mu\beta}}{\sqrt{-g}} [-g(g^{\nu\beta} g^{\alpha\xi} - g^{\alpha\beta} g^{\nu\xi})]_{,\xi} \tag{10}$$

Θ_0^0 is the energy density, Θ_{α}^0 are the momentum density components, and Θ_0^{α} are the components of energy current density. The Einstein energy and momentum density satisfy the local conservation laws as

$$\frac{\partial \Theta_{\mu}^{\nu}}{\partial x^{\nu}} = 0 \tag{11}$$

and energy and momentum components are given by

$$P^{\mu} = \int\int\int \Theta_{\mu}^{0} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z. \tag{12}$$

Further, Gauss's theorem furnishes

$$P^{\mu} = \frac{1}{16\pi} \int\int H_{\mu}^{0\alpha} \eta_{\alpha} \, \mathrm{d}S. \tag{13}$$

η_{α} stands for the 3-components of unit vector over an infinitesimal surface element dS. The quantities P^i for i=1,2,3 are the momentum components, while P^0 is the energy. Using the line element in eq. (1) with eqs (9), (10) and (12), we can easily see that the Einstein momentum four-vector is obtained as

$$P^{\mu} = 0. \tag{14}$$

3.3 *Møller's energy–momentum formulation*

The energy–momentum complex of Møller is given by

$$M_\mu^\nu = \frac{1}{8\pi}\chi_{\mu,\alpha}^{\nu\alpha} \tag{15}$$

satisfying the local conservation laws:

$$\frac{\partial M_\mu^\nu}{\partial x^\nu} = 0, \tag{16}$$

where the antisymmetric superpotential $\chi_\mu^{\nu\alpha}$ is

$$\chi_\mu^{\nu\alpha} = \sqrt{-g}[g_{\mu\beta,\gamma} - g_{\mu\gamma,\beta}]g^{\nu\gamma}g^{\alpha\beta}. \tag{17}$$

The locally conserved energy–momentum complex M_μ^ν contains contributions from the matter, non-gravitational fields. M_0^0 is the energy density and M_α^0 are the momentum density components. The momentum four-vector of Møller is given by

$$P_\mu = \int\int\int M_\mu^0 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z. \tag{18}$$

Use Gauss's theorem, this definition transforms into

$$P_\mu = \frac{1}{8\pi}\int\int \chi_\mu^{\nu\alpha}\mu_\alpha \, \mathrm{d}S, \tag{19}$$

where μ_α is the outward unit normal vector over the infinitesimal surface element $\mathrm{d}S$. P_i give momentum components P_1, P_2, P_3 and P_0 gives the energy. Using the metric in eq. (1), we found the required components of $\chi_\mu^{\nu\alpha}$ to be zero. From this point of view, by using eq. (19), we obtain

$$P_\mu = 0. \tag{20}$$

3.4 *Landau–Lifshitz's energy–momentum formulation*

Energy–momentum prescription of Landau–Lifshitz is given by

$$\Omega^{\mu\alpha} = \frac{1}{16\pi}S^{\mu\nu\alpha\beta}_{\ \ ,\nu\beta}, \tag{21}$$

where

$$S^{\mu\nu\alpha\beta} = -g(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha}). \tag{22}$$

Ω_0^0 is the energy density, Ω_μ^0 are the momentum density components, and Ω_0^μ are the components of energy current density. The Landau–Lifshitz energy–momentum complex satisfies the local conservation laws

$$\frac{\partial \Omega^{\mu\nu}}{\partial x^{\nu}} = 0 \tag{23}$$

in any coordinate system. The energy and momentum components are given by

$$P^{\mu} = \int\int\int \Omega^{\mu 0}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z. \tag{24}$$

Further, Gauss's theorem furnishes

$$P^{\mu} = \frac{1}{16\pi}\int\int S^{\mu\alpha 0\nu}_{,\nu}\eta_{\alpha}\, \mathrm{d}S, \tag{25}$$

where η_{α} stands for the 3-components of unit vector over an infinitesimal surface element $\mathrm{d}S$. The quantities P^{i} for i=1,2,3 are the momentum components, while P^{0} is the energy. Using the line element in eq. (1), we found the required components of $S^{\mu\nu\alpha\beta}$ to be zero. From this point of view, by using eq. (24), we obtain

$$P^{\mu} = 0. \tag{26}$$

3.5 *Papapetrou's energy–momentum formulation*

The energy–momentum complex of Papapetrou is given by

$$\Sigma^{\mu\nu} = \frac{1}{16\pi} N^{\mu\nu\alpha\beta}_{,\alpha\beta}, \tag{27}$$

where

$$N^{\mu\nu\alpha\beta} = \sqrt{-g}(g^{\mu\nu}\eta^{\alpha\beta} - g^{\mu\alpha}\eta^{\nu\beta} + g^{\alpha\beta}\eta^{\mu\nu} - g^{\nu\beta}\eta^{\mu\alpha}). \tag{28}$$

Σ^{0}_{0} is the energy density, Σ^{0}_{μ} are the momentum density components, and Σ^{μ}_{0} are the components of energy current density. The Papapetrou energy–momentum complex satisfies the local conservation laws

$$\frac{\partial \Sigma^{\mu\nu}}{\partial x^{\nu}} = 0 \tag{29}$$

in any coordinate system. The energy and momentum components are given by

$$P^{\mu} = \int\int\int \Sigma^{\mu 0}\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z. \tag{30}$$

Here P^{μ} gives momentum components P^{1}, P^{2}, P^{3} and P^{0} gives the energy distribution. Using the line element in eq. (1), we found the required components of $N^{\mu\nu\alpha\beta}$ to be zero. From this point of view, by using eq. (30), we obtain

$$P^{\mu} = 0. \tag{31}$$

3.6 *Qadir–Sharif's energy–momentum formulation*

The momentum four-vector of Qadir–Sharif is given by

$$P_\mu = \int F_\mu \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z, \tag{32}$$

where

$$F_0 = m\left[\left\{\ln\left(\frac{A}{\sqrt{g_{00}}}\right)\right\}_{,0} - \frac{g_{\mu\nu,0}g^{\mu\nu}_{,0}}{4A}\right], \quad F_a = m\left(\ln\sqrt{g_{00}}\right)_{,\mu} \tag{33}$$

and

$$A = \left(\ln\sqrt{-g}\right)_{,0}. \tag{34}$$

This force definition depends on the choice of frame, which is not uniquely fixed. The quantity, whose proper time derivative is F_μ, is called the momentum four-vector for the test particle. Using the line element in (1), with eqs (33), (34) then we get

$$F_\mu = 0. \tag{35}$$

Using this result, we can easily see that the Qadir–Sharif momentum four-vector is obtained as

$$P_\mu = 0. \tag{36}$$

3.7 *Weinberg's energy–momentum formulation*

The energy–momentum complex of Weinberg [7] is given by

$$W^{\mu\nu} = \frac{1}{16\pi}K^{\mu\nu\alpha}_{,\alpha}, \tag{37}$$

where

$$K^{\mu\nu\alpha} = \frac{\partial h^\beta_\beta}{\partial x_\mu}\eta^{\nu\alpha} - \frac{\partial h^\beta_\beta}{\partial x_\nu}\eta^{\mu\alpha} - \frac{\partial h^{\beta\mu}}{\partial x^\beta}\eta^{\nu\alpha} + \frac{\partial h^{\beta\nu}}{\partial x^\beta}\eta^{\mu\alpha} + \frac{\partial h^{\mu\alpha}}{\partial x_\nu} - \frac{\partial h^{\nu\alpha}}{\partial x_\mu} \tag{38}$$

and

$$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}. \tag{39}$$

The indices on $h_{\mu\nu}$ or $\partial/\partial x_\mu$ are raised or lowered with the help of η's. It is clear that

$$K^{\mu\nu\alpha} = -K^{\nu\mu\alpha} \tag{40}$$

W^{00} and W^{a0} are the energy and momentum density components respectively. The Weinberg energy and momentum complex $W^{\mu\nu}$ contains contributions from the matter, non-gravitational and gravitational fields, and satisfies the local conservation laws

$$\frac{\partial W^{\mu\nu}}{\partial x^{\nu}} = 0. \tag{41}$$

The momentum four-vector is given by

$$P^{\mu} = \int\int\int W^{\mu 0}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{42}$$

and the angular momentum is given by

$$J^{\mu\nu} = \int\int\int (x^{\mu}W^{0\nu} - x^{\nu}W^{0\mu})\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z. \tag{43}$$

Further, Gauss's theorem furnishes

$$P^{\mu} = \frac{1}{16\pi}\int\int K^{\mu\nu\alpha}\kappa_{\alpha}\,\mathrm{d}S \tag{44}$$

and the physically interesting components of the angular momentum are

$$J^{\mu\nu} = \frac{1}{16\pi}\int\int (x^{\mu}K^{\alpha 0\nu} - x^{\nu}K^{\alpha 0\mu} + \eta^{\alpha\mu}h^{0\nu} - \eta^{\alpha\nu}h^{0\mu})\kappa_{\alpha}\,\mathrm{d}S. \tag{45}$$

κ_{α} stands for the 3-components of unit vector over an infinitesimal surface element $\mathrm{d}S$. P_i give momentum(energy current) components P_1, P_2, P_3 and P_0 gives the energy. We are interested in determining the momentum four-vector. Now using the line element in (1) in eqs (38) and (39) we find that all components of $K^{\mu\nu\alpha}$ vanish. Thus, from eq. (44) we have

$$P^{\mu} = 0. \tag{46}$$

4. Summary and discussion

The subject of energy–momentum localization in the general theory of relativity has been very exciting and interesting; however it has been associated with some debate. Recently, some researchers have been interested in studying the energy content of the Universe in various models. Rosen and Johri *et al* considered Einstein and Landau–Lifshitz complexes respectively, studied the total energy of a closed Universe described by FRW line element and found that to be zero. Furthermore, Banerjee–Sen, Xulu and Aydoğdu–Saltı investigated the problem of the Bianchi Type-I Universes, using the general relativity and teleparallel versions of Einstein and Landau–Lifshitz complexes found that the total energy of this model is vanishing. Tryon suggested that in our Universe, all conserved quantities have to vanish. Tryon's Big Bang model predicted a homogeneous, isotropic and closed Universe inclusive of matter and anti-matter equally [13].

The object of the present paper is to show that it is possible to solve the problem of localization of energy and momentum in general relativity by using the energy and momentum complexes. In this paper, to compute the energy distribution of the Marder space–time, we have considered seven different energy and momentum complexes in general relativity: e.g. Bergmann–Thomson, Einstein, Møller, Landau–Lifshitz, Papapetrou, Qadir–Sharif and Weinberg. We found that the total energy and momentum components have zero net value at all investigated energy momentum densities, because the energy contributions from the matter and gravitational field inside an arbitrary two-surface in the case of anisotropic model of the Universe based on cylindrically symmetric Marder metric, cancel each other. Therefore, the total energy and momentum components have zero value and different energy–momentum complexes give same results in Marder space–time.

When we apply coordinate transformation $t \to \int A(t)\,\mathrm{d}t$, our energy–momentum results agree with for Bianchi Type-I solutions.

$$E_{\text{Marder}} = E_{\text{Bianchi I}} = 0, \quad M_{\text{Marder}} = M_{\text{Bianchi I}} = 0.$$

The results advocate the importance of energy–momentum complexes.

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References

[1] C Møller, *Ann. Phys. (NY)* **4**, 347 (1958)
C Møller, *Ann. Phys. (NY)* **12**, 118 (1961)
[2] A Papapetrou, *Proc. R. Irish. Acad.* **A11**, 11 (1948)
[3] L D Landau and E M Lifshitz, *The classical theory of fields*, 4th Edition (Pergamon Press, Oxford) (Reprinted in 2002)
[4] R C Tolman, *Relativity, thermodynamics and cosmology* (Oxford Univ. Press, London, 1934)
[5] P G Bergmann and R Thomson, *Phys. Rev.* **89**, 400 (1953)
[6] A Qadir and M Sharif, *Phys. Lett.* **A167**, 331 (1992)
[7] S Weinberg, *Gravitation and cosmology: Principle and applications of general theory of relativity* (John Wiley and Sons, Inc., New York, 1972)
[8] T Vargas, *Gen. Relativ. Gravit.* **36**, 1255 (2004)
[9] F I Mikhail, M I Wanas, A Hindawi and E I Lashin, *Int. J. Theor. Phys.* **32**, 1627 (1993)

[10] K S Virbhadra, *Phys. Rev.* **D41**, 1086 (1990)
K S Virbhadra, *Phys. Rev.* **D42**, 2919 (1990)
K S Virbhadra, *Phys. Rev.* **D60**, 104041 (1999)
N Rosen and K S Virbhadra, *Gen. Relativ. Gravit.* **25**, 429 (1993)
K S Virbhadra, *Pramana – J. Phys.* **45**, 215 (1995)
A Chamorro and K S Virbhadra, *Pramana – J. Phys.* **45**, 181 (1995)
[11] J M Aguirregabiria, A Chamorro and K S Virbhadra, *Gen. Relativ. Gravit.* **28**, 1393 (1996)
[12] C C Chang, J M Nester and C Chen, *Phys. Rev. Lett.* **83**, 1897 (1999)
[13] E P Tryon, *Nature (London)* **246**, 396 (1973)
[14] N Rosen, *Gen. Relativ. Gravit.* **26**, 319 (1994)
[15] F I Cooperstock, *Gen. Relativ. Gravit.* **26**, 323 (1994)
[16] V B Johri, D Kalligas, G P Singh and C W F Everitt, *Gen. Relativ. Gravit.* **27**, 323 (1995)
[17] N Banerjee and S Sen, *Pramana – J. Phys.* **49**, 609 (1997)
[18] S S Xulu, *Int. J. Mod. Phys.* **A15**, 4849 (2000)
[19] O Aydoğdu and M Saltı, *Astrophys. Space Sci.* **A299**, 227 (2005)
[20] M Saltı, *Czech. J. Phys.* **A56**, 177 (2006)
[21] R Bali and D R Jain, *Int. J. Theor. Phys.* **28**, 8 (1989)
[22] L Marder, *Proc. R. Soc. London* **A246**, 133 (1958)
[23] S Prakash, *Astrophys. Space Sci.* **111**, 383 (1985)
[24] C B Kılınç, *Astrophys. Space Sci.* **222**, 171 (1994)
[25] M A H MacCallum, *Einstein centenary survey* (Cambridge Univ. Press, England, 1979)