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Calculation of CWKB envelope in boson and fermion productions

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Abstract. We present the calculation of envelope of boson and of both low- and high-mass fermion production at the end of inflation when the coherently oscillating inflatons decay into bosons and fermions. We consider three different models of inflation and use CWKB technique to calculate the envelope to understand the structure of resonance band formation. We observe that though low-mass fermion production is not effective in pre-heating because of Pauli blocking, it is quite probable for high-mass fermion to take part in pre-heating.

Keywords. Boson production; fermion production; CWKB technique.

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1. Introduction

In cosmology, production of particles by decaying inflaton at the end of inflation is a way of studying quantum particle production by external classical fields. At the end of inflation, the inflaton field ϕ oscillates about the minimum of effective potential with a large amplitude $\phi = \frac{1}{10}M_P$. The coherent oscillation of this scalar field results in particle production by transferring the scalar field energy to thermal energy of produced particles; temperature at this stage is the reheating temperature. For bosons there is efficient and explosive creation through parametric resonance production. Recently, parametric creation of spin $1/2$ fermions are being studied though fermion production is limited by Pauli blocking. The most common way to study particle production is to write the field equation for boson and fermion field in FRW background and calculate the occupation number N_k.

Boson production is standardized but for fermion production there are wide discrepancies due to complicacy of the spinors. In this case, there are variant results in the literature. We test it through the CWKB (complex trajectory WKB) method developed by Biswas *et al* [1–5] by calculating the envelope and finding whether it fits correctly over the occupation number. The envelope is the curve joining the maximum of occupation number (N_k) vs. k. The standard approach is to calculate N_k using Runge–Kutta IV method and the envelope is calculated using the technique given in [6] and subsequently used by Greene and Kofman [7,8]. Most of the workers studying fermion particle production use this formula of envelope in their work (see ref. [9]).

We have discussed elsewhere [10] that for low-mass fermion production the envelope of Greene and Kofman [7] gives a correct fitting to N_k. But for high mass, fermion production results seem to be questionable because of the definition of vacuum adopted in [7] while calculating the envelope. We in our earlier work [10] developed a method to calculate the envelope using CWKB. In the present work we briefly describe the CWKB particle production and procedure to calculate the envelope. We then test the CWKB envelope for the production of fermions from low mass to high mass in three different models of inflation. It is found that the CWKB gives a nice fitting of all the results. To make the paper self-contained, in §2 we describe the basics of particle production using CWKB. In §3 we describe the boson production very briefly that would be required to discuss fermion production. In §4 we treat fermion production. Section 5 deals with the calculation of envelope. The concluding section discusses the numerical results obtained in the present work.

2. Particle production using CWKB technique

The complex time WKB method is a novel approach to calculate the particle production for both boson and fermion. Let us briefly describe the CWKB approach. We consider Schrödinger-like equation, not in space but in time.

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} + [k^2 - V(t)]\phi = 0. \tag{1}$$

Equation (1) is the temporal equation that one generally obtains from Klein–Gordon equation or Dirac equation (second-order form) in a FRW background. In CWKB, particle production is considered as a reflection in time using Feynman–Stückleberg prescription as shown in figure 1. In CWKB, pair production is considered as reflection in time. In CWKB, $V(t)$ in figure 1 is the site where the particle turns back and are the turning points of eq. (1) such that the effective energy $\omega^2(t) \equiv k^2 - V(t)$ becomes zero at the turning point. In the standard WKB description, at the turning point, since the solution is not defined, we go over to complex time and consider complex trajectory to avoid the turning points. In other words, this allows us to go inside the classically unallowed region through complex WKB paths.

We can write eq. (1) as

$$\ddot{\phi} + \omega^2(t)\phi = 0, \tag{2}$$

where $\omega^2(t) = k^2 - V(t)$. Now we consider eq. (2) and assume that $\omega(t)$ is not zero for real t but when considered as a function of complex t it has complex turning points given by $\omega(t_{1,2}) = 0$. According to the CWKB, the classical paths

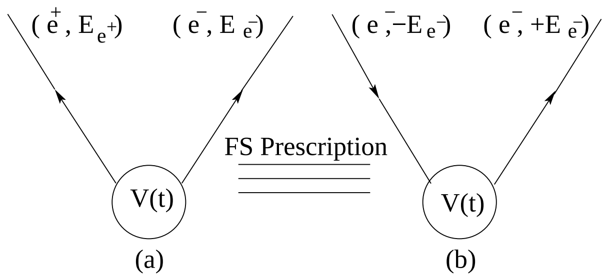

Figure 1. Feynmann–Stückleberg prescription. (**a**) A pair (e^+, e^-) is produced by the time-dependent potential $V(t)$. (**b**) F–S prescription replaces (e^+, E_{e^+}) to $(e^-, -E_{e^+})$ as if a negative energy electron moving backward in time suffers a reflection at $V(t)$-site to emerge as positive energy electron.

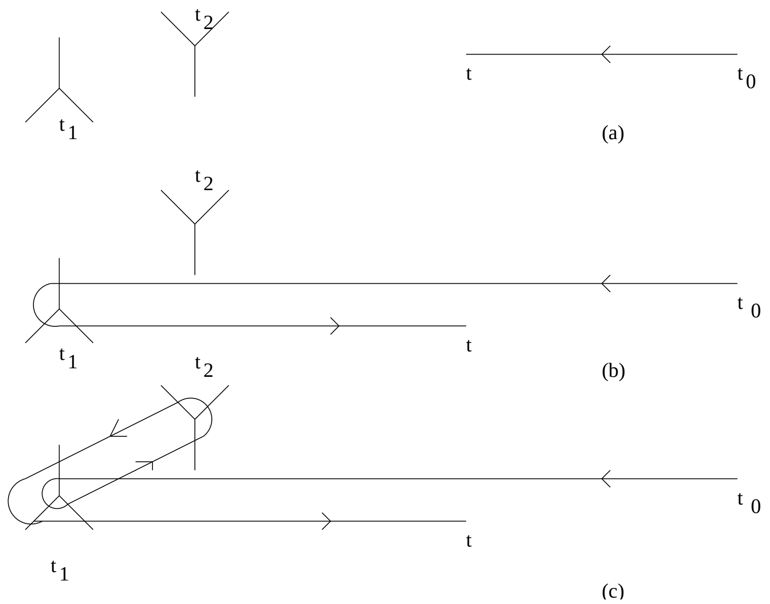

Figure 2. Trajectories with no reflection, one reflection and multiple reflections.

contributing in a complex semiclassical approximation joining two prescribed real points t' and t'' as shown in figure 2, are composed of two parts coming from the contribution of the direct trajectory and the reflected trajectory. With

$$S(t_f, t_i) = \int_{t_i}^{t_f} \omega(t)\, \mathrm{d}t, \tag{3}$$

the direct trajectory contribution is written as (we assume that $t' \gg t''$)

$$\mathrm{DT} \equiv \frac{1}{\sqrt{\omega(t)}} \mathrm{e}^{iS(t'',t')}. \tag{4}$$

The reflected trajectory starts from t' and moving backward in time gets reflected from the turning point t_1 and moving forward arrives at t''. This contribution is then multiplied by $0, 1, 2, 3, \ldots$ reflections between the two complex turning points t_1 and t_2. This contribution is written as

$$\mathrm{RT} \equiv -i\frac{1}{\sqrt{\omega(t)}}\mathrm{e}^{iS(t_1,t')-iS(t'',t_1)}\sum_{\mu=0}^{\infty}[-i\mathrm{e}^{iS(t_1,t_2)}]^{2\mu}. \tag{5}$$

Using

$$\sum_{\mu=0}^{\infty}[-i\mathrm{e}^{iS(t_1,t_2)}]^{2\mu} = \frac{1}{1+\exp(2iS(t_1,t_2))}, \tag{6}$$

and taking $t'' = t$ we get with the replacement $t' \to \infty$

$$\phi(t,\infty) \to \mathrm{e}^{iS(t,\infty)} + iR\mathrm{e}^{-iS(t,\infty)}, \tag{7}$$

where we have neglected the WKB pre-exponential factor for convenience. The reflection amplitude is given by

$$R = -\frac{\mathrm{e}^{2iS(t_1,\infty)}}{1+\mathrm{e}^{2iS(t_1,t_2)}}. \tag{8}$$

Using Feynman–Stückleberg prescription and the boundary condition that there is no particle at $t \to -\infty$, i.e.,

$$\phi(t \to -\infty) \sim T\mathrm{e}^{iS(t,t_0)}, \tag{9}$$

we identify

$$R_c = -\frac{\mathrm{e}^{2iS(t_1,\infty)}}{\sqrt{1+\mathrm{e}^{2iS(t_1,t_2)}}} = \sqrt{1+\mathrm{e}^{2iS(t_1,t_2)}}R, \tag{10}$$

as the pair production amplitude with $|R_c|^2 + |T_c|^2 = 1$. In eqs (8) and (10) R ($\equiv$ full S-matrix element) is related to disconnected propagator and R_c ($\equiv$ connected S-matrix element) is related to connected propagator [11]. The boundary conditions (7) and (9) are known as scattering boundary condition and is applied in time-dependent gauge.

Hence, for certain complex values of multiple reflection terms, (10) may exhibit poles for some parameter values if

$$S(t_1,t_2) = (N + \tfrac{1}{2})\pi \tag{11}$$

with N an integer. When the conditions (11) and $\omega(t_{1,2}) = 0$ are not satisfied or the reality condition on z are violated, the physical-region pole becomes a resonance. This resonance occurs in parameter space and we will call it parametric resonance. The poles add up to give nonperturbative contributions enhancing R_c. Thus in CWKB we have a transparent idea of resonance particle production.

With a view to apply eq. (1) to particle production at the end of inflation, we choose $V(t)$ to be a periodic potential with period T so that it satisfies the condition $V(T+t)=V(t)$. For periodic $V(t)$ there are two types of bounce points, namely turning points and reflection points from where WKB trajectories turn back resulting in particle production. We call the points where $V(t_j)=V_0$ as reflection points and the points where $k^2-V(t)\equiv\omega^2(t)=0$ as turning points. Production at the turning points (usually complex) are identified as spontaneous particle creation while particle production at reflection points are referred to as induced particle creation. The inflaton $\phi(t)$ which generates this $V(t)$ is zero at these points. There is another aspect of periodicity. In a particular barrier, between t_j and t_{j+1}, there will be some particles already produced with respect to the previous barrier and hence the boundary condition of 'no particle state' at $t\to-\infty$ will not be satisfied. This fact is to be taken into account while considering particle production in a given barrier. Due to the periodicity of $V(t)$ there are many reflection points t_j with $j=1,2,3\ldots$. Let us evaluate ϕ at a time t between t_j and t_{j+1}. Our aim is to write ϕ as

$$\phi=\sum_{\text{WKB paths}}\mathrm{e}^{iS}.$$

Suppose we have two complex turning points t_1 and t_2 given by $\omega(t_{1,2})=0$. We adopt the boundary conditions such that at $t\to-\infty$ we have the transmitted wave

$$\phi(t\to-\infty)=T_c\mathrm{e}^{iS(t,t_0)}, \tag{12}$$

and at $t\to+\infty$

$$\phi(t\to+\infty)=\mathrm{e}^{iS(t,t_0)}+R_c\mathrm{e}^{-iS(t,t_0)}. \tag{13}$$

Here R_c and T_c are respectively the reflection and the transmission amplitude. The above two equations have the interpretation that at $t\to-\infty$ we have no particle but at $t\to+\infty$ we have pair production in the *out* vacuum with respect to *in* vacuum. The same problem can be evaluated in terms of Bogolubov mode decomposition technique with

$$\psi(t\to+\infty)=\alpha_\omega\mathrm{e}^{iS(t,t_0)}+\beta_\omega\mathrm{e}^{-iS(t,t_0)}, \tag{14}$$

in which the Bogolubov coefficients are given in terms of transmission and reflection coefficients as [2]

$$|\alpha_\omega|^2=\frac{1}{|T_c|^2},\quad |\beta_\omega|^2=\frac{|R_c|^2}{|T_c|^2}. \tag{15}$$

In the above treatment the particle and antiparticle states are defined at $t\to\pm\infty$ with the corresponding vacua respectively as *out* and *in* vacuum. We now proceed toward the construction of CWKB eigenfunction. Let t_j and t_{j+1} be the two points where $V(t_j)=0$ and t is a point such that $t_j<t<t_{j+1}$ where we want to calculate the eigenfunction. We choose the right-moving and left-moving waves as follows (shown in figure 3):

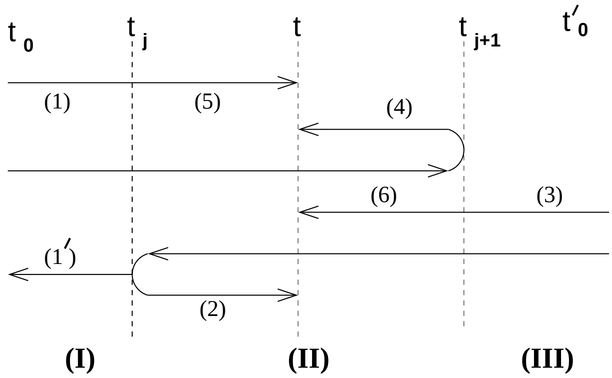

Figure 3. The CWKB trajectories that contribute to particle production. Here $t_0' \gg t_0$. Considered as reflection in time, $(3) - (6)$ and $(1') - (2)$ correspond to contribution when there are no particles at the initial stage, i.e., at time t_0. The trajectories $(1) - (5)$ and (4) contribute to ϕ when there are particles in the initial moment, more specifically, before any turning point.

Right-moving wave from t_0 to $t = \exp[-iS(t, t_0)]$,
Left moving wave from $t_0{}'$ to $t = \exp[iS(t, t_0')]$,

where $S(t_i, t_f)$ is given by eq. (3). For the right-moving wave the transmission and the reflection coefficients are T_k, R_k whereas for the left-moving wave we take the coefficients as T_k^*, R_k^*. We now avoid the subscript c on R and T. Let, in the region I, $t_{j-1} < t < t_j$, the CWKB solution before tunnelling is as represented by (1) and ($1'$) in figure 3

$$\psi_k^j(t) = \frac{\alpha_k^j}{\sqrt{2\omega}} \mathrm{e}^{-iS(t,t_0)} + \frac{\beta_k^j}{\sqrt{2\omega}} \mathrm{e}^{+iS(t,t_0)}. \tag{16}$$

The Bogolubov coefficients in a given barrier are now supposed to be constants. Between $t_j < t < t_{j+1}$, in the region II we take similarly

$$\psi_k^{j+1}(t) = \frac{\alpha_k^{j+1}}{\sqrt{2\omega}} \mathrm{e}^{-iS(t,t_0')} + \frac{\beta_k^{j+1}}{\sqrt{2\omega}} \mathrm{e}^{+iS(t,t_0')}, \tag{17}$$

where the coefficients α_k^{j+1} and β_k^{j+1} are constants for $t_j < t < t_{j+1}$. Henceforth we drop the WKB pre-exponential factor for convenience. The construction of $\psi_k^{j+1}(t)$, according to CWKB is now shown in figure 3. In the region $t_j < t < t_{j+1}$ the rightmoving part consists of two parts. Part (1) represented by $\alpha_k^j \exp[-iS(t, t_0)]$ after transmission gets multiplied by $1/T_k$ because of transmission at t_j and using (15) we find

$$\alpha_k^j \exp[-iS(t, t_0)] \to \frac{\alpha_k^j}{T_k} \exp[-iS(t, t_0)]. \tag{18}$$

There is also a contribution to the right-moving wave coming from the region III where $t > t_j$, as shown in figure 3, giving the part $\beta_k^j \exp[+iS(t, t_0)]$ in the region

$t < t_j$ represented by ($1'$) in figure 3. The amplitude of the left-moving part at t_j is $\beta_k^j \exp[+iS(t_j,t_0)]$. This amplitude part when continued in the region $t > t_j$ becomes β_k^j/T_k^* which again being reflected at t_j becomes $\beta_k^j R_k^*/T_k^*$ so that in the region $t > t_j$

$$\beta_k^j \exp[iS(t,t_0)] \rightarrow \frac{\beta_k^j R_k^*}{T_k^*} \exp[+iS(t_j,t_0)] \exp(-iS(t,t_j)). \tag{19}$$

Writing $S(t,t_j) = S(t,t_j) + S(t_j,t_0) - S(t_j,t_0) = S(t,t_0) - S(t_j,t_0)$, from (19) for the right-moving part we get

$$\phi_{k,\mathrm{RMP}}^j = \left[\frac{\alpha_k^j}{T_k} + \frac{\beta_k^j R_k^*}{T_k^*} \mathrm{e}^{2\,iS(t_j,t_0)}\right] \mathrm{e}^{-iS(t,t_0)}. \tag{20}$$

To get (20) we can also start from region III with $\beta_k^{j+2} \exp(iS(t,t_0'))$ that now gets multiplied by R_k^*/T_k^* because of transmission at t_{j+1} and reflection at t_j and then use the continuity condition

$$\beta_k^j \mathrm{e}^{iS(t_j,t_0)} = \beta_k^{j+2} \mathrm{e}^{iS(t_j,t_0')}.$$

We will get the same result as (20). Comparing (20) with the first term in (16) we get

$$\alpha_k^{j+1} = \left[\frac{\alpha_k^j}{T_k} + \frac{\beta_k^j R_k^*}{T_k^*} \mathrm{e}^{2iS(t_j,t_0)}\right]. \tag{21}$$

Now the left-moving part has two contributions given by the trajectory (3) and (4) of figure 3. The trajectory (3) coming from t_0' gives the left-moving part $\beta_k^{j+2} \exp(iS(t,t_0'))$ in the region $t < t_{j+1}$. Now we use the relation above to convert β_k^{j+2} in terms of β_k^j. This part then in $t > t_j$ becomes

$$\frac{\beta_k^j}{T_k^*} \mathrm{e}^{iS(t,t_0)}. \tag{22}$$

Another contribution comes from $\alpha_k \exp(-iS(t,t_0))$ which on transmission at t_j and reflection at t_{j+1} gets modified to

$$\frac{\alpha_k^j}{T_k} R_k \mathrm{e}^{-iS(t_{j+1},t_0)+iS(t,t_{j+1})}. \tag{23}$$

Now, $iS(t,t_{j+1}) = iS(t,t_{j+1}) + iS(t_{j+1},t_0) - iS(t_{j+1},t_0) = iS(t,t_0) - iS(t_{j+1},t_0)$ so that the left-moving part becomes

$$\phi_{k,\mathrm{LMP}}^{j+1} = \left[\frac{\beta_k^j}{T_k^*} + \frac{\alpha_k^j R_k}{T_k} \mathrm{e}^{-2iS(t_{j+1},t_0)}\right] \mathrm{e}^{+iS(t,t_0)}. \tag{24}$$

Hence comparing (24) with the second term in (16) we get

$$\beta_k^{j+1} = \left[\frac{\beta_k^j}{T_k^*} + \frac{\alpha_k^j R_k}{T_k} \mathrm{e}^{-2iS(t_{j+1},t_0)} \right]. \tag{25}$$

This result exactly coincides with Kofman *et al* [8]. We have in the expression of β_k^{j+1} the phase term as $S(t_{j+1}, t_0)$ instead of $S(t_j, t_0)$. It should be pointed out that we have not taken repeated reflections between the turning points t_j and t_{j+1} which will automatically be introduced when we calculate R_k and T_k through the technique of CWKB.

For fermions and bosons the Bogolubov coefficients satisfy the relation

$$|\alpha_k^j|^2 \pm |\beta_k^j|^2 = 1, \tag{26}$$

where $\pm$ sign respectively correspond to fermions and bosons. The number of particles produced at time after crossing t_j [4] is given by

$$n_k^{j+1} = |\beta_k^{j+1}|^2. \tag{27}$$

We will use expression (25) to calculate the CWKB envelope while calculating the occupation number using expression (66), where χ is obtained through Runge–Kutta calculation.

3. Boson production

For flat Friedmann background with scale factor $a(t)$ the temporal part of the eigenfunction with comoving momentum $\vec{k}$ satisfies the equation

$$\ddot{\chi}_k + 3\frac{\dot{a}}{a}\dot{\chi}_k + \left[\frac{k^2}{a^2} + m_\chi^2(0) - \xi R + g^2\phi^2\right]\chi_k = 0 \tag{28}$$

and is obtained with the Lagrangian $\mathcal{L}$ [7],

$$\begin{aligned}\mathcal{L} = {} & \frac{1}{2}\phi,\phi' - V(\phi) - \frac{1}{2}\chi,\chi' - \frac{1}{2}m_\chi^2(0)\chi^2 + \frac{1}{2}\xi R\chi^2 \\ & + \bar{\psi}(i\gamma^i\partial_i - m_\psi(0))\psi - \frac{1}{2}g^2\phi^2\chi^2 - h\bar{\psi}\psi\phi.\end{aligned} \tag{29}$$

Here g, h and ξ are small coupling constants, R is the Ricci scalar. The time dependence of background field ϕ and scale factor is obtained through the following two evolution equations:

$$H^2 = \frac{8\pi}{3M_P^2}\left[\frac{1}{2}\dot{\phi}^2 + V(\phi)\right], \tag{30}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = 0, \tag{31}$$

where $H = \dot{a}/a$ is the Hubble constant and $V(\phi)$ is the effective potential of the scalar field. In Minkowski space $(a(t) = 1)$ we parametrize the χ equation as

$$\ddot{\chi}_k + [k^2 + g^2\sigma^2 + 2g^2\sigma\Phi\sin(mt)]\chi_k = 0, \tag{32}$$

where periodicity $\phi(t) = \Phi\sin(mt)$ comes from inflaton field through the evolution equation (31). The above form is obtained with $V(\phi) \sim \frac{1}{2}(\phi - \sigma)^2$ and making the shift $\phi - \sigma \to \phi$ in the standard way and redefining $m_\chi = \sqrt{m_\chi^2(0) + g^2\sigma^2}$. Equation (32) is of the form

$$\ddot{\chi}_k + \omega_k^2(t)\chi_k = 0, \tag{33}$$

where $\omega_k^2 = k^2 + g^2\sigma^2 + 2g^2\sigma\sin(mt)$. Without the symmetry breaking we may take, $V(\phi) = \frac{1}{2}m^2\phi^2 - \frac{1}{2}g^2\phi^2\chi^2$, for chaotic inflation in which the amplitude of ϕ is $\sim M_p \gg \sigma$. Thus (32) becomes

$$\ddot{\chi}_k + [k^2 + g^2\Phi^2\sin^2(mt)]\chi_k = 0. \tag{34}$$

Comparing eq. (1) of previous section, we may identify $V(t) \equiv -g^2\Phi^2\sin^2(mt)$.

3.1 *Reflection and transmission coefficients*

In the vicinity of t_j, eq. (34) becomes

$$\frac{\mathrm{d}^2\chi_k}{\mathrm{d}t} + (k^2 + g^2\phi^2m^2(t - t_j)^2\chi_k = 0. \tag{35}$$

Putting $\tau = k_*(t - t_j), \lambda = k/k_*$, $k_*^2 = g\phi m$ we get,

$$\frac{\mathrm{d}^2\chi_k}{\mathrm{d}\tau} + (\lambda^2 + \tau^2) = 0. \tag{36}$$

The reflection and transmission coefficients are respectively,

$$(R_k)_{\text{boson}} = \frac{-i\mathrm{e}^{-i\phi_{kb}}}{\sqrt{1 + \mathrm{e}^{\pi\lambda^2}}} \quad \text{and} \quad (T_k)_{\text{boson}} = \frac{\mathrm{e}^{-i\phi_{kb}}}{\sqrt{1 + \mathrm{e}^{-\pi\lambda^2}}},$$

where ϕ_{kb} is the unknown phase given by

$$\phi_{kb} = \arg\Gamma\left(\frac{1 + i\lambda^2}{2}\right) + \frac{k^2}{2}\left(1 + \ln\frac{2}{\lambda^2}\right).$$

Number density of outgoing particles

$$\begin{aligned} n_k^{j+1} &= |\beta_k^{j+1}|^2 \\ &= \mathrm{e}^{-\pi\lambda^2} + \left(1 + 2\mathrm{e}^{-\pi\lambda^2}\right) n_k^j \\ &\quad -2\mathrm{e}^{-\pi\lambda^2/2}\sqrt{1 + \mathrm{e}^{-\pi\lambda^2}}\sqrt{n_k^j(1 + n_k^j)}\,\sin\theta_{\text{total}}^j, \end{aligned} \tag{37}$$

where the phase θ_{total}^j is

$$\theta^j_{\text{total}} = 2\theta^{j+1}_k - \phi_k + \arg\,\beta^j_k - \arg\,\alpha^j_k.$$

We have used

$$|\alpha^j_k|^2 - |\beta^j_k|^2 = 1 \quad \text{and} \quad |\alpha^j_k| = \sqrt{1 + n^j_k}.$$

For boson $n_k \gg 1$ and the growth index is given by $\mu^j_k = (1/2\pi)\ln(n^{j+1}_k/n^j_k)$. The evaluation of phase θ^j_{total}, though not hard, is very complicated and can be avoided through the calculation of envelope. This μ_k serves to characterize the resonance structure. It has been shown [7] that though in static case one observes distinguishable stability/instability band, in expanding case the resonance band is much broader and there are no remarkable stability/instability bands.

4. Fermion production

We first consider massless quantum Fermi field $\psi(t, \mathbf{x})$

$$[i\gamma^\mu \nabla_\mu - h\phi(t)]\,\psi = 0, \tag{38}$$

in the model $V(\phi) = \frac{1}{4}\lambda\phi^4 + h\bar{\psi}\phi\psi$. With the substitutions $\varphi = a\phi, \Psi = a^{3/2}\psi$ and $\eta = \sqrt{\lambda\tilde{\varphi}}^2 \int \frac{\mathrm{d}t}{a(t)}$, we will find with $\Psi = (\gamma^\mu\nabla_\mu + h\varphi)X$, the second-order form of temporal equation for the eigenmode X_k as

$$\ddot{X}_k + (\kappa^2 + qf^2 - i\sqrt{q}\dot{f})X_k = 0, \tag{39}$$

where we have used $X = X_k(\eta)\mathrm{e}^{ik\cdot x}R_r$, $\kappa^2 \equiv k^2/\lambda\tilde{\varphi}^2$, $\phi = \tilde{\varphi}f$ as background inflaton solution and $q \equiv h^2/\lambda$; R_r being the eigenvectors of Dirac matrix γ^0 such that $\gamma^0 R_r = R_r$. It is found that $f(\eta)$ is periodic with unit amplitude given by $f(\eta) = cn(\eta, \frac{1}{\sqrt{2}})$. We have also used $\mathrm{d}s^2 = a^2(\eta)(\mathrm{d}\eta^2 - \mathrm{d}\vec{x}^2)$ as FRW background. The above substitution for $V(\phi) = \frac{1}{4}\lambda\phi^4 + h\bar{\psi}\phi\psi$-inflation put the Dirac equation in Minkowski form. We do not require to solve the ϕ and $a(t)$ evolution equations separately. However, this is not possible for other models of inflation.

For massive fermion, we consider a fermionic field ψ satisfying the Dirac equation

$$(i\gamma^\mu\nabla_\mu - g\phi(t) - m_\psi) = 0. \tag{40}$$

The solutions are obtained using auxillary field $X(\vec{x}, t)$ so that $\psi = (i\gamma^\mu\nabla_\mu + g\phi(t) + m_\psi)X$ and decomposing it as $\exp(i\vec{K}\cdot\vec{x})X_k(t)R_r$ with R_r being eigenvector of γ^0 with eigenvalue $+1$ we can write the equation of motion for fermion modes X_k as

$$\begin{aligned} X_{\tau\tau}(\kappa) + [\kappa^2 &+ (M_X + \sqrt{q}\tilde{\phi})^2 a^2 - i(M_X + \sqrt{q}\tilde{\phi})a_\tau \\ &- i\sqrt{q}\,a\,\tilde{\phi}_\tau]X(\kappa) = 0. \end{aligned} \tag{41}$$

Here F_τ refers to differentiation of F with respect to $\tau = m_\phi\eta$ and $\phi = \tilde{\phi}\,M_P$. The inflaton and scale factor evolution equations are given by

$$\tilde{\phi}_{\tau\tau} + 2\frac{a_\tau}{a}\tilde{\phi}_\tau + a^2\tilde{\phi} = 0, \tag{42}$$

$$\left(\frac{a_\tau}{a}\right)^2 = \frac{4\pi}{3}\left(\tilde{\phi}_\tau^2 + a^2\tilde{\phi}^2\right). \tag{43}$$

In eq. (41) $\kappa = K/m_\phi$, $\sqrt{q} = gM_P/m_\phi$, $M_X = m_\psi/m_\phi$ and the model Lagrangian is $\mathcal{L} = \frac{1}{2}\phi_{;\mu}\phi^{;\mu} - \frac{1}{2}m_\phi\phi^2 + \bar{\psi}(i\gamma^\mu\nabla_\mu - m_\psi)\psi + \frac{1}{2}g^2\bar{\psi}\psi\phi$. We will use (39) and (41) for numerical calculation.

Before passing on to the expanding case, let us consider the massless fermion case in Minkowski background ($a = 1$) with $f^2 = \Phi^2\sin^2(mt)$, $\Phi =$ constant. Equation (39) reduces to

$$\ddot{\chi}_k + (k^2 + q\Phi^2\sin^2(mt) - i\sqrt{q}m\Phi\,\cos(mt))\chi_k = 0 \tag{44}$$

since in flat background $t = \eta$.

4.1 *Reflection and transmission coefficients*

Expanding around $t = t_j$ where t_j denotes reflection points we get

$$\ddot{\chi}_k + \left[k^2 + q^2\Phi^2m^2(t-t_j)^2 - i\sqrt{q}m\Phi\right]\chi_k = 0. \tag{45}$$

With

$$k_* = q^{1/4}\phi^{14}m^{1/2}, \quad \tau = k_*(t-t_j)$$

eq. (45) is of the form

$$\chi_k'' + \left[\lambda^2 + \tau^2\right]\chi_k = 0, \tag{46}$$

where $\lambda^2 = \left(\frac{k^2}{k_*^2} - i\right) = (\lambda_0^2 - i)$. The reflection and transmission coefficients in this case are well-known and are given by

$$(R_k)_{\text{fermion}} = \frac{\mathrm{e}^{-\pi\lambda_0^2/2 - i\phi_{kf}}}{\sqrt{1-\mathrm{e}^{\pi\lambda_0^2}}}$$

and

$$(T_k)_{\text{fermion}} = \frac{\mathrm{e}^{-i\phi_{kf}}}{\sqrt{1-\mathrm{e}^{-\pi\lambda_0^2}}},$$

where ϕ_{kf} is the unknown phase given by

$$\phi_{kf} = \arg\Gamma\left(\frac{1+i\lambda^2}{2}\right) + \frac{\lambda^2}{2}\left(1 + \ln\frac{2}{\lambda^2}\right).$$

Using eqs (25) and (27), and the fermion conservation equation $|\alpha_k^j|^2 + |\beta_k^j|^2 = 1$ with $|\beta_k^j| = \sqrt{n_k^j}$, the number density of outgoing particles is given by

$$
\begin{aligned}
n_k^{j+1} &= |\beta_k^{j+1}|^2 \\
&= \mathrm{e}^{-\pi\lambda_0^2} + \left(1 - 2\mathrm{e}^{-\pi\lambda_0^2}\right) n_k^j \\
&\quad +2\mathrm{e}^{-\pi\lambda_0^2/2}\sqrt{1-\mathrm{e}^{-\pi\lambda_0^2}}\sqrt{n_k^j(1-n_k^j)}\cos\theta_{\mathrm{total}}^j,
\end{aligned} \tag{47}
$$

where

$$\theta_{\mathrm{total}}^j = 2\theta_k^{j+1} - \phi_k + \arg\beta_k^j - \arg\alpha_k^j.$$

Hence

$$N_1 = n_k^1 = |\beta_k^1|^2 = \mathrm{e}^{-\pi\lambda_0^2}$$

with no particles in initial vacuum, i.e., $\alpha_k^0 = 1$ and $\beta_k^0 = 0$. For fermion due to Pauli blocking we always have $n_k^j \leq 1$ and is satisfied by our expression of n_k^{j+1}. We have obtained n_k^j with Runge–Kutta method using (69) and found that the step function nature of n_k^j of (47) is reproduced through numerical calculation. The calculation of n_k^j for fermions show that the occupation number exhibits high-frequency oscillation which are modulated by a long periodic behaviour. If we average over the high-frequency oscillations, the average n_k turns out as $n_k = E_k \sin^2(n\Lambda)$ with $n = t/T$ being the number of oscillations. This E_k is known as the envelope and Λ/T plays the role of μ_k (boson production) in fermion production. Let us now proceed to calculate this envelope for any general periodic inflaton. For $\lambda\phi^4$ inflation f is given by *cn* function; for $m^2\phi^2$ inflation $\phi \simeq \phi_0\cos(mt)/\sqrt{3\pi}t$ and for hybrid inflation (with an extra scalar Higgs field with χ^4 interaction along with the ϕ^2 inflation) the background inflaton is given by periodic JacobiCD function. All the three models have been studied in detail and the results will be placed shortly. In the present work we report some of the results through the envelope calculation.

5. Calculation of envelope

In evaluating n_k^{j+1} for boson or fermion using eqs (37) and (47), we note that the calculation of $\theta_{\mathrm{total}}^j$ is not very easy. However, we can avoid this phase-factor calculation if we evaluate the envelope function of particle production using CWKB technique. The envelope function can be found easily and it avoids complicated calculations. The envelope function for fermion production is now calculated in the following way. We write (25) as

$$
\begin{pmatrix} \alpha_k^{j+1} \\ \beta_k^{j+1} \end{pmatrix} = \begin{pmatrix} F & G^*\mathrm{e}^{2i\theta_0^j} \\ G^{-2i\theta_0^{j+1}} & F^* \end{pmatrix} \begin{pmatrix} \alpha_k^j \\ \beta_k^j \end{pmatrix}, \tag{48}
$$

where

$$\theta_a^b = \int_{t_a}^{t_b} (k^2 - V(t))\mathrm{d}t. \tag{49}$$

To satisfy fermions, the Bogolubov coefficients should satisfy the relation

$$|\alpha_k^j|^2 + |\beta_k^j|^2 = |\alpha_k^{j+1}|^2 + |\beta_k^{j+1}|^2 = 1. \tag{50}$$

To get a clear idea we write for one complete oscillation

$$\begin{pmatrix} \alpha_k^{j+1} \\ \beta_k^{j+1} \end{pmatrix} = \begin{pmatrix} F & G\mathrm{e}^{2i\theta_j^{j+1}} \\ G^*\mathrm{e}^{-2i\theta_j^{j+2}} & F_k^* \end{pmatrix} \times \begin{pmatrix} F & G\mathrm{e}^{2i\theta_0^j} \\ G^*\mathrm{e}^{-2i\theta_0^{j+1}} & F_k^* \end{pmatrix} \begin{pmatrix} \alpha_k^j \\ \beta_k^j \end{pmatrix} = O_2\, O_1 \begin{pmatrix} \alpha_k^j \\ \beta_k^j \end{pmatrix}. \tag{51}$$

As $\det O_1 = \det(O_2\, O_1) = 1$, we get from $\det O_1 = 1$

$$|\alpha_k^j|^2 - |\beta_k^j|^2 \mathrm{e}^{-2i\theta_j^{j+1}} = 1.$$

Hence

$$\mathrm{e}^{-2i\theta_j^{j+1}} = -1 \tag{52}$$

to satisfy the fermion conservation equation (50). We thus get

$$\begin{pmatrix} \alpha_k^{j+2} \\ \beta_k^{j+2} \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ -B_1^* & A_1^* \end{pmatrix} \begin{pmatrix} \alpha_k^j \\ \beta_k^j \end{pmatrix}, \tag{53}$$

where

$$A_1 = F^2 + G^2\mathrm{e}^{-2i\theta_0^j} \tag{54}$$

$$B_1 = FG\mathrm{e}^{2i\theta_0^j} - F^*G \tag{55}$$

and

$$F = \frac{1}{T_k} = \sqrt{(1-\mathrm{e}^{\pi\lambda_0^2})}\mathrm{e}^{i\varphi}, \quad G = \left|\frac{R_k}{T_k}\right| = \mathrm{e}^{-\pi\lambda_0^2}, \quad \theta_a^b = S(b,a). \tag{56}$$

Incidentally our result will be the same as in [9] for the expression of envelope provided we write

$$A_1 = F^2\mathrm{e}^{i(\theta_1^2+\theta_2^3)} + G^2\mathrm{e}^{-i(\theta_1^2-\theta_2^3)}$$
$$B_1 = \mathrm{e}^{2i\theta_1^2}(FG - F^*G\mathrm{e}^{-i(\theta_1^1-\theta_2^3)})$$

using the relation (52). After nth complete oscillations we have

$$\begin{pmatrix} \alpha_k^{j+2n} \\ \beta_k^{j+2n} \end{pmatrix} = \tilde{O}_n\tilde{O}_{n-1}\ldots\tilde{O}_2\tilde{O}_1 \begin{pmatrix} \alpha_k^j \\ \beta_k^j \end{pmatrix}, \tag{57}$$

where $\tilde{O}_1 = O_1O_2$. We notice that $\det\tilde{O}_1 = 1$. We now take the initial state as

$$(\alpha_k^j, \beta_k^j)^T \equiv (\alpha_k^0, \beta_k^0)^T = (1,0)^T$$

so that there is no particle in the initial state before entering the barrier. As the operator $\tilde{O}_1$ is repeated n times, we write after nth oscillations

$$\begin{pmatrix} \alpha_k^{2n} \\ \beta_k^{2n} \end{pmatrix} = O^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{58}$$

Here O^n has to be understood as product terms as in the above. The number of particles produced after n oscillations is now calculated as follows. One has simply to study the eigenvalue problem for $\tilde{O}_1$. The eigenvalues are $\lambda_{1,2} = \mathrm{e}^{\pm i\Lambda}$ with $\cos\Lambda = \mathrm{Re}\, A_1$. The corresponding eigenvectors are

$$|1\rangle = \begin{pmatrix} -\frac{B_1}{A_1-\lambda_1} \\ 1 \end{pmatrix} \tag{59}$$

$$|2\rangle = \begin{pmatrix} -\frac{B_1}{A_1-\lambda_2} \\ 1 \end{pmatrix} \tag{60}$$

and write the initial states as

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = C|1\rangle + D|2\rangle \tag{61}$$

and find

$$C = \frac{(A_1 - \mathrm{e}^{i\Lambda})(A_1 - \mathrm{e}^{-i\Lambda})}{2B_1 \cos\Lambda}. \tag{62}$$

We now use the condition $(A_1 - \mathrm{e}^{i\Lambda})(A_1^* - \mathrm{e}^{-i\Lambda)} + |B_1|^2 = 0$ to get

$$|C|^2 = \frac{|B_1|^2}{4\sin^2\Lambda}. \tag{63}$$

It is now easy to find the occupation number N_n after n oscillations as

$$\begin{aligned} N_n \equiv |\beta_k^{2n}|^2 = |C|^2 \sin^2\Lambda &= \frac{|B_1|^2}{1-(\mathrm{Re}\, A_1)^2} \sin^2(n\Lambda) \\ &= \frac{|B_1|^2}{\sin^2\Lambda} \sin^2(n\Lambda). \end{aligned} \tag{64}$$

Equation (64) is now devoid of phase θ_{tot}^j as anticipated earlier.

If we put $n = 1$ in eq. (64) we get $N_1 = |B_1|^2$ which we know very easily from CWKB and is given by $\mathrm{e}^{-\pi\lambda_0^2}$. Obviously $n = t/T$, and hence N_n has peaks at $n\Lambda = (2k+1)\pi/2$, the first peak occurs at $\Lambda = \pi/2n$. This is the resonance peak. This allows us to determine Λ from numerical calculation where $N_n \to 1$ first. The equation is identical with the expression of the envelope given by Greene and Kofman [7] with

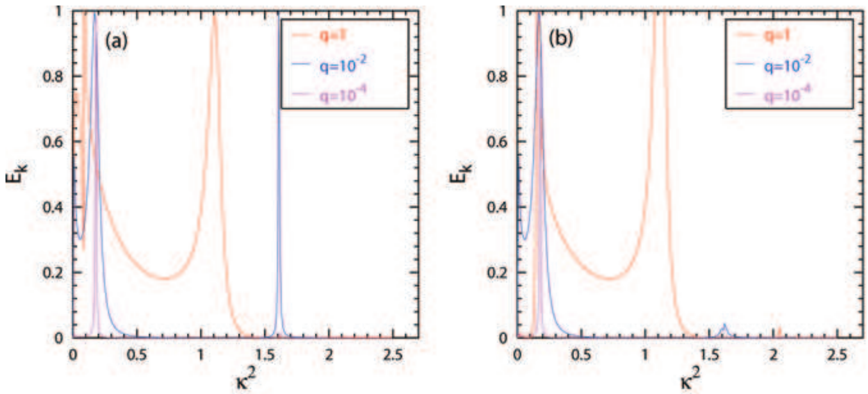

Figure 4. The resonance band in $\frac{1}{2}m_\phi^2\phi^2$-inflation with $q = \frac{g^2 M_P^2}{m_\phi^2} = 1,\ 10^{-2}$, 10^{-4} from broadest (red) to narrowest band; and the width of the resonance band decreasing with q values. Here $M_X = 0$ and the expansion of the Universe is not taken into account. Figure 4a is plotted with (70) and 4b with (71).

$$\Lambda = \nu_k T \quad \text{and} \quad E_k = \frac{|B_1|^2}{\sin^2\Lambda}. \tag{65}$$

Putting $n = 2$ in eq. (64) we get

$$N_2 = \frac{|B_1|^2}{1 - (\mathrm{Re}\,A_1)^2}\sin^2 2\Lambda = \frac{4|B_1|^2\sin^2\Lambda\cos^2\Lambda}{\sin^2\Lambda}$$

and thus $\cos^2\Lambda = N_2/4\,N_1$. Hence envelope function is given by

$$E_k = \frac{|B_1|^2}{\sin^2\Lambda} = \frac{N_1}{1 - \frac{N_2}{4\,N_1}}. \tag{66}$$

We will use this result for numerical calculation where N_1 and N_2 are obtained by Runge–Kutta method. The envelope so obtained will then be compared with the occupation number after ten oscillations.

6. Numerical results

The master equation in the case of fermion production is given by

$$\chi'' + [\Omega^2 \pm i(am)']\chi = 0, \tag{67}$$

where $\Omega^2 = \kappa^2 + a^2m^2$ and $m(\eta) = m_X + g\,\phi(\eta)$. We, for convenience, omit the subscript k on χ. In $\frac{1}{4}\lambda\phi^4$-type of inflation, for massless fermions ($m_\psi = 0$) the master equation is brought to equivalent Minkowski form (see §2)

$$\chi'' + (\omega^2 - i\sqrt{q}\dot{f})\chi = 0, \tag{68}$$

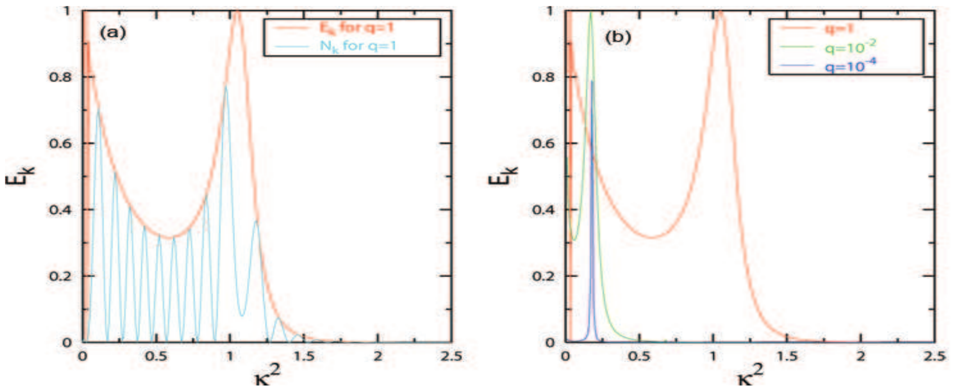

Figure 5. Envelope function E_k plotted with eq. (66) with $M_X = 0$, $\phi = \Phi_0 \cos mt$: (**a**) E_k and oscillation number N_k after ten oscillations for $q = 1$ and (**b**) E_k for different values of q. The expansion of the Universe is not taken into consideration.

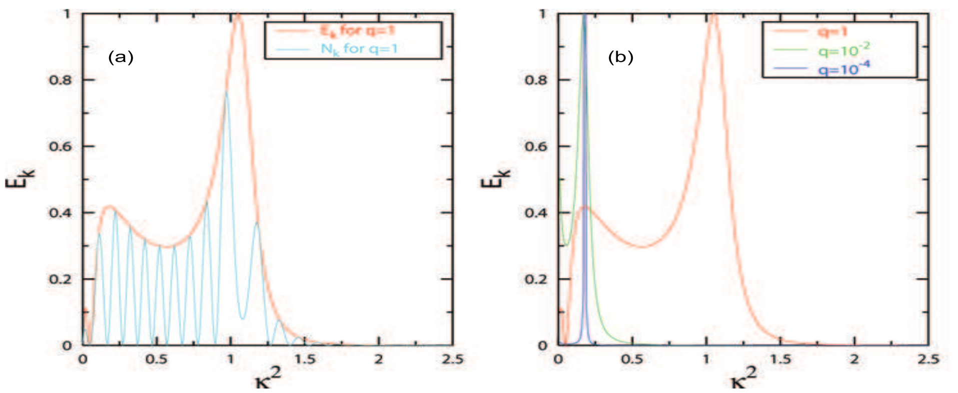

Figure 6. E_k plotted with $M_X = 0.1, \phi = \Phi_0 \cos mt$: (**a**) E_k and oscillation number N_k after ten oscillations for $q = 1$ and (**b**) E_k for different values of q.

where $\omega^2 = \kappa^2 + qf^2$ and $\phi(\eta) = f = cn(\tau, 1/\sqrt{2})$ being a periodic function of constant amplitude. The occupation number has already been derived earlier [10] and is given by

$$N_k = \frac{1}{2} - \frac{\kappa^2}{\Omega}\mathrm{Im}(\chi\chi^{*\prime}) - \frac{\sqrt{q}f}{2\Omega}. \tag{69}$$

The envelope function used by Greene and Kofman is [7]

$$(F_k)_{\mathrm{GK}} = \frac{1}{\sin^2(\nu_k T)}\frac{\kappa^2}{2\omega}\left(\mathrm{Im}\,\chi_k^{(1)}(T)\right)^2, \tag{70}$$

where $\chi_k^{(1)}$=first fundamental solution of eq. (68) with $\chi_k^{(1)} = 1\,,\, \dot{\chi}_k^{(1)} = 0$.

We use the envelope function as

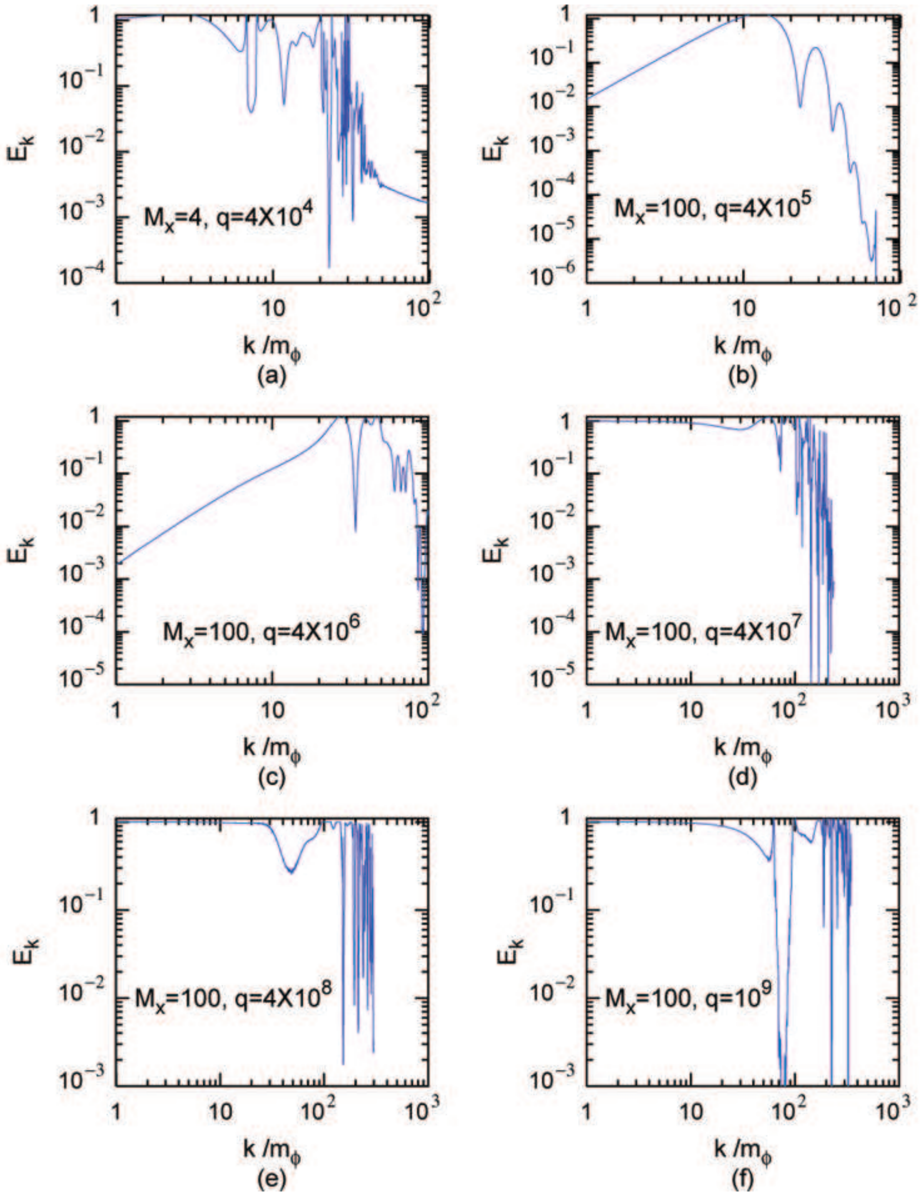

Figure 7. E_k plotted with different values of M_X and q using eq. (66) and taking into account the expansion of the Universe.

$$E_k = \frac{N_1}{1 - \frac{N_2}{4\,N_1}}. \tag{71}$$

Instead of testing the numerical calculation of N_k with an approximate analytical expression (as is done in other works in the literature), we test the expression of

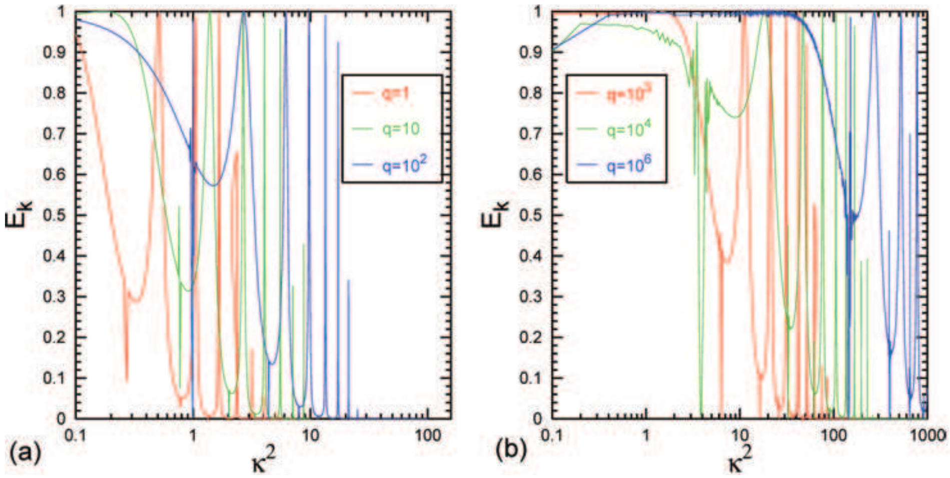

Figure 8. E_k plotted with different values of q using eq. (71) in hybrid inflation. At higher k values the width of the resonance peaks decreases. The study of instability chart (that we have carried out) also reveals this feature.

envelope which requires only N_1 and N_2. This fits with the occupation number after n oscillations (may be 8–10) will test both the analytical and the numerical calculation since in calculating the envelope and the occupation number we keep the initial vacuum the same, unlike other works in the literature and discussed in [10]. The advantage of CWKB is that we do not have to solve the mode equations to calculate the occupation number.

The numerical plots for $\frac{1}{4}\phi^4$ inflation has already been studied in details in our previous work [10] and since the curves for $\frac{1}{2}\phi^2$ inflation are more or less the same (without the expansion of the Universe) as in $\sim\phi^4$ inflation, we reproduce only the envelopes of $\sim\phi^2$ inflation using (70) and (71). In $(E_k)_{\rm GK}$ (figure 4a) we observe three resonance peaks whereas in $(E_k)_{\rm CWKB}$ we observe two peaks where $E_K \to 1$ at resonance. Both the calculations give almost the same result. To confirm that the CWKB envelope calculation fits remarkably with the occupation number after ten oscillations, we plot in figure 5a E_k and N_k in the same graph. For massive fermions using $M_X \geq 0, \phi = \Phi_0 \cos mt$ in eq. (67) the graphs plotted with eq. (66) agree satisfactorily with the occupation number. This is shown in figure 6a and 6b. In the case of $\frac{1}{2}m^2\phi^2$ inflation for massive fermion with $M_X \neq 0$ and considering expansion of the Universe we find $\phi \simeq (\Phi/\sqrt{3\pi}mt)\cos mt$. The graphs plotted with eq. (66) fit the occupation number very well (figure 7). We have also plotted graphs using the same equation for hybrid inflation with $M_X = 0$ and $\phi =$ JacobiCD function, the envelope fits very well. This is shown in figure 8. It may be noted in figure 7 that at small values of k/m_ϕ our envelope differs remarkably from the works in [9] for the same set of resonance parameters q. We have tested our envelopes through the calculation of occupation number. We have taken $a(0) = 1$, $\tilde{\phi}(0) = 0.28$, and $\tilde{\phi}_\tau(0) = -0.15$ in our calculation and solved the evolution equations numerically. For hybrid inflation we have taken

$$V(\phi,\chi) = \frac{M_P^4}{4\lambda} - \frac{1}{2}m_\chi^2\chi^2 + \frac{1}{4}\lambda\chi^4 + \frac{1}{2}g^2\chi^2\phi^2 + \frac{1}{2}m_\phi^2\phi^2. \tag{72}$$

Here ϕ is the inflaton that slow-rolls down a potential driving inflation, and χ is the other scalar Higgs field with symmetry breaking potential that triggers the end of inflation.

We have made detailed numerical calculation of occupation number, and envelope function for $V(\phi) \sim \phi^2$ and $\sim\phi^4$ inflation and in hybrid inflation as in (72). The details in each case and its relation to heating mechanism will be presented shortly. Our work is not simply a restudy in CWKB, but as noted earlier in our previous work, many results in the literature in this regard are to be questioned. In CWKB we can double check. We plot the envelope function and the occupation number for n oscillations in the same plot to show the correctness of our calculation. We will discuss all these aspects in our next work.

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