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Hierarchy of rational order families of chaotic maps with an invariant measure

M A JAFARIZADEH[1,2,3], M FOROUTAN[2,3,4] and S AHADPOUR[1,2,3]
[1]Department of Theoretical Physics and Astrophysics, Tabriz University, Tabriz 51664, Iran
[2]Institute for Studies in Theoretical Physics and Mathematics, Tehran 19395-1795, Iran
[3]Research Institute for Fundamental Sciences, Tabriz 51664, Iran
[4]Department of Chemistry, Faculty of Science, Tehran University, Tehran, Iran
E-mail: jafarizadeh@tabrizu.ac.ir; m.foroutan@ut.ac.ir; s.ahadpour@tabrizu.ac.ir

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Abstract. We introduce an interesting hierarchy of rational order chaotic maps that possess an invariant measure. In contrast to the previously introduced hierarchy of chaotic maps [1–5], with merely entropy production, the rational order chaotic maps can simultaneously produce and consume entropy. We compute the Kolmogorov–Sinai entropy of these maps analytically and also their Lyapunov exponent numerically, where the obtained numerical results support the analytical calculations.

Keywords. Kolmogorov–Sinai entropy; invariant measure; Lyapunov exponent; chaos.

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1. Introduction

There have been some attempts [1–6] at introducing the hierarchy of chaotic maps with an invariant measure in recent years. The objective of these papers is to describe the dynamic behavior of chaotic maps using Kolmogorov–Sinai entropy. The hierarchies of these chaotic maps are of interest as models for describing the behavior of dynamical systems. As an example, the random chaotic maps have attracted the attention of physicists as models of convection by temporarily irregular fluid flows [7]. Once a map is determined, the long-term statistical behavior is described by a probability density function, which can be obtained by solving the Frobenius–Perron equation [8] or can be estimated by measurement of the system. Therefore, the complexity, non-linearity and non-stationarity of physical, chemical, biological, physiological and financial systems [9–14] have been of main interest in introducing the new hierarchy of the chaotic maps. On the other hand, the sensitively to the initial condition, control parameter and ergodicity which have strong relationships with the requirement of pseudo-random coding and cryptography [15,16] are examples of interesting features of chaotic systems and it is natural to use chaos as a new source to construct new encryption systems [17].

In the present paper, we introduce the rational order families of chaotic maps as a new hierarchy of chaotic map with an invariant measure. The Kolmogorov–Sinai entropy of these chaotic maps can be calculated analytically by using their invariant measure. An interesting property of these chaotic maps is their ability in simultaneous production and consumption of entropy. Additionally, being a measurable dynamical system, it can be studied analytically.

The paper is organized as follows: In §2, we introduce the rational order families of chaotic maps. In §3, the invariant measures of these maps are given and in §4, we review the Kolmogorov–Sinai entropy and compute it for the rational order chaotic maps. Finally, in §5 we calculate the Lyapunov exponent numerically and compare the results of simulation with analytically calculated Kolmogorov–Sinai entropy. The last sections contain our conclusion and two appendices. In these appendices we have calculated the invariant measure of the rational order families of chaotic maps via two different methods.

2. Hierarchy of rational order families of chaotic maps with an invariant measure

We first review hierarchy of one-parameter chaotic maps which can be used in the construction of families of rational order chaotic maps with an invariant measure. The one-parameter chaotic maps [1] are defined as the ratio of polynomials of degree N:

$$\begin{aligned}\phi_N^1(x,a) &= \frac{a^2(1+(-1)^N\ {}_2F_1(-N,N,\frac{1}{2},x))}{(a^2+1)+(a^2-1)(-1)^N\ {}_2F_1(-N,N,\frac{1}{2},x)} \\ &= \frac{a^2(T_N(\sqrt{x}))^2}{1+(a^2-1)(T_N(\sqrt{x}))^2},\end{aligned}$$

$$\begin{aligned}\phi_N^2(x,a) &= \frac{a^2(1-(-1)^N\ {}_2F_1(-N,N,\frac{1}{2},(1-x)))}{(a^2+1)-(a^2-1)(-1)^N\ {}_2F_1(-N,N,\frac{1}{2},(1-x))} \\ &= \frac{a^2(U_N(\sqrt{(1-x)}))^2}{1+(a^2-1)(U_N(\sqrt{(1-x)}))^2},\end{aligned}$$

where N is an integer greater than 1. Also,

$${}_2F_1\left(-N,N,\tfrac{1}{2},x\right) = (-1)^N\cos(2N\arccos\sqrt{x}) = (-1)^N T_{2N}(\sqrt{x})$$

is the hypergeometric polynomials of degree N and $T_N(U_n(x))$ are Chebyshev polynomials of Type-I (Type-II), respectively. In this paper we are concerned about their conjugate maps which are defined as

$$\begin{cases} \tilde{\phi}_N^{(1)}(x,a) = h \circ \phi_N^{(1)}(x,a) \circ h^{-1} = \dfrac{1}{a^2}\tan^2(N\arctan\sqrt{x}), \\ \tilde{\phi}_N^{(2)}(x,a) = h \circ \phi_N^{(2)}(x,a) \circ h^{-1} = \dfrac{1}{a^2}\cot^2\left(N\arctan\dfrac{1}{\sqrt{x}}\right). \end{cases} \tag{2.1}$$

Conjugacy means that invertible map $h(x) = \frac{1-x}{x}$ maps $I = [0,1]$ into $[0,\infty)$.

Now, in order to generalize the above hierarchy of integer order chaotic maps to the hierarchy of rational order chaotic maps with an interesting property of simultaneous production and consumption of entropy, we need to replace x_{n+1} with a non-linear function of x_{n+1}, particularly a non-linear function of chaotic maps of the above type. But in order to have a single-valued map, we will take one of the inverse branches of the above non-linear functions for x_{n+1} in each step with a probability equal to the probabilities of occurrence of the branches in the iteration of the maps. As we will show in §3, the probability of occurrence of each branch is equal to the integral of invariant measure of the map over the corresponding domain of the same branch. Therefore, we can define these maps as

$$x_{n+1,k} = g_{i,k}^{-1} \circ g_j(x_n), \quad (i,j \in \{1,2\}) \text{ with probability } P_k, \tag{2.2}$$

where functions g_1 and g_2 can be chosen as one the functions given in (2.1) and P_k are probabilities of occurrence of inverse branches $g_{2,k}^{-1}$ of the map g_2 in its iteration. As we will see at the end of this section, the existence of an invariant measure will impose a relation between their parameters.

Of course the functions given in (2.1) are not the only choice for the functions g_1 and g_2 that lead to the hierarchy of rational order chaotic maps with an invariant measure. Obviously the following choices of the functions g_1 and g_2, i.e.,

$$\begin{array}{llll} \text{(a)} & \dfrac{1}{a^2}\tan^2(N\ \mathrm{arccot}\sqrt{x}), & \text{(b)} & \dfrac{1}{a^2}\cot^2(N\ \arctan\sqrt{x}), \\ \text{(c)} & \dfrac{1}{a^2}\cot^2(N\ \mathrm{arccot}\sqrt{x}), & \text{(d)} & \dfrac{1}{a}|\tan(N\ \arctan|x|)|, \\ \text{(e)} & \dfrac{1}{a}|\tan(N\ \mathrm{arccot}\,|x|)|, & \text{(f)} & \dfrac{1}{a}|\cot(N\ \arctan|x|)|, \\ \text{(g)} & \dfrac{1}{a}|\cot(N\ \mathrm{arccot}\,|x|)|, & \text{(h)} & \dfrac{1}{a}\tan(N\ \arctan x), \\ \text{(I)} & \dfrac{1}{a}\tan(N\ \mathrm{arccot}\,x), & \text{(J)} & \dfrac{1}{a}\cot(N\ \arctan x), \\ \text{(k)} & \dfrac{1}{a}\cot(N\ \mathrm{arccot}\,x) & & \end{array} \tag{2.3}$$

lead to the hierarchy of rational order chaotic maps of trigonometric types (with an invariant measure), where some of them are equivalent to each other up to conjugacy. Also with the choices of g_1 and g_2 as [4]

$$\text{(a)}\ \frac{1}{a^2}\mathrm{sc}^2(N\mathrm{sc}^{-1}(\sqrt{x})), \quad \text{(b)}\ \frac{1}{a^2}\mathrm{cs}^2(N\mathrm{cs}^{-1}(\sqrt{x}))\ , \tag{2.4}$$

we get the hierarchy of elliptic rational order chaotic maps of cs and sc types, where their invariant measures can be obtained for small enough values of module k of elliptic functions. Also it is possible to choose the functions g_1 and g_2 as one of the combined chaotic maps of ref. [3].

In this paper we will consider the hierarchy of rational order maps with

$$g_1(a_1, N_1, x_n) = \frac{1}{a_1}\tan(N_1 \arctan x_n)$$

and

$$g_2(a_2, N_2, x_{n+1}) = \frac{1}{a_2}\tan(N_2 \arctan x_{n+1}), \tag{2.5}$$

i.e., we have

$$x_{n+1,k_2} = \tan\left(\frac{\arctan(\frac{a_2}{a_1}\tan(N_1 \arctan x_n))}{N_2} + \frac{k_2\pi}{N_2}\right) \tag{2.6}$$

with probability P_{k_2}, $k_2 = 1, 2, ..., N_2$, where N_1 and N_2 are integers greater than 1 and a_1 and a_2 are control parameters. As we are going to see in §3, the maps (2.6) posses an invariant measure provided that we choose the parameters a_1 and a_2 in the form given in eqs (3.12) and (3.13), respectively. As an example, we consider the following map for $N_1 = 3$ and $N_2 = 2$:

$$x_{n+1,\pm} = \frac{a_1}{a_2} \times \frac{1-3x_n^2}{3x_n - x_n^3} \pm \sqrt{1 + \left(\frac{a_1}{a_2} \times \frac{1-3x_n^2}{3x_n - x_n^3}\right)^2} \tag{2.7}$$

with probabilities $P_\pm = \frac{1}{2}$.

3. Invariant measure

The dynamical systems, even the discrete-time and one-variable ones have different types of behaviour. The system can be in a fixed point and nothing changes, the trajectory of the system may also be a cycle with a certain period. Fixed point and periodic orbits may be stable or unstable. We are usually interested in an invariant measure μ, i.e. a probability measure that does not change under the dynamics. The probability measure μ on $[0, 1]$ is a Sinai–Rulle–Bowen (SRB) measure which is an invariant measure which describes statistically the stationary states of the system and absolutely continues with respect to Lebesgue measure.

Now, in order to determine the invariant measure of the analytical system described by the maps given in (2.6), we can write it as combination of the maps g_1 and g_{2,k_2}^{-1} (as the k_2-th inverse branch of g_2) in the following form:

$$x_{n+1,k_2} = g_{2,k_2}^{-1} \circ g_1(x_n) \quad \text{with probability } P_{k_2}, \quad k_2 = 1, ..., N_2 \tag{3.1}$$

with g_1 and g_2 given in (2.5).

Obviously the function $g_2(\cdot, a_2, N_2)$ maps its N_2 inverse branches $x_{n+1,k_2}, k_2 = 1, 2, ..., N_2$ with corresponding different domains $\Delta x_{n+1,k_2}$ ($\Delta x_{n+1,i} \cap \Delta x_{n+1,j} = \emptyset$ for $i \neq j = 1, 2, ..., N_2$) into the same region. Therefore, if we denote its value by y for different values of its argument, then the map (3.1) can be written as

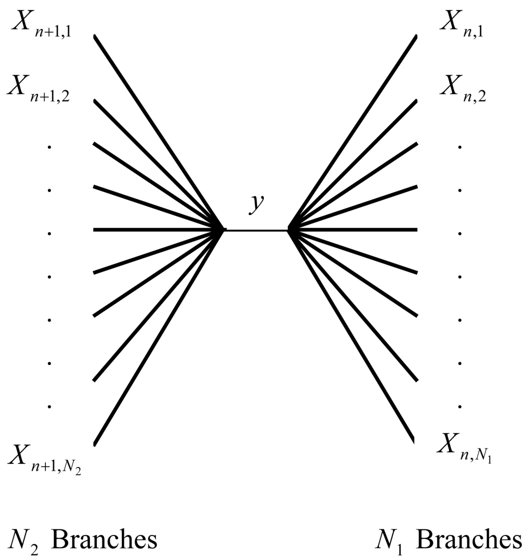

Figure 1. The schematic diagram of forward and backward branching of rational order chaotic map for describing simultaneous production and consumption of entropy.

$$g_2(x_{n+1}, a_2, N_2) = y = g_1(x_n, a_1, N_1), \tag{3.2}$$

irrespective of to which branch or domain, the output x_{n+1} belongs (see figure 1). But in order to have a single output or single-valued dynamical map, we have to consider only one possible x_{n+1} in each step with some probabilities or weights. Certainly the most natural weight of a given branch is the corresponding probability of its occurrence in infinite iteration of map $y = g_2(\cdot, a_2, N_2)$, where it can be written in terms of its invariant measure μ_{g_2} as

$$P(\text{occurrence of } k_2 - \text{the branch}) = \int_{\Delta x_{n+1,k_2}} \mu_{g_2}(x)\mathrm{d}x. \tag{3.3}$$

Therefore, the invariant measure of this map should satisfy the following Frobenius–Perron (FP) integral equations:

$$\mu(y) = \int_0^1 \delta(y - g_1(x_n, a_1, N_1))\mu(x_n)\mathrm{d}x_n \tag{3.4}$$

and

$$\mu(y) = \int_0^1 \delta(y - g_2(x_{n+1}, a_2))\mu(x_{n+1})\mathrm{d}x_{n+1}, \tag{3.5}$$

which are equivalent to

$$\mu(y) = \sum_{x_{n,k_1} \in g_1^{-1}(y)} \mu(x_{n,k_1}) \left| \frac{\mathrm{d}x_{n,k_1}}{\mathrm{d}y} \right| \tag{3.6}$$

and

$$\mu(y)=\sum_{x_{n+1,k_2}\in g_2^{-1}(y)}\mu(x_{n+1,k_2})\left|\frac{\mathrm{d}x_{n+1,k_2}}{\mathrm{d}y}\right|, \tag{3.7}$$

where

$$x_{n,k_1}=\tan\left(\frac{1}{N_1}\arctan(a_1y)+\frac{k_1\pi}{N_1}\right),\quad k_2=1,...,N_1$$

and

$$x_{n+1,k_2}=\tan\left(\frac{1}{N_2}\arctan(a_2y)+\frac{k_2\pi}{N_2}\right),\quad k_2=1,...,N_2.$$

The invariant measure $\mu(y)$ for $g_i(x)$ can be written as

$$\mu_{g_i(x)}(y)=\sum_{k_i=1}^{N_i}\frac{a_i}{N_i}\left(\frac{1+x_{n,ki}^2}{1+(a_iy)^2}\right)\mu_{g_i}(x),\quad i=1,2. \tag{3.8}$$

Assuming that $\mu(x)$ has the following form:

$$\mu(x)=\frac{\sqrt{\beta}}{\pi(1+\beta x^2)}, \tag{3.9}$$

where for $\beta=1$, it reduces to the invariant measure which has already applied to pushout measure [18], expression (3.6) reduces to

$$\frac{1+(a_iy)^2}{1+\beta y^2}=\sum_{k_i=1}^{N_i}\frac{a_i}{N_i}\left(\frac{1+x_{n,ki}^2}{1+\beta x_{n,ki}^2}\right),\quad i=1,2. \tag{3.10}$$

By comparing both sides of eq. (3.8), we can determine a_i $(i=1,2)$ as

$$a_i=\frac{\sum_{k=0}^{[N_i/2]}C_{2k}^{N_i}\beta^k}{\sum_{k=0}^{[(N_i-1)/2]}C_{2k+1}^{N_i}\beta^k}, \tag{3.11}$$

for even values of N_i, and

$$a_i=\frac{\sum_{k=0}^{[(N_i-1)/2]}C_{2k+1}^{N_i}\beta^k}{\sum_{k=0}^{[N_i/2]}C_{2k}^{N_i}\beta^k}, \tag{3.12}$$

for odd values of N_i (for proof, see Appendix A).

Therefore, a_1 and a_2 depend on the parameter β and integers N_1 and N_2, respectively. Also to make the paper more readable, we have derived the invariant measure of the map $y=\frac{1}{4}\tan(4\arctan x)$ by using Shure's invariant polynomials in Appendix B.

4. Kolmogorov–Sinai entropy

In this section we review first, the Shannon entropy and then talk about Kolmogorov–Sinai entropy (for more details, see [19]). Consider a dynamical system characterized by a certain iterative map. Let $B = (B_i, B_j, ..., B_n)$ be a decomposition of the unit interval along x_n. Now we subdivide each interval B_i into say Λ points, and perform ζ iterations on each one of them. So we make sure that transients have died out. Λ points by then will spread to other subintervals. A percentage of them will be perhaps located within the limits of B_j. After transients die out, the common area of $F^{\xi}(B_i)$ and B_j, e.g. $F^{\xi}(B_i) \cap B_j$ will be expressed in a non-normalized way, the number of elements of B_i reaching B_j after ξ iterations. So, in normalized form

$$W^{\xi}(B_j/B_i) = \frac{\mu(F^{\xi}(B_i) \cap B_j)}{\mu(B_j)}. \tag{4.1}$$

Here $\mu(\cdot) = \int_c \mu(x)\mathrm{d}x$, where c is the pertinent interval. The entropy of the chosen partition or, the average amount of information needed to locate the system in state space is given by the Shannonian entropy

$$S = -\sum_{i}^{\Lambda} \mu(B_i) \log_2 \mu(B_i) \text{ bits}. \tag{4.2}$$

The Λ values $\mu(B_i)$ may be calculated from the W_{ij} elements from the $(\Lambda - 1)$ equations of the linear system

$$\mu(B_i) = \sum_{j=1}^{\Lambda} \mu(B_j) W_{ij} \tag{4.3}$$

and the normalization condition

$$\sum_{j=1}^{\Lambda} \mu(B_j) = 1,$$

where the transition probability matrix W_{ij} describes the probability of jumping in one step (iteration) from the element B_i of the partition to the element B_j. The average amount of information created by the linguistic system per transition per unit time is given by the Kolmogorov–Sinai entropy for the chosen partition, namely,

$$S_k = \sum_{i=1}^{\Lambda} \sum_{j=1}^{\Lambda} \mu(B_i) W_{ij} \log_2 W_{ij} \text{ bits}. \tag{4.4}$$

The macroparameter however, characterizing the degree of grammatical coherence of the created Markovian chain is the mutual information or transinformation.

$$I(\xi) = \sum_{i=1}^{\Lambda} \sum_{j=1}^{\Lambda} \mu(F^{\xi}(B_i) \cap B_j) \log_2 \frac{\mu(F^{\xi}(B_i) \cap B_j)}{\mu(F^{\xi}(B_i)\mu(B_j))} \text{ bits}. \tag{4.5}$$

Equation (4.5) shows why $I(\xi)$ is called mutual information, since it stands for the information stored in a symbol along the sequence on it gives the information transformed between two symbols ξ steps apart. The number of decomposition of the unit interval goes to infinity, in such a way that, the size of each interval (B_i) goes to zero. The mutual entropy given in (4.5) reduces to the well-known Kolmogorov–Sinai (KS) entropy which is given by

$$h(\mu, g(x,a,N)) = \int \mu(x)\mathrm{d}x \ln\left|\frac{\mathrm{d}x_{n+1}}{\mathrm{d}x_n}\right| = \int_{-\infty}^{+\infty} \mu(x)\mathrm{d}x \ln\left|\frac{\mathrm{d}}{\mathrm{d}x} g(x,a,N)\right| \tag{4.6}$$

with $g(x,a,N) = (1/a)(\tan(N\arctan x))h(\mu, g(x,a,N))$ can be written as

$$h(\mu, g(x,a),N) = \int_{-\infty}^{+\infty} \frac{\sqrt{\beta}}{\pi(1+\beta x^2)}\mathrm{d}x \ln\left|\frac{N}{a} \times \frac{1+a^2y^2}{1+x^2}\right|. \tag{4.7}$$

Following the calculation in ref. [1], one can show that after a change of variable $\sqrt{\beta}x = \tan\theta$, and using the integral of type

$$\frac{1}{\pi}\int_0^{\pi} \ln|a+b\cos\theta| = \begin{cases} \ln\left|\dfrac{a+\sqrt{a^2-b^2}}{2}\right| & |a|>|b|, \\ \ln\left|\dfrac{b}{2}\right| & |a|\le|b|, \end{cases} \tag{4.8}$$

we get the following expression for the KS-entropy:

$$h(\mu, g(x,a,N)) = \frac{1}{\sqrt{\beta}} \ln\left[\frac{N}{a^3}\left(\frac{\sqrt{\beta}(\sum_{k=0}^{[N_i/2]} C_{2k}^{N_i} x^k)a + a(\sum_{k=0}^{[(N_i-1)/2]} C_{2k+1}^{N_i} x^k)}{(\sqrt{\beta}+1)(\sum_{k=0}^{[N_i/2]} C_{2k}^{N_i} x^k)}\right)^2\right]. \tag{4.9}$$

Now, we come to calculate the KS-entropy of fractional order maps. Before getting involved with the details of calculation, we first talk about simultaneous production and consumption of entropy in these maps. Figure 1 gives us an insight about how this is possible. In this figure, N_1 (N_2) corresponds to the number of branches $x_{n+1}(x_n)$ of the map at time $n+1(n)$. The left half of figure 1 shows the convergence of N_1 identical branches of x_{n+1} into a single branch, while the right half of the figure shows divergence to N_2 branches x_n, where they correspond to increase and decrease of entropy, respectively. It should be reminded that in the contraction of branches, entropy increases due to loss of information, while in the branch out it decreases due to the gain of information. Now in order to calculate KS-entropy of fractional order maps (2.6), we should notice that the right and left halves of figure 1 correspond to the right-hand and left-hand sides of eq. (3.3) (equivalent to (2.6)). In other words, there are N_1 convergent branch x_{n+1} and N_2 divergent branch x_n

in the left and right halves of figure 1. Therefore, according to figure 1 there are N_2 possible final states x_{n+1,k_2}, $k_2 = 1, 2, ..., N-2$ with corresponding weights given in (3.3), where we should take average over their corresponding KS-entropy given in (4.6). Hence, the KS-entropy of rational map (2.6) can be written as

$$\begin{aligned}
&h(\mu, \text{rational order map}) \\
&= \sum_{k_2=1}^{N_2} \int_{\Delta x_{n+1,k_2}} \mu(x_{n+1}, a_2)\mathrm{d}x_{n+1} \int_{-\infty}^{+\infty} \mu(x_n, a_1)\mathrm{d}x_n \ln\left|\frac{dx_{n+1}}{\mathrm{d}x_n}\right| \\
&= \sum_{k_2=1}^{N_2} \int_{\Delta x_{n+1,k_2}} \mu(x_{n+1}, a_2)\mathrm{d}x_{n+1} \int_{-\infty}^{+\infty} \mu(x_n, a_1)\mathrm{d}x_n \ln\left|\frac{\mathrm{d}x_{n+1}}{\mathrm{d}y}\right|\left|\frac{\mathrm{d}y}{\mathrm{d}x_n}\right| \\
&= \sum_{k_2=1}^{N_2} \int_{\Delta x_{n+1,k_2}} \mu(x_{n+1}, a_2)\mathrm{d}x_{n+1} \\
&\quad \times \int_{-\infty}^{+\infty} \mu(x_n, a_1)\mathrm{d}x_n \left(\ln\left|\frac{\mathrm{d}x_{n+1}}{\mathrm{d}y}\right| + \ln\left|\frac{\mathrm{d}y}{\mathrm{d}x_n}\right|\right) \\
&= \int_{-\infty}^{+\infty} \mu(x_n, a_1)\mathrm{d}x_n \ln\left|\frac{\mathrm{d}y}{\mathrm{d}x_n}\right| \\
&\quad - \int_{-\infty}^{+\infty} \mu(x_{n+1}, a_2)\mathrm{d}x_{n+1} \ln\left|\frac{\mathrm{d}y}{\mathrm{d}x_{n+1}}\right|,
\end{aligned} \tag{4.10}$$

where, in the last line above we have used the following normalization relation:

$$\int_{-\infty}^{+\infty} \mu(x, a_i)\mathrm{d}x = 1, \quad i = 1, 2.$$

Now, comparing the last line of (4.10) with (4.6), we get the following expression for KS-entropy of fractional order map (2.6):

$$\begin{aligned}
&h(\text{rational order map}) \\
&= h\left(\mu, \frac{1}{a_1}\tan(N_1 \arctan x_n)\right) - h\left(\mu, \frac{1}{a_2}\tan(N_2 \arctan x_n)\right)
\end{aligned} \tag{4.11}$$

with $h(\mu, \frac{1}{a_i}\tan(N_i \arctan x_n)), i = 1, 2$ given in (4.9).

Obviously, formula (4.11) implies the simultaneous production and consumption of the entropy, where the term with positive sign corresponds to the production of the entropy while the term with minus sign corresponds to the consumption of the entropy, respectively. Also, it is interesting to note that maximum value of the entropy is equal to $\ln_2(N_2/N_1)$ that corresponds to $a_1 = a_2 = 1$ (actually this is the main reason for naming these maps as rational order maps).

5. Lyapunov exponent and simulation

A useful numerical way to characterize chaotic phenomena in dynamic systems is by means of the Lyapunov exponents that describe the separation rate of systems

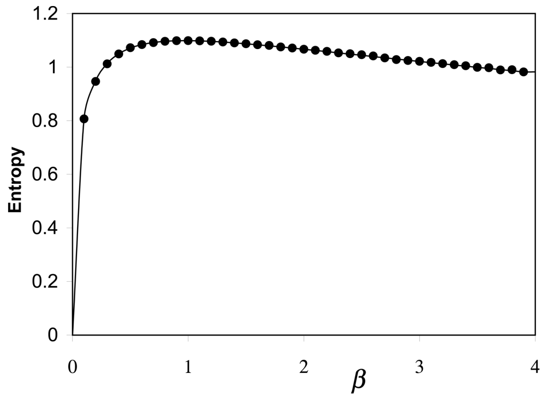

Figure 2. Lyapunov exponent (solid curve) and KS entropy (•) vs. the control parameter β.

whose initial conditions differ by a small perturbation. Suppose that there is a small change $\delta x(0)$ in the initial state $x(0)$. At step or time n this has changed to $\delta x(n)$ given by

$$\delta x(n) \approx \delta x(0) \left| \frac{\mathrm{d}x_n}{\mathrm{d}x_0} \right| = \delta x(0) \left| \frac{\mathrm{d}x_n}{\mathrm{d}x_{n-1}} \cdot \frac{\mathrm{d}x_{n-1}}{\mathrm{d}x_{n-2}} \cdots \frac{\mathrm{d}x_1}{\mathrm{d}x_0} \right|, \tag{5.1}$$

where we have used the chain rule to expand the derivative of $\mathrm{d}x_n/\mathrm{d}x_0$. In the limit of infinitesimal perturbations $\delta x(0)$ and infinite time we get an average exponential amplification, the Lyapunov exponent λ, where

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \ln \left| \frac{\delta x(n)}{\delta x(0)} \right| = \lim_{n\to\infty} \frac{1}{n} \ln \left| \frac{\mathrm{d}x_n}{\mathrm{d}x_0} \right| = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \ln \left| \frac{\mathrm{d}x_k}{\mathrm{d}x_{k-1}} \right|. \tag{5.2}$$

Similarly, the Lyapunov exponent of rational maps (2.6) can be obtained from formula (5.2) provided that we replace x_k, at random with $x_{k,\pm}$ at each step k with probabilities $P_\pm = \frac{1}{2}$. In this work we have simulated Lyapunov exponent of rational order map with $N_1 = 2$ and $N_2 = 3$ for different values of β, where the result that obtained supports the analytic calculation of KS-entropy given in (4.11) for particular cases of $N_1 = 2$ and $N_2 = 3$ (see figure 2).

6. Conclusion

We have given a new hierarchy of rational order families of chaotic maps which presents an interesting description of simultaneous production and consumption of entropies. Using the SRB measure, the Kolmogorov–Sinai entropies of chaotic maps have been calculated. It would be interesting to introduce these kinds of maps in higher dimensions which is under investigation.

Appendix A

Here in this appendix following the prescription of refs [1–5], we prove that the invariant measure given in (3.9) satisfies the corresponding FP equations of the maps $y_i = (1/a_i)\tan(N_i \arctan x_i),\ i = 1, 2$, provided that the parameters a_1 and a_2 can be expressed in terms of β as in formulas (3.11) and (3.12). To do so we we can write the right-hand side of eq. (3.8) as

$$\frac{a}{N}\sum_{k=1}^{N}\frac{1+x_k^2}{1+\beta x_k^2} = \frac{a}{\beta} + \frac{a(\beta-1)}{N\beta^2}\frac{\partial}{\partial\beta^{-1}}\ln\left(\prod_{k=1}^{N}(\beta^{-1}+x_k^2)\right), \tag{A.1}$$

where we have omitted the indices i and n. Hence, eq. (3.8) can be written as

$$\frac{1+a^2y^2}{1+\beta y^2} = \frac{a}{\beta} + \frac{a(\beta-1)}{N\beta^2}\frac{\partial}{\partial\beta^{-1}}\ln\left(\prod_{k=1}^{N}(\beta^{-1}+x_k^2)\right). \tag{A.2}$$

To evaluate the second term in the right-hand side of the above formulas we can write the equation $y = (1/a)\tan(N\arctan x)$ in the following form:

$$\begin{aligned}
0 &= \mathrm{a}y\cos(N\arctan x) - \sin(N\arctan x)\\
&= \frac{1}{(1+x^2)^{N/2}}\left(ay\sum_{k=0}^{[N/2]}C_{2k}^{N}(-1)^k x^{2k} - x\sum_{k=0}^{[N-1/2]}C_{2k+1}^{N}(-1)^k x^{2k}\right)\\
&= \frac{\text{constant}}{(1+x^2)^{N/2}}\prod_{k=1}^{N}(x-x_k),
\end{aligned}$$

where $x_k = \tan(\frac{1}{N}\arctan(ay)+\frac{k\pi}{N}), k = 1, ..., N$ are its roots. Therefore, we have

$$\begin{aligned}
&\frac{\partial}{\partial\beta^{-1}}\ln\left(\prod_{k=1}^{N}(\beta^{-1}+x_k^2)\right)\\
&= \frac{\partial}{\partial\beta^{-1}}\left(\ln\left(\prod_{k=1}^{N}(i\sqrt{\beta^{-1}}+x_k)\right)+\ln\left(\prod_{k=1}^{N}(-i\sqrt{\beta^{-1}}+x_k)\right)\right)\\
&= \frac{\partial}{\partial\beta^{-1}}\ln\left[(1-\beta^{-1})^{N/2}\big(ay\cos(N\arctan(-i\sqrt{\beta^{-1}}))\right.\\
&\qquad \left. -\sin(N\arctan(-i\sqrt{\beta^{-1}}))\big)\right]\\
&\quad +\frac{\partial}{\partial\beta^{-1}}\ln\left[(1-\beta^{-1})^{N/2}\big(ay\cos(N\arctan(i\sqrt{\beta^{-1}}))\right.\\
&\qquad \left. -\sin(N\arctan(i\sqrt{\beta^{-1}}))\big)\right]\\
&= \frac{\partial}{\partial\beta^{-1}}\ln\left[a^2y^2\left(\sum_{k=0}^{[N/2]}C_{2k}^{N}\beta^{-k}\right)^2+\beta^{-1}\left(\sum_{k=0}^{[(N-1)/2]}C_{2k+1}^{N}\beta^{-k}\right)^2\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\partial}{\partial\beta^{-1}} \ln\left[(1-\beta^{-1})^N \left(a^2y^2\cos^2(N\arctan(i\sqrt{\beta^{-1}}))\right.\right.\\
&\qquad \left.\left. -\sin^2(N\arctan(i\sqrt{\beta^{-1}}))\right)\right]\\
&= -\frac{N\beta}{\beta-1} + \frac{\beta N(1+a^2y^2)A(1/\beta)B(1/\beta)}{(1-\beta^{-1})\left((A(1/\beta))^2\beta y^2+(B(1/\beta))^2\right)} \quad , \qquad \text{(A.3)}
\end{aligned}
$$

with polynomials $A(x)$ and $B(x)$ defined as

$$
\begin{aligned}
A(x) &= \sum_{k=0}^{[N/2]} C_{2k}^{N} x^k,\\
B(x) &= \sum_{k=0}^{[(N-1)/2]} C_{2k+1}^{N} x^k. \qquad \text{(A.4)}
\end{aligned}
$$

To derive the above formula we have used the following identities:

$$
\begin{aligned}
\cos(N\arctan i\sqrt{x}) &= \frac{A(x)}{(1-x)^{N/2}},\\
\sin(N\arctan i\sqrt{x}) &= i\sqrt{x}\frac{B(x)}{(1-x)^{N/2}}, \qquad \text{(A.5)}
\end{aligned}
$$

inserting the results (A.3) in (A.2), we get

$$
\frac{1+a^2y^2}{1+\beta y^2} = \frac{1+a^2y^2}{\left(\frac{B(\frac{1}{\beta})}{aA(\frac{1}{\beta})} + \beta(\frac{aA(\frac{1}{\beta})}{B(\frac{1}{\beta})})y^2\right)}.
$$

Hence to get the final result we have to choose the parameter a as

$$
a = \frac{B(1/\beta)}{A(1/\beta)}.
$$

Appendix B

Here we derive the invariant measure of chaotic maps by using Shure's invariant polynomials in a way which is different from that of Appendix A. In order to make the paper more readable we consider only $N = 4$ case, i.e., the map,

$$
y = \frac{1}{a}\tan(4\arctan x) \qquad \text{(B.1)}
$$

which can be written as

$$
ay = \frac{4x(1-x^2)}{1-6x^2+x^4} \qquad \text{(B.2)}
$$

or

$$x^4 + \frac{4x^3}{ay} - 6x^2 - \frac{4x}{ay} + 1 = 0. \tag{B.3}$$

The needed Shure's invariant polynomials of variables $x_1, x_2, ..., x_4$ are defined as

$$\begin{aligned} &S_1 = x_1 + x_2 + x_3 + x_4, \\ &S_{11} = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4, \\ &S_{111} = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4, \\ &S_{1111} = x_1x_2x_3x_4 \end{aligned} \tag{B.4}$$

and

$$\begin{aligned} &S_2 = x_1^2 + x_2^2 + x_3^2 + x_4^2, \\ &S_{22} = x_1^2x_2^2 + x_1^2x_3^2 + x_1^2x_4^2 + x_2^2x_3^2 + x_2^2x_4^2 + x_3^2x_4^2, \\ &S_{222} = x_1^2x_2^2x_3^2 + x_1^2x_2^2x_4^2 + x_1^2x_3^2x_4^2 + x_2^2x_3^2x_4^2, \\ &S_{1111} = x_1^2x_2^2x_3^2x_4^2. \end{aligned} \tag{B.5}$$

The second set of Shure's invariant polynomials can be expressed in terms of the first one as

$$\begin{aligned} &S_2 = S_1^2 - 2S_{11}, \\ &S_{22} = S_{11}^2 - 2S_1S_{111} + 2S_{1111}, \\ &S_{222} = S_{111}^2 - 2S_{11}S_{1111}, \\ &S_{2222} = S_{1111}^2. \end{aligned} \tag{B.6}$$

Considering the variables x_1, x_2, x_3 and x_4 as roots of eq. (B.3), one can obtain the first set of Shure's invariant polynomials given in (B.4) as

$$S_1 = -\frac{4}{ay}, \quad S_{11} = -6, \quad S_{111} = \frac{4}{ay}, \quad S_{1111} = 1 \tag{B.7}$$

and using eq. (B.6), we have

$$S_2 = \frac{16}{(ay)^2} + 12, \quad S_{22} = \frac{32}{(ay)^2} + 38, \quad S_{222} = \frac{16}{(ay)^2} + 12, \quad S_{2222} = 1. \tag{B.8}$$

Again, writing the PF equation of map (B.1) and assuming that its invariant measure is of the form (B.3), we have

$$\mu(y) = \frac{a}{4} \times \left(\frac{1}{1 + a^2y^2}\right) \sum_{k=1}^{4} \frac{1 + x_k^2}{1 + \beta x_k^2}. \tag{B.9}$$

The summation on the right-hand side can be written as

$$\sum_{k=1}^{4} \frac{1 + x_k^2}{1 + \beta x_k^2} = \frac{(1 + x_1^2)(1 + \beta x_2^2)(1 + \beta x_3^2)(1 + \beta x_4^2)}{\prod_{k=1}^{4}(1 + \beta x_k^2)}, \tag{B.10}$$

where using eq. (B.8), the numerator of the above fraction becomes

$$4 + (3\beta + 1)S_2 + (2\beta^2 + 2\beta)S_{22} + (\beta^3 + 3\beta^2)S_{222} + 4\beta^3 S_{2222} \\ = 16(1 + \beta)(1 + 6\beta + \beta^2). \tag{B.11}$$

Also using again (B.8), for its denominator

$$1 + \beta S_2 + \beta^2 S_{22} + \beta^3 S_{222} + \beta^4 S_{2222}, \tag{B.12}$$

we get

$$\frac{16}{a^2 y^2}\beta(1 + \beta^2) + (\beta^2 + 6\beta + 1)^2. \tag{B.13}$$

Using the results obtained above, the invariant measure takes the following form:

$$\mu(y) = \frac{4\alpha(1 + \beta)(1 + 6\beta + \beta^2)}{\alpha^2 y^2 (\beta^2 + 6\beta + 1)^2 + 16\beta(\beta + 1)^2} \tag{B.14}$$

which should be equal to $1/(1 + \beta y^2)$, where this is possible only for the following choice of a, i.e.

$$a = \frac{4\beta + 4\beta^2}{1 + 6\beta + \beta^2}. \tag{B.15}$$

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