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Cosmic no-hair conjecture in scalar–tensor theories

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Abstract. We have shown that, within the context of scalar–tensor theories, the anisotropic Bianchi-type cosmological models evolve towards de Sitter Universe. A similar result holds in the case of cosmology in Lyra manifold. Thus the analogue of cosmic no-hair theorem of Wald [1] hold in both the cases. In fact, during inflation there is no difference between scalar–tensor theories, Lyra's manifold and general relativity (GR).

Keywords. Scalar–tensor theories; cosmic no-hair.

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1. Introduction

With regard to the question whether the Universe evolves to a homogeneous and isotropic state during an inflationary epoch, it has been conjectured by Gibbons and Hawking [2] and Hawking and Moss [3] that all expanding models with positive cosmological constant asymptotically approach the de Sitter solution. This has been termed as the cosmic 'no-hair' conjecture. The inflationary phase in early Universe can explain the present state of the Universe without using highly special initial conditions [1,4–6]. Several attempts at proving the conjecture in general relativity have been made and a general proof has been obtained for homogeneous cosmologies (Bianchi models) [1,4,7] and for some inhomogeneous models [8]. The range of validity of no-hair theorems is of great importance in several areas of physics such as, for example, inflationary cosmologies. In these one usually assumes that the Universe becomes dominated by a positive vacuum energy, i.e. cosmological constant $\Lambda > 0$, and for a period of time expands exponentially at the Hubble rate $H = \sqrt{3\Lambda}$. If the Universe undergoes a period of exponential expansion of more than 60 Hubble times it is possible to explain the cosmological horizon and flatness puzzles in a natural way [9].

The purpose of this paper is to prove version of the cosmic no-hair conjecture in a general class of scalar–tensor theories [10–13]. We assume that strong and dominant energy conditions hold.

2. Cosmic no-hair conjecture in a general scalar–tensor theory

The field equations of a general class of scalar–tensor (ST) theories in 'Einstein frame' [14–16] are

$$G_{ij} = R_{ij} - \frac{1}{2}g_{ij}R = -8\pi G T_{ij} + 2(\Phi_{,i}\,\Phi_{,j} - \frac{1}{2}g_{ij}\Phi_{,k}\,\Phi^{,k}) - \Lambda(\Phi)g_{ij}, \tag{2.1}$$

$$\Box\Phi = \Phi_{;i}^{;i} = -4\pi G\alpha(\Phi)T, \tag{2.2}$$

where $T = g^{ij}T_{ij}$, G is the gravitational constant, Φ is the scalar field. The metric tensor g_{ij} in the 'Einstein frame' is related to the metric tensor $\tilde{g}_{ij}$ in the 'Brans–Dicke frame' by the conformal transformation given by

$$\tilde{g}_{ij} = A^2(\Phi)g_{ij} \tag{2.3}$$

$$GA^2(\Phi) = \frac{1}{\phi} \tag{2.4}$$

$$\alpha^2(\Phi) = \frac{1}{[3 + 2\omega(\phi)]} \tag{2.5}$$

$$a(\Phi) = \frac{\mathrm{d}}{\mathrm{d}\Phi}[\log A(\Phi)], \quad \alpha(\Phi) \equiv \frac{\partial a}{\partial \Phi} \tag{2.6}$$

$$\Lambda(\Phi) = \frac{1}{A^2(\Phi)}\Lambda(\phi) \tag{2.7}$$

$$\Lambda(\phi) = A^2(\Phi)\Lambda(\Phi). \tag{2.7a}$$

The energy–momentum tensor T_{ij} is related by

$$T_{ij} = A^2(\Phi)\tilde{T}_{ij}. \tag{2.8}$$

The equation of motion is

$$T^i_j;_i = \alpha(\Phi)T\Phi_{,j} - g_{ij}[\Lambda(\Phi)]^{,i}. \tag{2.9}$$

In the asymptotic region $\Phi \to \Phi_0$ and $\Lambda(\Phi) \to \Lambda_0$ (constant). Use of eq. (2.2) in (2.1) leads to

$$R_{ij} = -8\pi G(T_{ij} - \tfrac{1}{2}g_{ij}T) + 2\Phi_{,i}\,\Phi_{,j} + \Lambda(\Phi)g_{ij}. \tag{2.10}$$

From eq. (2.1) the 'Hamiltonian constraint' equation is

$$G_{ij}n^i n^j = -8\pi G(T_{ij}n^i n^j) + \Lambda(\Phi) + 2(S_{ij}n^i n^j), \tag{2.11}$$

where

$$S_{ij} = \Phi_{,i}\,\Phi_{,j} - \tfrac{1}{2}g_{ij}\Phi_{,k}\,\Phi^{,k} \tag{2.12}$$

and n^i is the unit normal vector to the homogeneous hypersurface.

From eq. (2.10), the Raychaudhuri equation ('evolution equation') is

$$R_{ij}n^i n^j = -8\pi G(T_{ij} - \tfrac{1}{2}g_{ij}T)n^i n^j + \Lambda(\Phi) + (\Phi_{,i}\,\Phi_{,j}\,n^i n^j). \tag{2.13}$$

Equations (2.11) and (2.13) lead respectively to

$$K^2 - 3\Lambda(\phi) = \tfrac{3}{2}(\sigma_{ij}\sigma^{ij}) - \tfrac{3}{2}{}^{(3)}R + 2(S_{ij}n^i n^j) + 24\pi G(T_{ij}n^i n^j) \tag{2.14}$$

$$\dot{K} = \frac{\mathrm{d}k}{\mathrm{d}t} = \Lambda(\phi) - \frac{1}{3}K^2 - (\sigma_{ij}\sigma^{ij}) + (\phi_{,i}\,n^i)^2$$
$$-8\pi G\left(T_{ij} - \frac{1}{2}g_{ij}T\right)n^i n^j. \tag{2.15}$$

Here K_{ij} is the extrinsic curvature, and K is the trace of K_{ij}; ${}^{(3)}R$ is the three-scalar curvature, σ_{ij} is the shear,

$$K_{ij} = \sigma_{ij} + \tfrac{1}{3}Kh_{ij}$$

$$h_{ij} = g_{ij} + n_i n_j \quad \text{is the special metric.}$$

Clearly $\sigma_{ij}\sigma^{ij}$ is non-negative. Due to dominant energy condition $T_{ij}n^i n^j$ is positive and due to strong condition $(T_{ij} - \frac{1}{2}g_{ij}T)n^i n^j$ is positive. Also $(\Phi_{,i}\,n^i)^2$ is positive for a real scalar field Φ.

Therefore, from eq. (2.14), for ${}^{(3)}R \leq 0$,

$$\tfrac{1}{3}K^2 - \Lambda(\Phi) \geq 0. \tag{2.18}$$

From eq. (2.15)

$$\frac{\mathrm{d}K}{\mathrm{d}t} \leq \Lambda(\phi) - \frac{1}{3}K^2 \leq 0. \tag{2.19}$$

Here $\Lambda(\phi) > 0$.

If the space–time is initially expanding, then $K \geq 0$ for all time. Also we assume that $\Lambda(\phi) \to \Lambda_0$ (constant) asymptotically in time. Then eq. (2.17) becomes

$$\frac{\mathrm{d}K}{\mathrm{d}t} \leq \Lambda_0 - \frac{1}{3}K^2 \leq 0 \tag{2.20}$$

and K lies in the limits

$$\sqrt{3\Lambda_0} \leq K \leq [\sqrt{3\Lambda_0}/\tanh(t/L)], \tag{2.21}$$

where

$$L = \sqrt{3/\Lambda_0}. \tag{2.22}$$

Just as in general relativity (GR), K exponentially approaches $(3\Lambda_0)^{1/2}$. It follows from eqs (2.14) and (2.19) that

$$\sigma_{ij}\sigma^{ij} \leq \tfrac{2}{3}[K^2 - 3\Lambda_0] \leq [2\Lambda_0/\sinh^2(t/L)]. \tag{2.23}$$

Then from the above equation it immediately follows that the shear of the homogeneous hypersurface rapidly approaches zero. Thus we can see that for $t \geq L$ the contributions of matter energy density (non-vacuum) and the ST scalar field ϕ rapidly approach zero. Hence, as in GR [1], the Universe will appear like de Sitter Universe and the ST theory is indistinguishable from GR in the inflationary era. It can be shown that in the case of Bianchi Type-IX the same conclusions, as drawn by Wald [1] for GR, hold for the ST theory also. If Λ is large, as Wald [1] has shown, the Bianchi Type-IX Universe will approach de Sitter Universe. Otherwise it is possible that the Universe will recollapse.

3. Cosmic no-hair conjecture in Brans–Dicke theory and Lyra manifold

In the Brans–Dicke theory [17] the conjecture is the same as in GR [18]. In the Lyra manifold [11,12,19] it has been shown by Singh [20] that the conjecture is valid as in GR under the assumption of strong and dominant energy conditions.

4. Conclusion

Under the assumptions of strong and dominant energy conditions we have been able to show that anisotropic Bianchi-type models evolve towards de Sitter Universe in the context of scalar–tensor theories and Lyra's manifold. Thus during inflation there is no difference between scalar–tensor theories, Lyra's manifold and general relativity.

There is a further necessity to study the evolution of anisotropic cosmological models and cosmic no-hair conjecture in ST theory with scalar potential $V(\phi)$ [21,22] and non-minimally coupled scalar fields. These studies can be made on the lines suggested in this paper.

The cosmological models with accelerated expansion are currently of great interest in the context of two main themes viz. (i) inflation, which concerns the very early Universe and (ii) the accelerated cosmological expansion at the present epoch as evidenced, for instance, by Supernova observations. Our results have asymptotic approach towards a de Sitter Universe as the late time behaviour of an inflationary model. In fact, our result is an extension of Wald's no-hair theorem [1] to the general class of scalar–tensor theories of gravitation. It has come to our notice later on that Hayle and Narlikar [23] have also obtained a result similar to those in [2,3].

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