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Spectral inverse problem for *q*-deformed harmonic oscillator

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Abstract. The supersymmetric quantization condition is used to study the wave functions of SWKB equivalent q-deformed harmonic oscillator which are obtained by using only the knowledge of bound-state spectra of q-deformed harmonic oscillator. We have also studied the nonuniqueness of the obtained interactions by this spectral inverse method.

Keywords. q-Oscillator; supersymmetric WKB; wave functions.

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1. Introduction

One often regards the normal business of a quantum physicist as computing measurable quantities for known interparticle forces and confronting them with experimental data. In quantum theory [1], the spectral inverse problem consists of determining the interparticle forces from the information obtained more or less directly by experiments. Studies along this line have continued and we are convinced that these will be continued both at sophisticated and pedagogical levels [2].

In an interesting work, Engelke [3] reduced the WKB energy quantization condition to the Abel's integral equation and used it to derive information on the potential function from its bound-state energy spectrum only. This simple minded inversion process provides a basis to develop an intuitive physical feeling for the causal relationship between the potential shape and energy spectrum and turns out to be a useful technique for some real problems for which only the bound-state spectrum is known [4]. Bonatsos *et al* [5] also have determined WKB-equivalent potential which gives the same spectrum as the q-deformed harmonic oscillator with the symmetry $SU_q(2)$ according to the method of Chadan and Sabatier [6] and Wheeler [7].

With Engelke [3] and Bonatsos *et al* [5], we address in this paper a similar deep-rooted problem and examine the question of determining the wave functions without direct use of the Schrödinger equation via the formalism of supersymmetric quantum mechanics. Here the supersymmetric WKB (SWKB) quantization condition [8] is used to study the so-called spectral inverse problem. Wave functions of the SWKB-equivalent q-deformed harmonic oscillator which are obtained from the knowledge of the bound-state energy spectra of q-deformed harmonic oscillator are constructed. The analysis presented is based essentially on a repackaging of the conventional theory of integral equations. The formalism of supersymmetric quantum mechanics (SSQM) [9] is used to achieve the results. Here the WKB quantization condition is modified in SSQM [8]. The basic philosophy of supersymmetric quantum mechanics and associated algorithms for solving potential problems have been extensively discussed in [10].

This method does not have straightforward physical meaning because of the nonuniqueness of the deformation procedure. Even within the standard physical concepts, exact knowledge of the spectrum is not enough for the reconstruction of the potential. For a given potential with infinite number of bound states, one can associate another potential with many independent parameters and the same spectrum [11]. Therefore, the physics behind such deformations is not completely fixed. One should precisely describe what kind of interaction between the excitations leads to peculiar change of the spectrum. Some analysis of the inverse problem can be found in refs [5,12]. Here also q-analogs of the harmonic oscillators have been used for the description of small violation of the statistics of identical particles [13] and for SWKB quantization condition for the construction of artificial symmetric potentials and corresponding wave functions where we deform the shape in such a way that the problem remains exactly solvable.

In §§2 and 3, we will quote the main results of q-deformed harmonic oscillator and that of the supersymmetric WKB quantization condition. We will devote §4 to inject the idea of ref. [3] and finally in §5 we will draw the conclusion.

2. q-Deformed harmonic oscillator

The quantum group $SU_q(2)$ was first introduced by Sklyamin [14], Kulish and Reshetikhin [15] on Yang Baxter equations. These equations are well-known to play a crucial role in classical and quantum integrable systems [16]. Biedenharn [17] and Macfarlane [18] introduced q-deformed harmonic oscillator as a building block of quantum algebras.

In the case of q-deformed harmonic oscillator, the creation and annihilation operators a^+ and a satisfy the commutation relation

$$aa^+ - q^{-1}a^+a = q^N, \tag{1}$$

where N is the number operator, satisfying

$$[N, a^+] = a^+, \quad [N, a] = -a. \tag{2}$$

The relevant Fock space is defined as

$$a|0\rangle = 0, \quad |n\rangle = \frac{(a^+)^n}{([n]!)^{1/2}}\bigg| 0\rangle, \tag{3}$$

where the q-factorial is defined as

$$[n]! = [n][n-1]\cdots[1] \tag{4}$$

and the q-numbers are defined by

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}}. \tag{5}$$

When q is real ($q = \mathrm{e}^{\beta}$), the q-numbers take the form

$$[k] = \frac{\sinh(\beta k)}{\sinh(\beta)} \tag{6}$$

and when q is imaginary ($q = \mathrm{e}^{i\beta}$), the q-numbers take the form

$$[k] = \frac{\sin(\beta k)}{\sin(\beta)}. \tag{7}$$

It is clear that in both cases $[k] \to k$ in the limit $q \to 1$.

The Hamiltonian of the q-deformed harmonic oscillator [17] is

$$H_q = \frac{p_q^2}{2m} + \frac{1}{2}m\omega^2 Q_q^2 \tag{8}$$

so that

$$H_q = \frac{\hbar\omega}{2}\left(aa^+ + a^+a\right), \tag{9}$$

where the q-momentum (P_q) and q-position (Q_q) operators are directly written in terms of the q-boson operators a and a^+ introduced in (1) and (2) as

$$P_q = i\sqrt{\frac{\hbar m\omega}{2}}\left(a - a^+\right)$$

and

$$Q_q = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^+\right).$$

The eigenvalues in the Fock space defined above are

$$E_n = \frac{\hbar\omega}{2}([n] + [n+1]). \tag{10}$$

Hence the energy levels are no longer uniformly spaced as q is not equal to one. From eqs (6), (7) and (10), we find that the q-deformed harmonic oscillator has a spectrum given by

$$E_n = \frac{\hbar\omega}{2}\frac{\sin(\beta(n+\frac{1}{2}))}{\sin(\beta/2)}, \tag{11}$$

when q is a phase ($q = \mathrm{e}^{i\beta}$), and when q is real ($q = \mathrm{e}^{\beta}$), the spectrum is

$$E_n = \frac{\hbar\omega}{2}\frac{\sinh(\beta(n+\frac{1}{2}))}{\sinh(\beta/2)}. \tag{12}$$

3. Supersymmetric WKB (SWKB)

In field theory supersymmetric transformations make use of the graded Lie algebra and put together the bosonic and fermionic elements in a supermultiplet [19]. This remarkable theory has been realised by Witten [9] in the context of quantum mechanics in which a particle moving in a potential in zero dimension. For any Hamiltonian with one degree of freedom, a companion Hamiltonian has been constructed such that the resulting system as a whole is supersymmetric. The Hamiltonian $\mathcal{H}$ written in the form

$$\mathcal{H} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix} \tag{13}$$

is said to be supersymmetric if the corresponding potentials $V_\pm(x)$ are related by superpotential $W(x)$ as

$$V_\pm(x) = W^2(x) \pm \frac{\hbar}{\sqrt{2m}} \frac{\mathrm{d}W(x)}{\mathrm{d}x}, \tag{14}$$

where

$$W(x) = -\frac{\hbar}{\sqrt{2m}} \frac{\mathrm{d}}{\mathrm{d}x} \ln \psi_0^{(-)}(x) \tag{15}$$

and $\psi_0^{(-)}(x)$ is the ground-state wave function of

$$H_- = -\frac{\hbar^2}{2m} \frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_-(x). \tag{16}$$

Note that supersymmetry (SUSY) is unbroken if the ground-state energy $E_0^{(-)}$ of H_- is zero. Obviously, $H_+ = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_+(x)$ has a nonzero ground-state energy. The partner H_+ shares the same eigenvalue spectrum as that of H_- and their wave functions are simply related [20]. Thus, the diagonal elements H_+ and H_- in (13) can be viewed as 'fermionic' and 'bosonic' components of $\mathcal{H}$.

Semiclassical method for SSQM was first considered by Comtet *et al* [8], who derived a modified quantization rule, now called the supersymmetric WKB (SWKB) quantization condition. In the lowest order, the SWKB quantization condition for a one-dimensional problem is

$$\int_a^b \sqrt{E_n^{(-)} - W^2(x)}\,\mathrm{d}x = \frac{nh}{2\sqrt{2m}}, \quad n = 0, 1, 2, \ldots \tag{17}$$

where a and b are the classical turning points defined by

$$W^2(a) = W^2(b) = E_n^{(-)}. \tag{18}$$

Looking at (17) we see that the integral in the supersymmetric case does not involve the full potential $V_-(x)$ of H_-, but only the leading term $W^2(x)$. The classical turning points are also determined accordingly. Further, we note that the standard

WKB formula for energy quantization is characterized by boson zero point energy. In supersymmetric theories this term is cancelled exactly by the fermionic term so that we have n in place of $n+\frac{1}{2}$ on the right-hand side of (17). For unbroken SUSY, the bosonic oscillator spectrum can be written as

$$E_n = E_n^{(-)} + E_0. \tag{19}$$

Equation (18) is only the condition for classical turning points of zeroth order of $\hbar$. For the determination of ground-state wave function we write (18) as

$$E_n^{(-)} = W^2(x). \tag{20}$$

Again, the SWKB semiclassical quantization condition is exact not only for ground state ($n = 0$) and for large n, but also for all intermediate states [8]. So, to determine the higher-order state the full equation for determining the classical turning points [21] is

$$E_n^{(-)} - W^2(x) + \frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}W(x)}{\mathrm{d}x} - E_0 = 0, \tag{21}$$

which is the consistency condition for supersymmetric quantum mechanics and Schrödinger factorization method [22]. From the perturbation theoretic point of view, the term $\frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}W(x)}{\mathrm{d}x}$ comes from the fermionic 'loop' integrals and $W^2(x)$ is a bosonic contribution. In the general supersymmetric theories, the cancellation between bosonic and fermionic contributions are automatically guaranteed by the supersymmetry [18]. Here $W(x)$ is an arbitrary function which is to be determined by q-analogy of the harmonic oscillator.

4. Wave functions from SWKB quantization condition

Consider a symmetric potential $V_-(x)$ defined on $-\infty < x < +\infty$ with an absolute minimum at $x = 0$ and $V_-(-x) = V_-(x)[W^2(-x) = W^2(x)]$. Equation (17) reads

$$2\int_0^a \sqrt{E_n^{(-)} - W^2(x)}\mathrm{d}x = N\big(E_n^{(-)}\big), \tag{22}$$

where we have used $-a$ for a, and a for b. Also

$$N\big(E_n^{(-)}\big) = \frac{nh}{2\sqrt{2m}}. \tag{23}$$

Integrating by parts, we convert (22) to an Able's integral equation [23] like

$$N\big(E_n^{(-)}\big) = \int_0^x \frac{x'\frac{\mathrm{d}W^2(x)}{\mathrm{d}x'}\mathrm{d}x'}{\sqrt{E_n^{(-)} - W^2(x)}}. \tag{24}$$

As in Engelke [3] and scaled by $W^2 = E_n^{(-)}y$, the solution of (24) is

$$x\left(E_n^{(-)}\right) = \frac{\hbar}{\sqrt{2m}} \frac{1}{\sqrt{E_n^{(-)}}} \int_0^1 \frac{\frac{\mathrm{d}n(E_n^{(-)},y)}{\mathrm{d}y}}{\sqrt{1-y}} \mathrm{d}y. \tag{25}$$

For q-deformed harmonic oscillator with $q = \mathrm{e}^{i\beta}$, using (19) and after making scaled $W^2 = E_n^{(-)} y$ we get from (11)

$$n\left(E_n^{(-)}, y\right) = -\frac{1}{2} + \frac{1}{\beta} \sin^{-1}\left[\left(1 + \frac{E_n^{(-)}}{E_0} y\right) \sin\left(\frac{\beta}{2}\right)\right]. \tag{26}$$

With the help of (26), (25) can be written as

$$x\left(E_n^{(-)}\right) = \frac{1}{\omega} \sqrt{\frac{E_n^{(-)}}{2m}} \frac{\sin(\beta/2)}{\beta/2} I(\beta), \tag{27}$$

where the integration

$$I(\beta) = \int_0^1 \frac{\mathrm{d}y}{\sqrt{(1-y)\left[1 - \left(1 + \frac{E_n^{(-)}}{E_0} y\right)^2 \sin^2 \frac{\beta}{2}\right]}}. \tag{28}$$

Equation (27) is an expression of the potential form of SWKB-equivalent q-deformed harmonic oscillator. It has the same spectra as the q-deformed harmonic oscillator for $q = \mathrm{e}^{i\beta}$ because we have used the energy which is given in (11) as input.

As the superpotential is related to the ground-state wave function, we demand that the integration $I(\beta)$ is for ground state only and outside the integral the term $E_n^{(-)}$ behaves as an energy function. For ground state ($E_n^{(-)} = 0$) the value of the integral in (28) becomes

$$I(\beta) = 2 \sec\left(\frac{\beta}{2}\right). \tag{29}$$

Substituting this value of $I(\beta)$ in (27) and squaring we obtain the energy function

$$E_n^{(-)} = \frac{1}{2} m\omega^2 \left(\frac{\beta}{2}\right)^2 \cot\left(\frac{\beta}{2}\right)^2 x^2. \tag{30}$$

With the help of (20), we get from (30)

$$W^2(x) = \frac{1}{2} m\omega^2 \tilde{Q}_q^2, \tag{31}$$

where

$$\tilde{Q}_q = \left(\frac{\beta}{2}\right) \cot\left(\frac{\beta}{2}\right) x.$$

Equation (31) also gives the ground-state wave function of SWKB equivalent q-deformed harmonic oscillator in the form

$$\psi_{q,0}^{(-)}(x) = \exp\left(-\frac{1}{2}\alpha^2 \tilde{Q}_q^2\right) = \psi_{q,0}^{(-)}(\tilde{Q}_q), \tag{32}$$

where $\alpha^2 = m\omega/\hbar$. In the ground-state wave function $\psi_{q,0}^{(-)}(\tilde{Q}_q)$, the position co-ordinate x is replaced by $\beta/2 \cot(\beta/2)x$, and the corresponding Hamiltonian operator is

$$\tilde{H}_{\mathrm{SWKB}} = \frac{p_q^2}{2m} + \frac{1}{2}m\omega^2\tilde{Q}_q^2. \tag{33}$$

The form of this operator is similar to a q-deformed harmonic oscillator but differs in position operator. So (33) gives a new potential which is not exactly the same as q-deformed harmonic oscillator but having the same spectra of it. Hence we see that by spectral inverse method, the derived interaction may not be unique. This nonuniqueness is also present in inverse quantum scattering theory of nuclear physics [6]. So in q-parameter space, SWKB equivalent q-deformed harmonic oscillator behaves like an ordinary harmonic oscillator with frequency ω. Now, if we take the limit $q \to 1(\beta \to 0)$, this SWKB Hamiltonian $\tilde{H}_{\mathrm{SWKB}}$ and the wave funcion $\psi_{q,0}^{(-)}(\tilde{Q}_q)$ will be an odinary harmonic oscillator Hamiltonian and its ground-state wave function respectively, i.e. $Lt_{q\to 1}\tilde{H}_{\mathrm{SWKB}} \to H_{\mathrm{h.o.}}$ and $Lt_{q\to 1}\psi_{q,0}^{(-)}(\tilde{Q}_q) \to \psi_0^{(-)}(x)_{\mathrm{h.o.}}$.

In figure 1 we have plotted this ground-state wave functions for $\beta = 1$ and 1.5 and the ground-state wave function of the harmonic oscillator. Here we have used the atomic unit, i.e. $\hbar\omega = 1$ and $\hbar^2/2m = 1$. From this figure we realize that as $\beta \to 0(q \to 1)$, the ground-state wave function of SWKB equivalent q-deformed harmonic oscillator tends to coincide with the ground-state wave function of an ordinary harmonic oscillator.

The Hamiltonian in (33) can be written in factorized form as

$$\tilde{H}_{\mathrm{SWKB}+} = \tilde{A}^+(\tilde{Q}_q)\tilde{A}^-(\tilde{Q}_q) + E_0 \tag{34}$$

and its supersymmetric partner

$$\tilde{H}_{\mathrm{SWKB}-} = \tilde{A}^-(\tilde{Q}_q)\tilde{A}^+(\tilde{Q}_q) + E_0, \tag{35}$$

where the raising and lowering operators are

$$\tilde{A}^{\pm} = \pm\frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}}{\mathrm{d}\tilde{Q}_q} + W(\tilde{Q}_q).$$

Here $E_0 = \hbar\omega/2$ is the factorization energy. So it is proved that harmonic oscillator, q-deformed harmonic oscillator and SWKB-equivalent harmonic oscillator have the same ground-state energy. Here, the SUSY Hamiltonian of SWKB equivalent q-deformed harmonic oscillator is

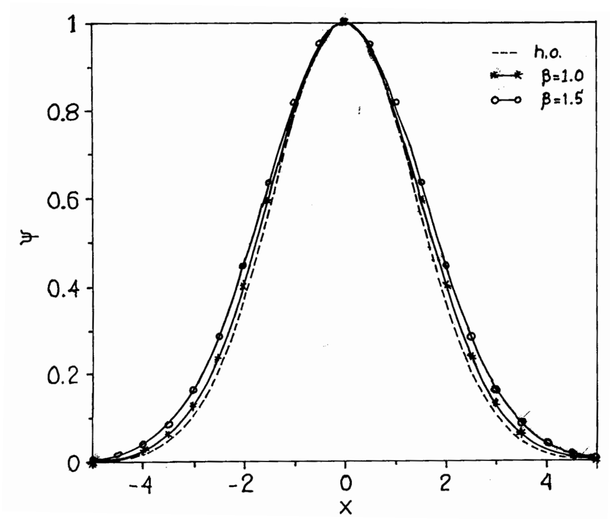

Figure 1. Ground-state wave function of the harmonic oscillator and that of the SWKB equivalent q-deformed harmonic oscillator for $q = \mathrm{e}^{i\beta}$ are given for $\beta = 1, 1.5$ (the parameters are $\hbar\omega = 1$ and $(\hbar^2/2m) = 1$).

$$\mathcal{H}_{\mathrm{SWKB}} = \begin{pmatrix} \tilde{H}_{\mathrm{SWKB}+} & 0 \\ 0 & \tilde{H}_{\mathrm{SWKB}-} \end{pmatrix}$$
$$= \begin{pmatrix} \tilde{A}^{+}(\tilde{Q}_q)\tilde{A}^{-}(\tilde{Q}_q) + E_0 & 0 \\ 0 & \tilde{A}^{-}(\tilde{Q}_q)\tilde{A}^{+}(\tilde{Q}_q) + E_0 \end{pmatrix}. \tag{36}$$

Again for q-deformed harmonic oscillator for $q = \mathrm{e}^{\beta}$, using (19) and after making scaled $W^2 = E_n^{(-)} y$ we obtain from (12)

$$n\big(E_n^{(-)}, y\big) = -\frac{1}{2} + \frac{1}{\beta}\sinh^{-1}\left[\left(1 + \frac{E_n^{(-)}}{E_0}y\right)\sinh\left(\frac{\beta}{2}\right)\right]. \tag{37}$$

With the help of (37), (25) can be written as

$$x\big(E_n^{(-)}\big) = \frac{1}{\omega}\sqrt{\frac{E_n^{(-)}}{2m}}\frac{\sinh(\beta/2)}{\beta/2}I(\beta), \tag{38}$$

where the integration

$$I(\beta) = \int_0^1 \frac{\mathrm{d}y}{\sqrt{(1-y)\left[1 - \left(1 + \frac{E_n^{(-)}}{E_0}y\right)^2 \sinh^2(\beta/2)\right]}}. \tag{39}$$

Equation (38) is also an expression of the potential form of SWKB-equivalent q-deformed harmonic oscillator for $q = \mathrm{e}^{\beta}$. It has also the same spectrum as the

q-deformed harmonic oscillator for $q = \mathrm{e}^{\beta}$ because we have used the energy which is given in (12) as input.

As before the case for $q = \mathrm{e}^{i\beta}$, the value of the integral in (39) for this case becomes

$$I(\beta) = 2\,\mathrm{sech}\left(\frac{\beta}{2}\right). \tag{40}$$

Substituting this value of $I(\beta)$ in (38) and squaring we obtain the energy function

$$E_n^{(-)} = \frac{1}{2}m\omega^2\left(\frac{\beta}{2}\right)^2 \coth\left(\frac{\beta}{2}\right)^2 x^2. \tag{41}$$

With the help of (20), we obtain from (41)

$$W^2(x) = \frac{1}{2}m\omega^2 Q_q^{*2}, \tag{42}$$

where

$$Q_q^* = \left(\frac{\beta}{2}\right)\coth\left(\frac{\beta}{2}\right)x.$$

Equation (42) also gives the ground-state wave function of SWKB equivalent potential of the q-deformed harmonic oscillator in the form

$$\psi_{q,0}^{(-)}(x) = \exp\left(-\frac{1}{2}\alpha^2 Q_q^{*2}\right) = \psi_{q,0}^{(-)}(Q_q^*). \tag{43}$$

In this ground-state wave function the position co-ordinate x is also replaced by $(\beta/2)\coth(\beta/2)x$. This ground-state wave function corresponds to the SWKB equivalent Hamiltonian operator of q-deformed harmonic oscillator

$$H_{\mathrm{SWKB}}^* = \frac{p_q^2}{2m} + \frac{1}{2}m\omega^2 Q_q^{*2}. \tag{44}$$

The form of this operator is also similar to a q-deformed harmonic oscillator but differs in position operator which is scaled. So we obtain a new potential which is not exactly the same as q-deformed harmonic oscillator. Hence, by the inverse spectral method, the derived interaction is also not unique. Again, in q-parameter space, SWKB equivalent q-deformed harmonic oscillator behaves like an ordinary harmonic oscillator with frequency ω. Now, if we take the limit $q \to 1(\beta \to 0)$, the SWKB Hamiltonian H_{SWKB}^* and the wave function $\psi_{q,0}^{(-)}(Q_q^*)$ will be an ordinary harmonic oscillator Hamiltonian and its ground-state wave function respectively, i.e. $Lt_{q\to 1}H_{\mathrm{SWKB}}^* \to H_{\mathrm{h.o.}-}$ and $Lt_{q\to 1}\psi_{q,0}^{(-)}(Q_q^*) \to \psi_0^{(-)}(x)_{\mathrm{h.o.}}$.

In figure 2 we have plotted this ground-state wave function for $\beta = 1$, 1.5 and the ground-state wave function of the harmonic oscillator in atomic unit for $q = \mathrm{e}^{\beta}$. The results are the same as in the case of $q = \mathrm{e}^{i\beta}$.

The Hamiltonian in (44) can be written in factorized form as

$$H_{\mathrm{SWKB}+}^* = A^{*+}(Q_q^*)A^{*-}(Q_q^*) + E_0 \tag{45}$$

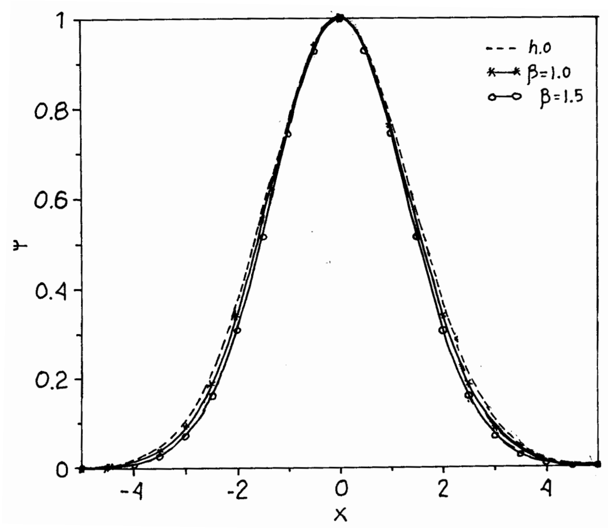

Figure 2. Ground-state wave function of the harmonic oscillator and that of the SWKB equivalent q-deformed harmonic oscillator for $q = e^{\beta}$ are given for $\beta = 1, 1.5$ (the parameters are $\hbar\omega = 1$ and $(\hbar^2/2m) = 1$).

and its supersymmetric partner

$$H^*_{\rm SWKB} = H^*_{\rm SWKB-} = A^{*-}(Q^*_q)A^{*+}(Q^*_q) + E_0, \tag{46}$$

where the raising and lowering operators are

$$A^{*\pm} = \pm \frac{\hbar}{\sqrt{2m}} \frac{\rm d}{{\rm d}Q^*_q} + W(Q^*_q).$$

Here $E_0 = \hbar\omega/2$ is the factorization energy. So it is proved that harmonic oscillator, q-deformed harmonic oscillator and SWKB-equivalent harmonic oscillator have the same ground-state energy. For this case, the SUSY Hamiltonian of SWKB equivalent q-deformed harmonic oscillator is

$$\begin{aligned}\mathcal{H}^*_{\rm SWKB} &= \begin{pmatrix} H^*_{\rm SWKB+} & 0 \\ 0 & H^*_{\rm SWKB-} \end{pmatrix} \\ &= \begin{pmatrix} A^{*+}(Q^*_q)A^{*-}(Q^*_q) + E_0 & 0 \\ 0 & A^{*-}(Q^*_q)A^{*+}(Q^*_q) + E_0 \end{pmatrix}. \end{aligned} \tag{47}$$

For higher excited state wave functions for both imaginary and real q, we venture to suggest that $E_0 = \frac{\hbar\omega}{2}([n]+[n+1])$. Here $W^2(\tilde{Q}^*_q)$ will be given in terms of the nth excited state wave function of SWKB-equivalent q-deformed harmonic oscillator for $q = e^{i\beta}$ and $q = e^{\beta}$. Hence, for both cases of q, with the help of (30) and (41), (21) can be written in general form by replacing x by $\tilde{Q}^*_q$, i.e., in transformed space

$$W^2(\tilde{Q}^*_q) - \frac{\hbar}{\sqrt{2m}} \frac{{\rm d}W(\tilde{Q}^*_q)}{{\rm d}\tilde{Q}^*_q} + \frac{\hbar\omega}{2}([n] + [n+1]) = \frac{1}{2} m\omega^2 \tilde{Q}^{*2}_q, \tag{48}$$

where the function $W(\tilde{Q}_q^*)$ is written in terms of the nth excited state wave functions of SWKB-equivalent q-deformed harmonic potential as

$$W(\tilde{Q}_q^*) = -\frac{\hbar}{\sqrt{2m}}\frac{\mathrm{d}}{\mathrm{d}\tilde{Q}_q^*}\ln\psi_n^{(-)}(\tilde{Q}_q^*), \tag{49}$$

$$\psi_n^{(-)}(\tilde{Q}_q^*) = f_n(\alpha\tilde{Q}_q^*)\exp\left(-\frac{1}{2}\alpha^2\tilde{Q}_q^{*2}\right), \tag{50}$$

where $f_n(\alpha,\tilde{Q}_q^*)$ is an unknown function to be determined from the non-linear form of the Schrödinger equation (48). Here $\tilde{Q}_q^*$ is $\tilde{Q}_q$ when q is imaginary and that is Q_q^* when q is real. Combining (48) to (50) we get the differential equation

$$\begin{aligned}&\frac{\mathrm{d}^2}{\mathrm{d}\tilde{Q}_q^{*2}}f_n(\alpha\tilde{Q}_q^*) - 2\alpha^2\tilde{Q}_q^*\frac{\mathrm{d}}{\mathrm{d}\tilde{Q}_q^*}f_n(\alpha\tilde{Q}_q^*)\\ &-\alpha^2(1-([n]+[n+1]))f_n(\alpha\tilde{Q}_q^*) = 0.\end{aligned} \tag{51}$$

Analytically and numerically one can check that in the limit $\beta \to 0 (q \to 1)$, the above equation will be a well-known differential equation of the Hermite polynomials. Hence, $Lt_{q\to 1}f_n(\alpha,\tilde{Q}_q^*) \to f(\alpha,x)$. So one can consider $f_n(\alpha,\tilde{Q}_q^*)$ as the q-deformed Hermite polynomial. Solving the above differential equation numerically for both cases of q, one can generate the q-Hermite polynomials for $n = 1,2,3,...$ and corresponding wave functions of SWKB equivalent q-deformed harmonic oscillator. Hence the above completes our conjecture.

5. Conclusion

Spiridonov [24] has discussed the general deformation of the SSQM and also showed that superpartner Hamiltonian satisfy non-trivial braid-type intertwining relations which remove degeneracies of the original SUSY spectra. Deformed supersymmetric algebra preserves semipositiveness of the vacuum energy. Peculiar set of q-SUSY potentials arising within the Landau-level like problem obey q-deformed dynamical conformal symmetry algebra $SU_q(1,1)$. It is interesting to know the most general exactly solvable q-deformed potential and its wave functions. There are some ways to approach this problem. For example, one can take the particle wave function as a q-hypergeometric function multiplied by some weight factor. This would correspond to the transformation of q-hypergeometric equation to the standard Schrödinger equation for some potentials. Another path for q-deformation of known model is found by the old factorization method at $q = 1$. In this paper we have dealt with the q-deformed harmonic oscillator and prescribed a technique to construct the wave functions of SWKB-equivalent q-deformed harmonic oscillators which are derived from the information of q-deformed harmonic oscillator. Here q-analogs of the harmonic oscillators have been used for the description of small violation of the statistics of identical particles [13] and SWKB quantization condition for the construction of artificial symmetric potential and corresponding wave functions.

We conclude that within the framework of SSQM, the approach of Engelke [3] is of considerable interest to construct the wave functions of any shape-invariant potential forms as well as that of the constructed SWKB equivalent q-deformed harmonic oscillator from the knowledge of the respective bound-state spectra of q-deformed harmonic oscillator. Here, we propose that the supersymmetric WKB quantization condition [8] is also a technique to study the inverse problem.

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