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Exact solutions to a class of nonlinear Schrödinger-type equations

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Abstract. A class of nonlinear Schrödinger-type equations, including the Rangwala–Rao equation, the Gerdjikov–Ivanov equation, the Chen–Lee–Lin equation and the Ablowitz–Ramani–Segur equation are investigated, and the exact solutions are derived with the aid of the homogeneous balance principle, and a set of subsidiary higher order ordinary differential equations (sub-ODEs for short).

Keywords. Homogeneous balance principle; nonlinear Schrödinger equation; Rangwala–Rao equation; Gerdjikov–Ivanov equation; Chen–Lee–Lin equation; Ablowitz–Ramani–Segur equation; exact solution.

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1. Introduction

In this paper, we consider a class of nonlinear Schrödinger-type equations which are of interest in plasma physics, wave propagation in nonlinear optical fibres, Ginzburg–Landau theory of superconductivity, etc.

The first one is the Rangwala–Rao equation [1,2]:

$$u_{xt} - \beta_1 u_{xx} + u + iT\beta_2|u|^2 u_x = 0, \quad (T = \pm 1), \tag{1}$$

where β_1, β_2 are real constants. Rangwala and Rao [1] introduced eq. (1) as the integrability condition when they studied the mixed derivative nonlinear Schrödinger equations and looked for the Bäcklund transformation and solitary wave solutions. Zhang [2] called eq. (1) as Rangwala–Rao equation, and derived the exact solitary wave solution of the form as $u = \mathrm{e}^{-i\omega t}\mathrm{e}^{i\psi(x-vt)}a(x-vt)$ with the aid of undetermined assumption method and certain explicit exact solutions of the Lienard equation.

The second one is the Gerdjikov–Ivanov equation [2,3]:

$$iu_t + u_{xx} + \beta|u|^2 u + 2\delta^2|u|^4 u + 2i\delta u^2 u_x^* = 0. \tag{2}$$

The third one is the Chen–Lee–Lin equation [2,4]:

$$iu_t + u_{xx} + i\delta|u|^2 u_x = 0. \tag{3}$$

The fourth one is the Ablowitz–Ramani–Segur equation [2,5]:

$$iu_t = u_{xx} + 8|u|^4 u - 4iu^2 u_x^*. \tag{4}$$

Equations (2)–(4) are introduced in refs [3], [4] and [5] respectively, and the exact solitary wave solutions of eqs (2)–(4) were derived in ref. [2].

In recent years, various powerful methods have been presented to derive the nonlinear transformations and exact solutions to the nonlinear PDEs in mathematical physics, such as the inverse scattering transform [6], the Hirota's bilinear operators [7], the Jacobi elliptic function expansion [8,9], the tanh-function expansion [10–14], F-expansion method [15–21] and so on. In [22,23], the homogeneous balance principle was put forward by one of the authors of this paper and has been applied to derive the nonlinear transformations and exact solutions (especially the solitary wave solution) [24–26], auto-Bäcklund transformations (auto-BT) [27] and similarity reductions [28] to nonlinear partial differential equations (PDEs) in mathematical physics. In ref. [29], Zhang extended the application of the homogeneous balance principle, and used the homogeneous balance principle to derive Bäcklund transformations and exact solutions for nonlinear partial differential equations that have more nonlinear terms and more highest-order partial derivative terms. In this paper, we use the homogeneous balance principle to explore the exact solutions of the Rangwala–Rao equation, the Gerdjikov–Ivanov equation, the Chen–Lee–Lin equation and the Ablowitz–Ramani–Segur equation with the aid of a set of subsidiary higher order ordinary differential equations.

The rest of this paper is organized as follows: in §2, a set of the subsidiary higher order ordinary differential equations are investigated and their exact solutions are presented; by using the homogeneous balance principle and a set of the subsidiary higher order ordinary differential equations, the exact solutions of the Rangwala–Rao equation, the Gerdjikov–Ivanov equation, the Chen–Lee–Lin equation, the Ablowitz–Ramani–Segur equation are derived respectively in §3; in §4, some remarks and conclusions are made.

2. The description of the subsidiary higher order ordinary differential equations

The first sub-ODE is considered in the form

$$F'^2(\xi) = \rho^2 F^2(\xi) + 2\rho\sigma F^{p+2}(\xi) + \sigma^2 F^{2p+2}(\xi), \tag{5}$$

where $p > 0, \rho$ and σ are constants.

Equation (5) admits solutions as

$$F(\xi) = \left\{-\frac{\rho}{\sigma}\left[\frac{1}{2} - \frac{1}{2}\tanh\left(\frac{1}{2}p|\rho|\xi\right)\right]\right\}^{1/p}, \quad \text{if} \quad \rho < 0, \sigma > 0 \tag{6}$$

and

$$F(\xi) = \left\{ -\frac{\rho}{\sigma} \left[\frac{1}{2} + \frac{1}{2} \tanh\left(\frac{1}{2} p\rho\xi \right) \right] \right\}^{1/p}, \quad \text{if} \quad \rho > 0, \sigma < 0. \tag{7}$$

The second sub-ODE is considered in the form

$$F'^2(\xi) = \rho F^2(\xi) + \sigma F^{p+2}(\xi) + \left(\frac{\sigma^2}{4\rho} - \rho \right) F^{2p+2}(\xi), \tag{8}$$

where $p > 0, \rho > 0$ and σ are constants.

Equation (8) admits solution

$$F(\xi) = \left[\frac{\operatorname{sech}^2\left(\frac{1}{2} p\sqrt{\rho}\xi\right)}{2 - (1 + (\sigma/2\rho))\operatorname{sech}^2\left(\frac{1}{2} p\sqrt{p}\xi\right)} \right]^{1/p}, \quad (-2\rho \le \sigma \le 2\rho). \tag{9}$$

The third sub-ODE is considered in the form

$$F'^2(\xi) = -\rho F^2(\xi) + \sigma F^{p+2}(\xi) + \left(\rho - \frac{\sigma^2}{4\rho} \right) F^{2p+2}(\xi), \tag{10}$$

where $p > 0, \rho > 0$ and σ are constants.

Equation (10) admits solutions

$$F(\xi) = \left[\frac{1}{\pm \sin(p\sqrt{\rho}\xi) + (\sigma/2\rho)} \right]^{1/p}, \quad \text{if} \quad \sigma \ge 2\rho. \tag{11}$$

The fourth sub-ODE is considered in the form

$$F'^2(\xi) = \frac{4}{p^2}[F^{p+2}(\xi) - \sigma F^{2p+2}(\xi)], \tag{12}$$

where $p > 0, \sigma > 0$, and eq. (12) admits the following solution:

$$F(\xi) = \left[\frac{1}{\xi^2 + \sigma} \right]^{1/p}. \tag{13}$$

It should be noted that ODEs (5), (8), (10) and (12) could admit other solutions. For example, the sinusoidal type solutions, the negative solutions for odd integer p, and so on. But for the sake of simplicity we neglect these cases here.

3. The exact solutions to the nonlinear Schrödinger-type equations

3.1 *The exact solutions to the Rangwala–Rao equation*

Firstly, we consider the Rangwala–Rao equation (1). Suppose the exact solutions of eq. (1) is of the form

$$u(x,t) = \mathrm{e}^{-i\omega t}\mathrm{e}^{i\psi(x-vt)}a(x - vt), \tag{14}$$

where a, ψ are undetermined functions with one variable only, ω, v are constants determined later, and set the relation of a, ψ as

$$\psi'(\xi) = -\frac{\omega}{2(v+\beta_1)} + \frac{T\beta_2}{2(v+\beta_1)}a^2(\xi), \tag{15}$$

where $\xi = x - vt$.

Substituting (14) into (1) and considering (15) simultaneously yields

$$a''(\xi) - \frac{4(v+\beta_1)-\omega^2}{4(v+\beta_1)^2}a(\xi) - \frac{T\beta_2\omega}{2(v+\beta_1)^2}a^3(\xi) + \frac{3T^2\beta_2^2}{16(v+\beta_1)^2}a^5(\xi) = 0. \tag{16}$$

According to the homogeneous balance principle, we suppose that the exact solutions of eq. (16) are of the form

$$a(\xi) = \mu F(\xi), \tag{17}$$

where $\mu > 0$ is a constant determined later, $F(\xi)$ satisfies one of the subsidiary ODEs in §2.

Case 3.1.1. Substituting (17) into (16) and considering eq. (8) ($p = 2$) simultaneously, the left-hand side of eq. (16) becomes a polynomial in $F(\xi)$

$$\begin{aligned} &a''(\xi) - \frac{4(v+\beta_1)-\omega^2}{4(v+\beta_1)^2}a(\xi) - \frac{T\beta_2\omega}{2(v+\beta_1)^2}a^3(\xi) + \frac{3T^2\beta_2^2}{16(v+\beta_1)^2}a^5(\xi) \\ &= \left[\rho - \frac{4(v+\beta_1)-\omega^2}{4(v+\beta_1)^2}\right]\mu F + \left[2\sigma - \frac{T\beta_2\omega}{2(v+\beta_1)^2}\mu^2\right]\mu F^3 \\ &+ \left[3\left(\frac{\sigma^2}{4\rho} - \rho\right) + \frac{3T^2\beta_2^2}{16(v+\beta_1)^2}\mu^4\right]\mu F^5. \end{aligned} \tag{18}$$

Setting the coefficients of the polynomial in eq. (18) to zero yields

$$\begin{aligned} &\rho - \frac{4(v+\beta_1)-\omega^2}{4(v+\beta_1)^2} = 0, \\ &2\sigma - \frac{T\beta_2\omega}{2(v+\beta_1)^2}\mu^2 = 0, \\ &3\left(\frac{\sigma^2}{4\rho} - \rho\right) + \frac{3T^2\beta_2^2}{16(v+\beta_1)^2}\mu^4 = 0. \end{aligned}$$

Solving the algebraic equations obtained above yields

$$\begin{aligned} &\rho = \frac{4(v+\beta_1)-\omega^2}{4(v+\beta_1)^2}, \\ &\mu = \sqrt[4]{\frac{[4(v+\beta_1)-\omega^2]^2}{(v+\beta_1)\beta_2^2}}, \\ &\sigma = \frac{T\beta_2\omega}{2(v+\beta_1)^2}\sqrt{\frac{[4(v+\beta_1)-\omega^2]^2}{(v+\beta_1)\beta_2^2}}, \end{aligned}$$

where $\omega^2 < 4(v+\beta_1), v+\beta_1 > 0$, ω, v are constants.

Then the exact solution of eq. (16) is expressed as

$$a_{11}(\xi) = \sqrt[4]{\frac{[4(v+\beta_1)-\omega^2]^2}{(v+\beta_1)\beta_2^2}}\sqrt{\frac{\operatorname{sech}^2(\sqrt{\rho}\xi)}{2-(1+(\sigma/2\rho))\operatorname{sech}^2(\sqrt{\rho}\xi)}}, \tag{19}$$

where

$$\rho = \frac{4(v+\beta_1)-\omega^2}{4(v+\beta_1)^2}, \quad \sigma = \frac{T\beta_2\omega}{4(v+\beta_1)^2}\sqrt{\frac{[4(v+\beta_1)-\omega^2]^2}{(v+\beta_1)\beta_2^2}},$$
$$\omega^2 < 4(v+\beta_1), \quad v+\beta_1 > 0, \quad \omega, v \text{ are constants.}$$

Substituting (19) into (15) yields

$$\psi_{11}(\xi) = C - \frac{\omega}{2(v+\beta_1)}\xi + \frac{T\beta_2}{2(v+\beta_1)}\sqrt{\frac{[4(v+\beta_1)-\omega^2]^2}{(v+\beta_1)\beta_2^2}} \times \int_0^{\xi}\frac{\operatorname{sech}^2(\sqrt{\rho}\xi)}{2-(1+(\sigma/2\rho))\operatorname{sech}^2(\sqrt{\rho}\xi)}\mathrm{d}\xi, \tag{20}$$

where

$$\xi = x - vt, \quad \rho = \frac{4(v+\beta_1)-\omega^2}{4(v+\beta_1)^2}, \quad \sigma = \frac{T\beta_2\omega}{4(v+\beta_1)^2}\sqrt{\frac{[4(v+\beta_1)-\omega^2]^2}{(v+\beta_1)\beta_2^2}},$$

$\omega^2 < 4(v+\beta_1), v+\beta_1 > 0, \omega, v, C$ are constants.

Substituting (19) and (20) into (14), the exact solution of eq. (1) can be expressed as

$$u_{11}(x,t) = \mathrm{e}^{-i\omega t}\mathrm{e}^{i\psi_{11}(x-vt)}\sqrt[4]{\frac{[4(v+\beta_1)-\omega^2]^2}{(v+\beta_1)\beta_2^2}} \times \sqrt{\frac{\operatorname{sech}^2\left(\sqrt{\rho}(x-vt)\right)}{2-(1+(\sigma/2\rho))\operatorname{sech}^2\left(\sqrt{\rho}(x-vt)\right)}},$$

where

$$\psi_{11}(x-vt) = C - \frac{\omega}{2(v+\beta_1)}(x-vt) + \frac{T\beta_2}{2(v+\beta_1)}\sqrt{\frac{[4(v+\beta_1)-\omega^2]^2}{(v+\beta_1)\beta_2^2}} \times \int_0^{(x-vt)}\frac{\operatorname{sech}^2(\sqrt{\rho}\xi)}{2-(1+(\sigma/2\rho))\operatorname{sech}^2(\sqrt{\rho}\xi)}\mathrm{d}\xi,$$

and

$$\rho = \frac{4(v+\beta_1)-\omega^2}{4(v+\beta_1)^2}, \quad \sigma = \frac{T\beta_2\omega}{4(v+\beta_1)^2}\sqrt{\frac{[4(v+\beta_1)-\omega^2]^2}{(v+\beta_1)\beta_2^2}},$$

$\omega^2 < 4(v+\beta_1), v+\beta_1 > 0, \omega, v, C$ are constants.

Case 3.1.2. Similar to Case 3.1.1, with the aid of eq. (10) ($p = 2$), the exact solution of eq. (1) can be expressed as

$$u_{12}(x,t) = \mathrm{e}^{-i\omega t}\mathrm{e}^{i\psi_{12}(x-vt)} a \sqrt[4]{\frac{[\omega^2 - 4(v+\beta_1)]^2}{(v+\beta_1)\beta_2^2}} \times \sqrt{\frac{1}{\pm\sin\left(2\sqrt{\rho}(x-vt)\right) + (\sigma/2\rho)}},$$

where

$$\psi_{12}(x-vt) = C - \frac{\omega}{2(v+\beta_1)}(x-vt) + \frac{T\beta_2}{2(v+\beta_1)}\sqrt{\frac{[\omega^2 - 4(v+\beta_1)]^2}{(v+\beta_1)\beta_2^2}} \times \int_0^{(x-vt)} \frac{1}{\pm\sin\left(2\sqrt{\rho}\xi\right) + (\sigma/2\rho)}\mathrm{d}\xi,$$

and

$$\rho = \frac{\omega^2 - 4(v+\beta_1)}{4(v+\beta_1)^2},\ \sigma = \frac{T\beta_2\omega}{4(v+\beta_1)^2}\sqrt{\frac{[\omega^2 - 4(v+\beta_1)]^2}{(v+\beta_1)\beta_2^2}},$$

$\omega^2 \geq 4(v+\beta_1) > 0,\ T\beta_2\omega > 0, \omega, v, C$ are constants.

Case 3.1.3. Similar to Case 3.1.1, with the aid of eq. (12) ($p = 2$), the exact solution of eq. (1) can be expressed as

$$u_{13}(x,t) = \mathrm{e}^{-i\omega t}\mathrm{e}^{i\psi_{13}(x-vt)}\sqrt{\frac{1}{(x-vt)^2 + \sigma}\ \frac{4(v+\beta_1)^2}{T\beta_2\omega}},$$

where

$$\psi_{13}(x-vt) = -\frac{\omega}{2(v+\beta_1)}(x-vt) + \frac{2(v+\beta_1)}{\sqrt{\sigma}\omega}\ \arctan\left(\frac{x-vt}{\sqrt{\sigma}}\right) + C,$$

and $\omega^2 = 4(v+\beta_1), \sigma = (v+\beta_1)/4, v+\beta_1 > 0, T\beta_2\omega > 0, v, C$ are constants.

3.2 *The exact solutions to the Gerdjikov–Ivanov equation*

Similar to §3.1, we suppose the exact solution of eq. (2) is of the form

$$u(x,t) = \mathrm{e}^{-i\omega t}\mathrm{e}^{i\psi(x-vt)}a(x-vt), \tag{14}$$

where a, ψ are undetermined functions with one variable only, ω, v are constants determined later, and set the relation of a, ψ as

$$\psi'(\xi) = -\frac{v}{2} + a^2(\xi), \tag{21}$$

where $\xi = x - vt$.

Substituting (14) into (2) and considering (21) simultaneously yields ODE for $a(\xi)$

$$a''(\xi) - \left(\omega - \frac{v^2}{4}\right) a(\xi) + 2va^3(\xi) + 3a^5(\xi) = 0. \tag{22}$$

The exact solutions of eq. (22) can be derived with the aid of subsidiary ODEs ($p = 2$) in §2. Then we obtain the exact solutions of eq. (2) as

Case 3.2.1

$$u_{21}(x,t) = \sqrt[4]{\frac{\left(\omega - (v^2/4)\right)^2}{\omega}} \times e^{-i\omega t} e^{i\psi_{21}(x-vt)} \sqrt{\frac{\operatorname{sech}^2\left(\sqrt{\rho}(x-vt)\right)}{2 - (1 + (\sigma/2\rho))\operatorname{sech}^2\left(\sqrt{\rho}(x-vt)\right)}},$$

where

$$\psi_{21}(x - vt) = C - \frac{v}{2}(x - vt) + \sqrt{\frac{\left(\omega - (v^2/4)\right)^2}{\omega}} \times \int_0^{(x-vt)} \frac{\operatorname{sech}^2\left(\sqrt{\rho}\xi\right)}{2 - (1 + (\sigma/2\rho))\operatorname{sech}^2\left(\sqrt{\rho}\xi\right)} \mathrm{d}\xi,$$

and

$$\rho = \omega - v^2/4, \quad \sigma = -\frac{v\left(\omega - (v^2/4)\right)}{\sqrt{\omega}},$$

$\omega > v^2/4$, ω, v, C are constants.

Case 3.2.2

$$u_{22}(x,t) = \sqrt[4]{\frac{\left((v^2/4) - \omega\right)^2}{\omega}} e^{-i\omega t} e^{i\psi_{22}(x-vt)} \times \sqrt{\frac{1}{\pm\sin\left(2\sqrt{\rho}(x-vt)\right) + (\sigma/2\rho)}}$$

where

$$\psi_{22}(x - vt) = C - \frac{v}{2}(x - vt) + \sqrt{\frac{\left((v^2/4) - \omega\right)^2}{\omega}} \times \int_0^{(x-vt)} \frac{\mathrm{d}\xi}{\pm\sin\left(2\sqrt{\rho}\xi\right) + (\sigma/2\rho)},$$

and $\rho = (v^2/4) - \omega, \sigma = -v\sqrt{\frac{((v^2/4)-\omega)^2}{\omega}}, v^2 > 4\omega > 0, v < 0, \omega, v, C$ are constants.

Case 3.2.3

$$u_{23}(x,t) = \mathrm{e}^{-i\omega t}\mathrm{e}^{i\psi_{23}(x-vt)}\sqrt{-\frac{1}{v((x-vt)^2+\sigma)}}$$

where

$$\psi_{23}(x-vt) = C - \frac{v}{2}(x-vt) - \frac{1}{v\sqrt{\sigma}}\arctan\left(\frac{\xi}{\sqrt{\sigma}}\right),$$

and $\omega = v^2/4$, $\sigma = 1/v^2$, $v < 0$, ω, v, C are constants.

3.3 *The exact solutions to the Chen–Lee–Lin equation*

Similar to §3.1, we suppose the exact solution of eq. (3) is of the form

$$u(x,t) = \mathrm{e}^{-i\omega t}\mathrm{e}^{i\psi(x-vt)}a(x-vt), \tag{14}$$

where a, ψ are undetermined functions with one variable only, ω, v are constants determined later, and set the relation of a, ψ as

$$\psi'(\xi) = \frac{v}{2} - \frac{\alpha}{4}a^2(\xi), \tag{23}$$

where $\xi = x - vt$.

Substituting (14) into (3) and considering (23) simultaneously yields ODE for $a(\xi)$

$$a''(\xi) + \left(\omega + \frac{v^2}{4}\right)a(\xi) - \frac{v\alpha}{2}a^3(\xi) + \frac{3\alpha^2}{16}a^5(\xi) = 0. \tag{24}$$

The exact solutions of eq. (24) can be derived with the aid of subsidiary ODEs ($p = 2$) in §2. Then we obtain the exact solutions of eq. (3) as

Case 3.3.1

$$u_{31}(x,t) = 2\sqrt[4]{-\frac{(\omega+(v^2/4))^2}{\omega\alpha^2}}\mathrm{e}^{-i\omega t}\mathrm{e}^{i\psi_{31}(x-vt)} \times\sqrt{\frac{\operatorname{sech}^2\left(\sqrt{\rho}(x-vt)\right)}{2-(1+(\sigma/2\rho))\operatorname{sech}^2\left(\sqrt{\rho}(x-vt)\right)}},$$

where

$$\psi_{31}(x-vt) = C + \frac{v}{2}(x-vt) - \alpha\sqrt{-\frac{(\omega+(v^2/4))^2}{\omega\alpha^2}} \times\int_0^{(x-vt)}\frac{\operatorname{sech}^2\left(\sqrt{\rho}\xi\right)\mathrm{d}\xi}{2-(1+(\sigma/2\rho))\operatorname{sech}^2\left(\sqrt{\rho}\xi\right)},$$

and

$$\rho = -\left(\omega + \frac{v^2}{4}\right), \quad \sigma = v\alpha\sqrt{-\frac{(\omega + (v^2/4))^2}{\omega\alpha^2}},$$

$\omega < 0, v^2 < -4\omega, \omega, v, C$ are constants.

Case 3.3.2

$$u_{32}(x,t) = 2\sqrt[4]{-\frac{(\omega + (v^2/4))^2}{\omega\alpha^2}}\mathrm{e}^{-i\omega t}\mathrm{e}^{i\psi_{32}(x-vt)} \times\sqrt{\frac{1}{\pm\sin\left(2\sqrt{\rho}(x-vt)\right) + (\sigma/2\rho)}},$$

where

$$\psi_{32}(x-vt) = C + \frac{v}{2}(x-vt) - \alpha\sqrt{-\frac{(\omega + (v^2/4))^2}{\omega\alpha^2}} \times\int_0^{(x-vt)} \frac{\mathrm{d}\xi}{\pm\sin\left(2\sqrt{\rho}\xi\right) + (\sigma/2\rho)},$$

and

$$\rho = \omega + \frac{v^2}{4}, \quad \sigma = v\alpha\sqrt{-\frac{(\omega + (v^2/4))^2}{\omega\alpha^2}},$$

$\omega < 0, \quad \omega + \frac{v^2}{4} > 0, \omega, v, C$ are constants.

Case 3.3.3

$$u_{33}(x,t) = \mathrm{e}^{-i\omega t}\mathrm{e}^{i\psi_{33}(x-vt)}\sqrt{\frac{4}{v\alpha}}\sqrt{\frac{1}{(x-vt)^2 + \sigma}},$$

where

$$\psi_{33}(x-vt) = C + \frac{v}{2}(x-vt) - \frac{4}{\sqrt{\sigma}v^2\alpha}\ \arctan\left(\frac{x-vt}{\sqrt{\sigma}}\right),$$

and

$$\omega = -\frac{v^2}{4}, \quad \sigma = \frac{1}{v^2}, \quad v\alpha > 0, v, C \text{ are constants.}$$

3.4 *The exact solutions to the Ablowitz–Ramani–Segur equation*

By using the same method in §3.1, we suppose the exact solution of eq. (4) is of the form

$$u(x,t) = \mathrm{e}^{-i\omega t}\mathrm{e}^{i\psi(x-vt)}a(x-vt), \tag{14}$$

where a, ψ are undetermined functions with one variable only, ω, v are constants determined later, and set the relation of a, ψ as

$$\psi'(\xi) = \frac{v}{2} - \frac{\delta}{2}a^2(\xi), \tag{25}$$

where $\xi = x - vt$.

Substituting (14) into (4) and considering (25) simultaneously yields ODE for $a(\xi)$

$$a''(\xi) + \left(\omega + \frac{v^2}{4}\right)a(\xi) + (\beta + \delta v)a^3(\xi) + \frac{3}{4}a^5(\xi) = 0. \tag{26}$$

The exact solutions of eq. (26) can be derived with the aid of subsidiary ODEs ($p = 2$) in §2. Then we obtain the exact solutions of eq. (4) as

Case 3.4.1

$$u_{41}(x,t) = \sqrt[4]{\frac{16\left(\omega + (v^2/4)\right)^2}{(\beta+\delta v)^2 - (4\omega + v^2)}}\mathrm{e}^{-i\omega t}\mathrm{e}^{i\psi_{41}(x-vt)} \times \sqrt{\frac{\mathrm{sech}^2\left(\sqrt{\rho}(x-vt)\right)}{2 - (1 + (\sigma/2\rho))\,\mathrm{sech}^2\left(\sqrt{\rho}(x-vt)\right)}},$$

where

$$\psi_{41}(x-vt) = C + \frac{v}{2}(x-vt) - \frac{\delta}{2}\sqrt{\frac{16\left(\omega + (v^2/4)\right)^2}{(\beta+\delta v)^2 - (4\omega+v^2)}} \times \int_0^{(x-vt)} \frac{\mathrm{sech}^2\left(\sqrt{\rho}\xi\right)\mathrm{d}\xi}{2 - (1 + (\sigma/2\rho))\,\mathrm{sech}^2\left(\sqrt{\rho}\xi\right)},$$

and

$$\rho = -\left(\omega + \frac{v^2}{4}\right), \quad \sigma = -\frac{1}{2}(\beta+\delta v)\sqrt{\frac{16\left(\omega + (v^2/4)\right)^2}{(\beta+\delta v)^2 - (4\omega+v^2)}},$$

$(\beta+\delta v)^2 - (4\omega + v^2) > 0, \omega + \frac{v^2}{4} < 0, \omega, v, C$ are constants.

Case 3.4.2

$$u_{42}(x,t) = 2\sqrt[4]{\frac{\left(\omega + (v^2/4)\right)^2}{(\beta+\delta v)^2 - (4\omega + v^2)}}\mathrm{e}^{-i\omega t}\mathrm{e}^{i\psi_{42}(x-vt)} \times \sqrt{\frac{1}{\pm\sin\left(2\sqrt{\rho}(x-vt)\right) + (\sigma/2\rho)}},$$

where

$$\psi_{42}(x-vt) = C + \frac{v}{2}(x-vt) - 2\delta\sqrt{\frac{\left(\omega+(v^2/4)\right)^2}{(\beta+\delta v)^2-(4\omega+v^2)}} \times \int_0^{(x-vt)} \frac{\mathrm{d}\xi}{\pm\sin\left(2\sqrt{\rho}\xi\right)+(\sigma/2\rho)},$$

and

$$\rho = \omega + \frac{v^2}{4}, \quad \sigma = -2(\beta+\delta v)\sqrt{\frac{\left(\omega+(v^2/4)\right)^2}{(\beta+\delta v)^2-(4\omega+v^2)}},$$

$(\beta+\delta v)^2-(4\omega+v^2)>0, \beta+\delta v<0, \omega+\frac{v^2}{4}>0, \omega, v, C$ are constants.

Case 3.4.3

$$u_{43}(x,t) = \mathrm{e}^{-i\omega t}\mathrm{e}^{i\psi_{43}(x-vt)}\sqrt{-\frac{2}{(\beta+\delta v)\left((x-vt)^2+\sigma\right)}},$$

where

$$\psi_{43}(x-vt) = C + \frac{v}{2}(x-vt) + \frac{\delta}{(\beta+\delta v)\sqrt{\sigma}} \arctan\left(\frac{x-vt}{\sqrt{\sigma}}\right),$$

and $\omega = -v^2/4, \beta+\delta v<0, \sigma = 1/(\beta+\delta v)^2, v, C$ are constants.

4. The remarks and conclusions

By using the homogeneous balance principle, and a set of the subsidiary higher order ordinary differential equations, the exact solutions to a class of the nonlinear Schrödinger-type equations, including the Rangwala–Rao equation, the Gerdjikov–Ivanov equation, the Chen–Lee–Lin equation and the Ablowitz–Ramani–Segur equation are derived respectively. The method in this paper can be applied to the other nonlinear evolution equations including the nonlinear Schrödinger-type equations, higher-order nonlinear evolution equations.

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