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Construction of exact complex dynamical invariant of a two-dimensional classical system

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Abstract. We present the construction of exact complex dynamical invariant of a two-dimensional classical dynamical system on an extended complex space utilizing Lie algebraic approach. These invariants are expected to play a vital role in understanding the complex trajectories of both classical and quantum systems.

Keywords. Complex Hamiltonian; exact complex invariant.

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1. Introduction

In recent years, considerable progress has been made to develop the theory of dynamical invariants for both time-dependent and time-independent classical systems [1–3]. The existence and subsequently the construction, if possible, of these additional invariants for a dynamical system help a lot in understanding the detailed properties of the corresponding system. Therefore, the importance of invariants in the domains of a variety of fields [4,5] like plasma physics, laser physics, accelerator physics etc. clearly offers a motivation to look for system(s) that can generate such structures in an unambiguous manner. To this effect several methods have been developed in the literature (see [2]) for such construction with varying degrees of applicability.

Although real Hamiltonians and their associated invariants have been studied extensively, same efforts have not been made for the study of complex Hamiltonian systems. It may be mentioned that not only the complex Hamiltonians but also the real Hamiltonians are found to admit complex invariants, e.g. harmonic oscillator system possesses a complex invariant, namely $u = \ln(p + im\omega x) - i\omega t$ [6].

The complex Hamiltonians have been in use for a long time to study many physical systems such as the optical model of nucleus. Non-Hermitian PT-symmetric complex Hamiltonians are also used to study delocalization of transitions in condensed matter systems such as vortex flux line depinning in Type-II superconductors [7], to study population biology [8], to study the complex trajectories particularly in laser physics [9] etc. The complex Hamiltonians are now gaining importance for explaining several phenomena [10] such as the resonance scattering in atomic, molecular and nuclear physics and also in some chemical reactions. Therefore, the study of the associated complex and/or real dynamical invariants become desirable and if they exist and are available, they can provide insight into the analysis of complex trajectories. In fact the quantum mechanics of such systems has been studied extensively by many authors [11–14] in recent years, and the classical mechanics of such systems needs to be investigated. Very recently, Kaushal and Sweta Singh [15,16] have investigated the construction of complex invariants of one-dimensional complex Hamiltonian systems.

Although there are several schemes [9,15,17] for the complexification of Hamiltonian $H(x,p)$, in the present work we follow the approach used by Xavier and de Aguiar [9] to develop an algorithm for the computation of semiclassical coherent-state propagator for a particle going through a simple barrier potential, which is given by the expression

$$x = x_1 + ip_2; \quad p = p_1 + ix_2, \tag{1.1}$$

for canonical variables x and p in one dimension. In fact the transformation, eq. (1.1), makes both x and p separately complex by extending each of them to the corresponding complex planes, i.e. inserting an imaginary component in each. An important aspect of such a characterization is the manifestation of (x_1, p_1) and (x_2, p_2) as canonical pairs which in turn provides a link between the complex Hamiltonian and a pair of real Hamiltonians and can be helpful in establishing the integrability of $H(x,p)$ [18]. It is worth mentioning that the transformations (1.1) have been successfully used for solving Schrödinger equation for a large number of one-dimensional complex potentials [13,14]. Also the PT symmetry of non-Hermitian $H(x,p)$ appears to be a special case of the general transformation (1.1) under certain limits (in the sense that under PT-symmetry this transformation reduces to a restriction on the variables (x_1, p_1, x_2, p_2), such that $(x_1, p_1, x_2, p_2) \to (-x_1, p_1, -x_2, p_2; i \to -i)$).

Further, from the physics point of view, the imaginary parts x_2, p_2 respectively are some components of momentum and coordinate and their presence in (1.1) give some sort of momentum-coordinate coupling in the dynamical system and hence for dimensional consistency, eq. (1.1) needs modification, i.e.

$$x = x_1 + idp_2; \qquad p = p_1 + id^{-1}x_2,$$

and in the present study for simplicity we consider $d = 1$.

Most of the studies on this front are carried out for one-dimensional systems. Here, in the present work we carried out the extended phase-plane approach in two dimensions with a view to obtain exact complex invariants of classical dynamical systems. Lie algebraic method [19] is explored for such constructions as this has been widely used in literature for the construction of exact and real invariants for a variety of classical dynamical systems and can easily be extended for quantum systems also.

The organization of the paper is as follows: in §2, the method of complexification and construction of invariants of dynamical systems is described. In §3, we apply the results of §2 to obtain a complex invariant of a dynamical system and finally concluding remarks are given in §4.

2. Construction of complex invariants

Consider a two-dimensional real phase-space (x, y, p_x, p_y), which can be transformed into a complex space $(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4)$, by extending eq. (1.1) as

$$\begin{aligned} &x = x_1 + ip_3; \quad y = x_2 + ip_4, \\ &p_x = p_1 + ix_3; \quad p_y = p_2 + ix_4, \end{aligned} \tag{2.1}$$

in two dimensions. Therefore, the Hamiltonian $H(x, y, p_x, p_y)$ of a two-dimensional system in complex space can be expressed, using eq. (2.1), as

$$H = H_1(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) + iH_2(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4). \tag{2.2}$$

Clearly, from eq. (2.1) we get

$$\begin{aligned} &\frac{\partial}{\partial x} = \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial p_3}; \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial x_2} - i\frac{\partial}{\partial p_4}, \\ &\frac{\partial}{\partial p_x} = \frac{\partial}{\partial p_1} - i\frac{\partial}{\partial x_3}; \quad \frac{\partial}{\partial p_y} = \frac{\partial}{\partial p_2} - i\frac{\partial}{\partial x_4}. \end{aligned} \tag{2.3}$$

The Hamilton's equations of motion for complex H in eq. (2.2) can be written as

$$\begin{aligned} &\dot{x}_1 = \frac{\partial H_1}{\partial p_1} + \frac{\partial H_2}{\partial x_3}; \quad \dot{p}_3 = \frac{\partial H_2}{\partial p_1} - \frac{\partial H_1}{\partial x_3}, \\ &\dot{x}_2 = \frac{\partial H_1}{\partial p_2} + \frac{\partial H_2}{\partial x_4}; \quad \dot{p}_4 = \frac{\partial H_2}{\partial p_2} - \frac{\partial H_1}{\partial x_4}, \\ &\dot{p}_1 = -\left(\frac{\partial H_1}{\partial x_1} + \frac{\partial H_2}{\partial p_3}\right); \quad \dot{x}_3 = -\left(\frac{\partial H_2}{\partial x_1} - \frac{\partial H_1}{\partial p_3}\right), \\ &\dot{p}_2 = -\left(\frac{\partial H_1}{\partial x_2} + \frac{\partial H_2}{\partial p_4}\right); \quad \dot{x}_4 = -\left(\frac{\partial H_2}{\partial x_2} - \frac{\partial H_1}{\partial p_4}\right). \end{aligned} \tag{2.4}$$

If the Hamiltonian H in eq. (2.2) is to be analytic function of complex variables, then H_1 and H_2 satisfy the Cauchy–Riemann conditions [18] and after employing such analyticity conditions, eq. (2.4) becomes

$$\begin{aligned} &\dot{x}_1 = 2\frac{\partial H_1}{\partial p_1}; \quad \dot{p}_1 = -2\frac{\partial H_1}{\partial x_1}; \quad \dot{x}_2 = 2\frac{\partial H_1}{\partial p_2}; \quad \dot{p}_2 = -2\frac{\partial H_1}{\partial x_2}, \\ &\dot{x}_3 = 2\frac{\partial H_1}{\partial p_3}; \quad \dot{p}_3 = -2\frac{\partial H_1}{\partial x_3}; \quad \dot{x}_4 = 2\frac{\partial H_1}{\partial p_4}; \quad \dot{p}_4 = -2\frac{\partial H_1}{\partial x_4}. \end{aligned} \tag{2.5}$$

Note that (x_1, p_1), (x_2, p_2), (x_3, p_3) and (x_4, p_4) constitute canonical pairs.

Now consider a complex phase-space function $I(x, y, p_x, p_y, t)$ as

$$I = I_1(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4, t) + iI_2(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4, t). \tag{2.6}$$

Thus for the function I to be the time-dependent dynamical invariant of the system in complex phase-space, we write the invariance condition as

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\partial I}{\partial t} + [I, H] = 0, \tag{2.7}$$

where $[\cdot, \cdot]$ is the Poisson bracket, which in view of the definition eq. (2.1) turns out as

$$\begin{aligned}[A, B]_{(x,p)} = {} & [A, B]_{(x_1,p_1)} - i[A, B]_{(x_1,x_3)} - i[A, B]_{(p_3,p_1)} \\ & -[A, B]_{(p_3,x_3)} + [A, B]_{(x_2,p_2)} - i[A, B]_{(x_2,x_4)} \\ & -i[A, B]_{(p_4,p_2)} - [A, B]_{(p_4,x_4)},\end{aligned} \tag{2.8}$$

which indicates that the computation of Poisson bracket in the case of complex Hamiltonian systems becomes a bit tedious.

With a view to demonstrate the underlying elegance of the Lie algebraic approach at the classical level, we briefly describe this in order to construct complex invariants of the dynamical systems. In the Lie algebraic approach, one can express the complex Hamiltonian $H(x, y, p_x, p_y, t)$ of the system as

$$H = \sum_n h_n(t)\Gamma_n(x, y, p_x, p_y, t), \tag{2.9}$$

where the set of functions $\{\Gamma_1, ..., \Gamma_n\}$ is not explicitly time-dependent and $h_n(t)$ are complex coefficient functions of time. The Γ_n's in eq. (2.9) generate a closed dynamical algebra, which implies

$$[\Gamma_n, \Gamma_m] = \sum_l C^l_{nm}\Gamma_l, \tag{2.10}$$

where C^l_{nm} are the complex structure constants of the algebra. If the Γ_n's in eq. (2.9) are not sufficient to close the algebra then the set of Γ_n must be extended by adding new Γ_l's, such that $\Gamma_l = [\Gamma_n, \Gamma_m]$, until the closure is obtained along with additional $h_l(t)$'s which are taken to be zero.

Since the complex dynamical invariant I is also a part of Lie algebra, then one can express this as

$$I(t) = \sum_k \lambda_k(t)\Gamma_k(x, y, p_x, p_y), \tag{2.11}$$

where $\lambda_k(t)$'s are time-dependent complex coefficients. Thus by using eq. (2.9) and (2.11) for H and I respectively in eq. (2.7), we get a system of linear, first-order differential equations, namely

$$\dot{\lambda}_r + \sum_n \left[\sum_m C^r_{nm} h_m(t) \right] \lambda_n = 0, \tag{2.12}$$

in λ_n's. Therefore, the solutions of these differential equations in turn provide classical complex invariant of a given system from eq. (2.11).

In the next section we will use the prescription given in the present section to obtain complex invariant of a classical complex Hamiltonian system.

3. Illustrative example

Consider the case of a simple harmonic oscillator in two dimensions, for which the Hamiltonian is written as

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}\omega^2(t)(x^2 + y^2). \tag{3.1}$$

Using the complexification eq. (2.1), the above Hamiltonian can be expressed as

$$\begin{aligned} H &= \frac{1}{2}p_1^2 - \frac{1}{2}x_3^2 + \frac{1}{2}p_2^2 - \frac{1}{2}x_4^2 + ip_2x_4 + ip_1x_3 + i\omega^2(t)p_3x_1 \\ &\quad + i\omega^2(t)p_4x_2 + \frac{1}{2}\omega^2(t)x_1^2 - \frac{1}{2}\omega^2(t)p_3^2 + \frac{1}{2}\omega^2(t)x_2^2 - \frac{1}{2}\omega^2(t)p_4^2 \\ &= \sum_{m=1}^{12} h_m(t)\Gamma_m(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4), \end{aligned} \tag{3.2}$$

and the various Γ's and $h(t)$'s for the above complex H are expressed as

$$\begin{aligned} &\Gamma_1 = \frac{1}{2}p_1^2; \quad \Gamma_2 = \frac{1}{2}x_3^2; \quad \Gamma_3 = \frac{1}{2}p_2^2; \quad \Gamma_4 = \frac{1}{2}x_4^2, \\ &\Gamma_5 = p_1x_3; \quad \Gamma_6 = p_2x_4; \quad \Gamma_7 = p_3x_1; \quad \Gamma_8 = p_4x_2; \\ &\Gamma_9 = \frac{1}{2}x_1^2; \quad \Gamma_{10} = \frac{1}{2}p_3^2; \quad \Gamma_{11} = \frac{1}{2}x_2^2; \quad \Gamma_{12} = \frac{1}{2}p_4^2, \\ &h_1 = h_3 = 1; \quad h_2 = h_4 = -1; \quad h_5 = h_6 = i; \quad h_7 = h_8 = i\omega^2(t), \\ &h_9 = h_{11} = \omega^2(t); \quad h_{10} = h_{12} = -\omega^2(t). \end{aligned} \tag{3.3}$$

The dynamical algebra in this case is not closed unless one adds eight more phase-space functions (Γ_l)'s. The additional (Γ_l)'s and their corresponding h_l's are given as

$$\begin{aligned} &\Gamma_{13} = p_1p_3; \quad \Gamma_{14} = p_1x_1; \quad \Gamma_{15} = x_1x_3; \quad \Gamma_{16} = p_3x_3, \\ &\Gamma_{17} = p_2p_4; \quad \Gamma_{18} = p_2x_2; \quad \Gamma_{19} = p_4x_4; \quad \Gamma_{20} = x_2x_4, \\ &h_{13} = h_{14} = h_{15} = h_{16} = h_{17} = h_{18} = h_{19} = h_{20} = 0. \end{aligned} \tag{3.4}$$

Now in the light of modified definition of Poisson bracket for complex systems, eq. (2.8), we get large number of nonvanishing Poisson brackets, namely

$$[\Gamma_1,\Gamma_7] = -\Gamma_{13} + i\Gamma_{14};\quad [\Gamma_1,\Gamma_9] = -\Gamma_{14};\quad [\Gamma_1,\Gamma_{10}] = i\Gamma_{13};\quad [\Gamma_1,\Gamma_{13}] = 2i\Gamma_1,$$
$$[\Gamma_1,\Gamma_{14}] = -2\Gamma_1;\quad [\Gamma_1,\Gamma_{15}] = -\Gamma_5;\quad [\Gamma_1,\Gamma_{16}] = i\Gamma_5;\quad [\Gamma_2,\Gamma_7] = \Gamma_{15} + i\Gamma_{16},$$
$$[\Gamma_2,\Gamma_9] = i\Gamma_{15};\quad [\Gamma_2,\Gamma_{10}] = \Gamma_{16};\quad [\Gamma_2,\Gamma_{13}] = \Gamma_5;\quad [\Gamma_2,\Gamma_{14}] = i\Gamma_5,$$
$$[\Gamma_2,\Gamma_{15}] = 2i\Gamma_2;\quad [\Gamma_2,\Gamma_{16}] = 2\Gamma_2;\quad [\Gamma_3,\Gamma_8] = -\Gamma_{17} + i\Gamma_{16};\quad [\Gamma_3,\Gamma_{11}] = -\Gamma_{18},$$
$$[\Gamma_3,\Gamma_{12}] = i\Gamma_{17};\quad [\Gamma_3,\Gamma_{17}] = 2i\Gamma_3;\quad [\Gamma_3,\Gamma_{18}] = -2\Gamma_3;\quad [\Gamma_3,\Gamma_{19}] = i\Gamma_6,$$
$$[\Gamma_3,\Gamma_{20}] = -\Gamma_6;\quad [\Gamma_4,\Gamma_8] = i\Gamma_{19} + \Gamma_{20};\quad [\Gamma_4,\Gamma_{11}] = i\Gamma_{20};\quad [\Gamma_4,\Gamma_{12}] = \Gamma_{19};$$
$$[\Gamma_4,\Gamma_{17}] = \Gamma_6;\quad [\Gamma_4,\Gamma_{18}] = i\Gamma_6;\quad [\Gamma_4,\Gamma_{19}] = 2\Gamma_4;\quad [\Gamma_4,\Gamma_{20}] = 2i\Gamma_4,$$
$$[\Gamma_5,\Gamma_7] = i\Gamma_{13} + \Gamma_{14} + i\Gamma_{15} - \Gamma_{16};\quad [\Gamma_5,\Gamma_9] = i\Gamma_{14} - \Gamma_{15};\quad [\Gamma_5,\Gamma_{10}] = \Gamma_{13} + i\Gamma_{16},$$
$$[\Gamma_5,\Gamma_{13}] = 2\Gamma_1 + i\Gamma_1;\quad [\Gamma_5,\Gamma_{14}] = 2i\Gamma_1 - \Gamma_5;\quad [\Gamma_5,\Gamma_{15}] = -2\Gamma_{14} + i\Gamma_5,$$
$$[\Gamma_5,\Gamma_{16}] = 2i\Gamma_2 + \Gamma_5;\quad [\Gamma_6,\Gamma_8] = i\Gamma_{17} + \Gamma_{18} - \Gamma_{19} + i\Gamma_{20};\quad [\Gamma_6,\Gamma_{11}] = i\Gamma_{18} - \Gamma_{20},$$
$$[\Gamma_6,\Gamma_{12}] = \Gamma_{17} + i\Gamma_{19};\quad [\Gamma_6,\Gamma_{17}] = 2\Gamma_3 + i\Gamma_6;\quad [\Gamma_6,\Gamma_{18}] = 2i\Gamma_3 - \Gamma_6,$$
$$[\Gamma_6,\Gamma_{19}] = 2i\Gamma_4 + \Gamma_6;\quad [\Gamma_6,\Gamma_{20}] = -2\Gamma_4 + i\Gamma_6;\quad [\Gamma_7,\Gamma_{13}] = -i\Gamma_7 + 2\Gamma_{10},$$
$$[\Gamma_7,\Gamma_{14}] = \Gamma_7 - 2i\Gamma_9;\quad [\Gamma_7,\Gamma_{15}] = -i\Gamma_7 - 2\Gamma_9;\quad [\Gamma_7,\Gamma_{16}] = -\Gamma_7 - 2i\Gamma_{10},$$
$$[\Gamma_8,\Gamma_{17}] = -i\Gamma_8 + 2\Gamma_{12};\quad [\Gamma_8,\Gamma_{18}] = \Gamma_8 - 2i\Gamma_{11};\quad [\Gamma_8,\Gamma_{19}] = -\Gamma_8 - 2i\Gamma_{12},$$
$$[\Gamma_8,\Gamma_{20}] = -i\Gamma_8 - 2\Gamma_{11};\quad [\Gamma_9,\Gamma_{13}] = \Gamma_7;\quad [\Gamma_9,\Gamma_{14}] = 2\Gamma_9;\quad [\Gamma_9,\Gamma_{15}] = -2i\Gamma_9,$$
$$[\Gamma_9,\Gamma_{16}] = -i\Gamma_7;\quad [\Gamma_{10},\Gamma_{13}] = -2i\Gamma_{10};\quad [\Gamma_{10},\Gamma_{14}] = -i\Gamma_7;\quad [\Gamma_{10},\Gamma_{15}] = -\Gamma_7,$$
$$[\Gamma_{10},\Gamma_{16}] = -2\Gamma_{10};\quad [\Gamma_{11},\Gamma_{17}] = \Gamma_8;\quad [\Gamma_{11},\Gamma_{18}] = 2\Gamma_{11};\quad [\Gamma_{11},\Gamma_{19}] = -i\Gamma_8,$$
$$[\Gamma_{11},\Gamma_{20}] = -2i\Gamma_{11};\quad [\Gamma_{12},\Gamma_{17}] = -2i\Gamma_{12};\quad [\Gamma_{12},\Gamma_{18}] = -i\Gamma_8;\quad [\Gamma_{12},\Gamma_{19}] = -2\Gamma_{12},$$
$$[\Gamma_{12},\Gamma_{20}] = -\Gamma_8;\quad [\Gamma_{13},\Gamma_{14}] = -\Gamma_{13} - i\Gamma_{14};\quad [\Gamma_{13},\Gamma_{15}] = -\Gamma_{14} - \Gamma_{16},$$
$$[\Gamma_{13},\Gamma_{16}] = -\Gamma_{13} + i\Gamma_{16};\quad [\Gamma_{14},\Gamma_{15}] = -i\Gamma_{14} - \Gamma_{15};\quad [\Gamma_{14},\Gamma_{16}] = -i\Gamma_{13} - i\Gamma_{15};$$
$$[\Gamma_{15},\Gamma_{16}] = \Gamma_{15} - i\Gamma_{16};\quad [\Gamma_{17},\Gamma_{18}] = -\Gamma_{17} - i\Gamma_{18};\quad [\Gamma_{17},\Gamma_{19}] = -\Gamma_{17} + i\Gamma_{19};$$
$$[\Gamma_{17},\Gamma_{20}] = -\Gamma_{18} - \Gamma_{19};\quad [\Gamma_{18},\Gamma_{19}] = -i\Gamma_{17} + i\Gamma_{20};\quad [\Gamma_{18},\Gamma_{20}] = -i\Gamma_{18} - \Gamma_{20};$$
$$[\Gamma_{19},\Gamma_{20}] = i\Gamma_{19} - \Gamma_{20}. \tag{3.5}$$

Thus we obtained a set of partial differential equations using eq. (3.5) in eq. (2.12), which are

$$\dot{\lambda}_1 = -4(i\lambda_{13} - \lambda_{14}), \tag{3.6}$$

$$\dot{\lambda}_2 = 4(i\lambda_{15} + \lambda_{16}), \tag{3.7}$$

$$\dot{\lambda}_3 = -4(i\lambda_{17} - \lambda_{18}), \tag{3.8}$$

$$\dot{\lambda}_4 = 4(\lambda_{19} + i\lambda_{20}), \tag{3.9}$$

$$\dot{\lambda}_5 = 2(\lambda_{13} + i\lambda_{14} + \lambda_{15} - i\lambda_{16}), \tag{3.10}$$

$$\dot{\lambda}_6 = 2(\lambda_{17} + i\lambda_{18} - i\lambda_{19} + \lambda_{20}), \tag{3.11}$$

$$\dot{\lambda}_7 = -2\omega^2(\lambda_{13} + i\lambda_{14} + \lambda_{15} - i\lambda_{16}), \tag{3.12}$$

$$\dot{\lambda}_8 = -2\omega^2(\lambda_{17} + i\lambda_{18} - i\lambda_{19} + \lambda_{20}), \tag{3.13}$$

$$\dot{\lambda}_9 = -4\omega^2(\lambda_{14} - i\lambda_{15}), \tag{3.14}$$

$$\dot{\lambda}_{10} = -4\omega^2(i\lambda_{13} + \lambda_{16}), \tag{3.15}$$

$$\dot{\lambda}_{11} = -4\omega^2(\lambda_{18} - i\lambda_{20}), \tag{3.16}$$

$$\dot{\lambda}_{12} = -4\omega^2(i\lambda_{17} + i\lambda_{19}), \tag{3.17}$$

$$\dot{\lambda}_{13} = -2\omega^2(i\lambda_1 + \lambda_5 - \lambda_7 + i\lambda_{10}), \tag{3.18}$$

$$\dot{\lambda}_{14} = -2\omega^2(\lambda_1 - i\lambda_5 + i\lambda_7 - \lambda_9), \tag{3.19}$$

$$\dot{\lambda}_{15} = 2\omega^2(i\lambda_2 - \lambda_5 + \lambda_7 + i\lambda_9), \tag{3.20}$$

$$\dot{\lambda}_{16} = -2\omega^2(\lambda_2 + i\lambda_5 - i\lambda_7 - \lambda_{10}), \tag{3.21}$$

$$\dot{\lambda}_{17} = -2\omega^2(i\lambda_3 + \lambda_6 - \lambda_8 + i\lambda_{12}), \tag{3.22}$$

$$\dot{\lambda}_{18} = -2\omega^2(\lambda_3 - i\lambda_6 + i\lambda_8 - \lambda_{11}), \tag{3.23}$$

$$\dot{\lambda}_{19} = -2\omega^2(\lambda_4 + i\lambda_6 - i\lambda_8 - \lambda_{12}), \tag{3.24}$$

$$\dot{\lambda}_{20} = 2\omega^2(i\lambda_4 - \lambda_6 + \lambda_8 + i\lambda_{11}). \tag{3.25}$$

In fact, it is difficult to solve these 20 coupled partial differential equations for complex λ's. Thus, here we make certain choices for λ's which facilitate one to find solutions of the above equations.

From eqs (3.10) and (3.12), we get $\omega^2\dot{\lambda}_5 + \dot{\lambda}_7 = 0$, and consider $\dot{\lambda}_5 = 0$, which immediately gives

$$\lambda_5 = c_5 \quad \text{and} \quad \lambda_7 = c_7, \tag{3.26}$$

where c_5 and c_7 are complex integration constants. Again from eqs (3.11) and (3.13), we obtain

$$\lambda_6 = c_6 \quad \text{and} \quad \lambda_8 = c_8. \tag{3.27}$$

Hence both eqs (3.10) and (3.12) become

$$i\lambda_{13} - \lambda_{14} + i\lambda_{15} + \lambda_{16} = 0,$$

and the above equation, after using eqs (3.6) and (3.7), reduces to $\dot{\lambda}_1 = \dot{\lambda}_2$, which on integration gives

$$\lambda_1 = \rho(t) + c_1 \quad \text{and} \quad \lambda_2 = \rho(t) + c_2, \tag{3.28}$$

where $\rho(t)$ is some arbitrary complex function of time and c_1 and c_2 are arbitrary complex constants.

Similarly, from eqs (3.11) and (3.13), after using eqs (3.8) and (3.9), we can find

$$\lambda_3 = \xi(t) + c_3 \quad \text{and} \quad \lambda_4 = \xi(t) + c_4. \tag{3.29}$$

Again, $\xi(t)$ is some arbitrary complex function of time and c_3 and c_4 are arbitrary complex constants.

Now, in order to find solutions for λ_9 and λ_{10}, subtract eqs (3.14) and (3.15) and then using eqs (3.10) or (3.12), we arrive at $\dot{\lambda}_9 = \dot{\lambda}_{10}$, which gives

$$\lambda_9 = \eta(t) + c_9 \quad \text{and} \quad \lambda_{10} = \eta(t) + c_{10}. \tag{3.30}$$

Here $\eta(t)$ is another arbitrary complex function of time and c_9 and c_{10} are complex constants. In the same spirit, subtracting eq. (3.16) from eq. (3.17) and then with the help of eq. (3.11) or (3.12), we get

$$\lambda_{11} = \phi(t) + c_{11} \quad \text{and} \quad \lambda_{12} = \phi(t) + c_{12}, \tag{3.31}$$

where $\phi(t)$ is one more arbitrary function of time and c_{11} and c_{12} are again complex constants.

Now for finding the solutions of λ_{13} and λ_{14}, subtract i times eq. (3.19) from eq. (3.18) and after using eq. (3.30), we get

$$i\dot{\lambda}_{13} + \dot{\lambda}_{14} = 2(2\eta + c_9 + c_{10}). \tag{3.32}$$

On the other hand, time derivative of eq. (3.6) is written as

$$\ddot{\lambda}_1 = 4(-i\lambda_{13} + \lambda_{14}) = \ddot{\rho}. \tag{3.33}$$

Hence using eqs (3.32) and (3.33), one immediately get

$$\lambda_{13} = \frac{i}{8}(\dot{\rho} - 8\sigma) + c_{13}, \tag{3.34}$$

and

$$\lambda_{14} = \frac{1}{8}(\dot{\rho} + 8\sigma) + c_{14}, \tag{3.35}$$

where $\sigma = \int (2\eta(t) + c_9 + c_{10})\mathrm{d}t$.

Similarly, from eqs (3.20) and (3.21) with eq. (3.7), from eqs (3.22) and (3.23) with eq. (3.8), and from eqs (3.24) and (3.25) with eq. (3.9), we obtain solutions for $(\lambda_{15}, \lambda_{16})$, $(\lambda_{17}, \lambda_{18})$ and $(\lambda_{19}, \lambda_{20})$ respectively as

$$\lambda_{15} = -\frac{i}{8}(\dot{\rho} - 8\sigma) + c_{15}, \tag{3.36}$$

$$\lambda_{16} = \frac{1}{8}(\dot{\rho} + 8\sigma) + c_{16}, \tag{3.37}$$

$$\lambda_{17} = \frac{i}{8}(\dot{\xi} - 8\theta) + c_{17}, \tag{3.38}$$

$$\lambda_{18} = \frac{1}{8}(\dot{\xi} + 8\theta) + c_{18}, \tag{3.39}$$

$$\lambda_{19} = \frac{1}{8}(\dot{\xi} + 8\theta) + c_{19}, \tag{3.40}$$

and

$$\lambda_{20} = -\frac{i}{8}(\dot{\xi} - 8\theta) + c_{20}, \tag{3.41}$$

where $\theta = \int(2\xi(t) + c_{11} + c_{12})\mathrm{d}t$.

Thus, we have solved eqs (3.6) to (3.25) in terms of arbitrary functions ρ, ξ, η and ϕ and complex constants, c_i's $(i = 1, ..., 20)$.

If one put back these solutions for λ_i $(i = 1, ..., 20)$ in eqs (3.6)–(3.25), we obtain a number of constraint relations among c_i's, and ρ, ξ, η, ϕ, which limit the choices of these arbitrary complex quantities. These relations are given as

$$\begin{aligned} &ic_{13} = c_{14}; \quad ic_{15} = -c_{16}; \quad ic_{17} = c_{18}, \\ &ic_{20} = -c_{19}; \quad ic_{15} = c_{14}; \quad ic_{13} = -c_{16}, \\ &ic_{20} = c_{18}; \quad ic_{17} = -c_{19}; \quad c_1 - c_2 = 2ic_5, \\ &c_3 - c_4 = 2ic_6; \quad c_9 - c_{10} = 2ic_7; \quad c_{11} - c_{12} = 2ic_8, \end{aligned} \tag{3.42}$$

and the equations determining arbitrary functions ρ, ξ, η and ϕ are written as

$$\begin{aligned} &\ddot{\rho} + 16\omega^2(\rho + c_1 - ic_5) = 0, \\ &\ddot{\xi} + 16\omega^2(\xi + c_3 - ic_6) = 0, \\ &\ddot{\eta} + 8\omega^2(2\eta + c_9 + c_{10}) = 0, \\ &\ddot{\phi} + 8\omega^2(2\phi + c_{11} + c_{12}) = 0. \end{aligned} \tag{3.43}$$

Therefore, after substituting the solutions of λ_i's in eq. (2.11), the final form of the invariant for a two-dimensional complex oscillator becomes

$$\begin{aligned} I = {} & \frac{1}{2}\rho(p_1^2 + x_3^2) + \frac{1}{2}(c_1 p_1^2 + c_2 x_3^2) + \frac{1}{2}\xi(p_2^2 + x_4^2) + \frac{1}{2}(c_3 p_2^2 + c_4 x_4^2) \\ & + \frac{1}{2}\eta(x_1^2 + p_3^2) + \frac{1}{2}(c_9 x_1^2 + c_{10} p_3^2) + \frac{1}{2}\phi(x_2^2 + p_4^2) \\ & + \frac{1}{2}(c_{11} x_2^2 + c_{12} p_4^2) + \frac{i}{8}(\dot{\rho} - 8\sigma)(p_1 p_3 - x_1 x_3) + \frac{1}{8}(\dot{\rho} + 8\sigma) \\ & \times (x_1 p_1 + x_3 p_3) + \frac{i}{8}(\dot{\xi} - 8\theta)(p_2 p_4 - x_2 x_4) + \frac{1}{8}(\dot{\xi} + 8\theta) \\ & \times (p_2 x_2 + p_4 x_4) + c_5 p_1 x_3 + c_6 p_2 x_4 + c_7 p_3 x_1 + c_8 p_4 x_2 + c_{13} p_1 p_3 \\ & + c_{14} p_1 x_1 + c_{15} x_1 x_3 + c_{16} p_3 x_3 + c_{17} p_2 p_4 \\ & + c_{18} p_2 x_2 + c_{19} p_4 x_4 + c_{20} x_2 x_4. \end{aligned} \tag{3.44}$$

which conforms to condition eq. (2.7) in view of the Poisson bracket eq. (2.8).

4. Conclusion

In this work, a modest attempt has been made to obtain exact complex second constant of motion of a two-dimensional complex simple harmonic oscillator on an extended complex phase-space characterized by eq. (2.1). The transformations (1.1) (or eq. (2.1)) had been a part of many studies [9,17] and can give PT symmetric Hamiltonians under certain boundary conditions. Just as the invariants of real Hamiltonian systems have played a vital role in understanding the underlying dynamics of the systems, we hope that the complex invariants could also be helpful in exploring some deep insight into features of complex dynamical systems including the real one. Lie algebraic method is used for the construction of complex invariants, which has been used extensively for the construction of time-dependent invariants of both classical and quantum systems. The degrees of freedom become just double after complexification of a system which make the construction of complex invariants a bit tedious in two or higher dimensions and needs some alternative and systematic approach which could address such complexities.

References

[1] J Hietarinta, *Phys. Rep.* **147**, 87 (1987)
[2] R S Kaushal, *Classical and quantum mechanics of noncentral potentials* (Narosa Publishing House, New Delhi, 1998)
[3] S C Mishra and Fakir Chand, *Pramana – J. Phys.* **66**, 601 (2006)
[4] K J Whiteman, *Rep. Prog. Phys.* **40**, 1033 (1977)
[5] R Rajaraman and E J Weinberg, *Phys. Rev.* **D11**, 2950 (1975)
R Damburg, R Propin and S Graffi, *Int. J. Quant. Chem.* **21**, 191 (1982)
[6] H Goldstien, C Poole and J Safco, *Classical mechanics* (Pearson Edu. Inc., Singapore, 2002)
[7] N Hatano and D R Nelson, *Phys. Rev. Lett.* **77**, 570 (1996); *Phys. Rev.* **B56**, 8651 (1997)
[8] D R Nelson and N M Snerb, *Phys. Rev.* **E58**, 1383 (1998)

[9] A L Xavier Jr. and M A M de Aguiar, *Ann. Phys. (N.Y.)* **252**, 458 (1996); *Phys. Rev. Lett.* **79**, 3323 (1997)
[10] N Moiseyev, *Phys. Rep.* **302**, 211 (1998)
[11] C M Bender and S Boettcher, *Phys. Rev. Lett.* **80**, 5243 (1998)
[12] C M Bender, S Boettcher and P N Meisinger, *J. Math. Phys.* **40**, 2201 (1999)
[13] R S Kaushal, *J. Phys. A: Math. Gen.* **34**, L709 (2001)
[14] R S Kaushal and Parthasarthi, *J. Phys. A: Math. Gen.* **35**, 8743 (2002)
[15] R S Kaushal and Sweta Singh, *Ann. Phys. (N.Y.)* **288**, 253 (2001)
[16] Sweta Singh and R S Kaushal, *Phys. Scr.* **67**, 181 (2003)
[17] N N Rao, B Buti and S B Khadkikar, *Pramana – J. Phys.* **27**, 497 (1986)
[18] R S Kaushal and H J Korch, *Phys. Lett.* **A276**, 47 (2000)
[19] R S Kaushal and S C Mishra, *J. Math. Phys.* **34**, 5843 (1993)