PRAMANA — journal of physics | © Indian Academy of Sciences | Vol. 67, No. 5 November 2006 pp. 803–811

Lepton family symmetries for neutrino masses and mixing

ERNEST MA
Physics Department, University of California, Riverside, CA 92521, USA
E-mail: ma@phyun8.ucr.edu

Abstract. I review some of the recent progress (up to December 2005) in applying non-Abelian discrete symmetries to the family structure of leptons, with particular emphasis on the tribimaximal mixing *ansatz* of Harrison, Perkins and Scott.

Keywords. Neutrino mixing; discrete symmetry.

PACS Nos 14.60.Pq; 11.30.Hv

1. Some basics

Using present data from neutrino oscillations, the 3×3 neutrino mixing matrix is largely determined, together with the two mass-squared differences [1]. In the Standard Model of particle interactions, there are 3 lepton families. The charged-lepton mass matrix linking left-handed (e, μ, τ) to their right-handed counterparts is in general arbitrary, but may always be diagonalized by two unitary transformations:

$$\mathcal{M}_l = U_{\rm L}^l \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} (U_{\rm R}^l)^\dagger. \tag{1}$$

Similarly, the neutrino mass matrix may also be diagonalized by two unitary transformations if it is Dirac:

$$\mathcal{M}_\nu^{\rm D} = U_{\rm L}^\nu \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} (U_{\rm R}^\nu)^\dagger, \tag{2}$$

or by just one unitary transformation if it is Majorana:

$$\mathcal{M}_\nu^{\rm M} = U_{\rm L}^\nu \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} (U_{\rm L}^\nu)^{\rm T}. \tag{3}$$

Notice that the charged leptons have individual names, the neutrinos are only labeled as $1, 2, 3$, waiting to be named. The observed neutrino mixing matrix is a mismatch between $U_{\rm L}^l$ and $U_{\rm L}^\nu$, i.e.

$$U_{l\nu} = (U_{\rm L}^l)^\dagger U_{\rm L}^\nu \simeq \begin{pmatrix} 0.83 & 0.56 & < 0.2 \\ -0.39 & 0.59 & -0.71 \\ -0.39 & 0.59 & 0.71 \end{pmatrix} \simeq \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}. \quad (4)$$

This approximate pattern has been dubbed tribimaximal by Harrison, Perkins, and Scott [2]. Notice that the three vertical columns are evocative of the mesons (η_8, η_1, π^0) in their $SU(3)$ decompositions.

How can the HPS form of $U_{l\nu}$ be derived from a symmetry? The difficulty comes from the fact that any symmetry defined in the basis $(\nu_e, \nu_\mu, \nu_\tau)$ is automatically applicable to (e, μ, τ) in the complete Lagrangian. To do so, usually one assumes the canonical seesaw mechanism and studies the Majorana neutrino mass matrix

$$\mathcal{M}_\nu = -\mathcal{M}_\nu^{\rm D} \mathcal{M}_N^{-1} (\mathcal{M}_\nu^{\rm D})^{\rm T} \quad (5)$$

in the basis where $\mathcal{M}_l$ is diagonal; but the symmetry apparent in $\mathcal{M}_\nu$ is often incompatible with a diagonal $\mathcal{M}_l$ with three very different eigenvalues.

Consider just two families. Suppose we want maximal $\nu_\mu - \nu_\tau$ mixing, then we should have

$$\mathcal{M}_\nu = \begin{pmatrix} a & b \\ b & a \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a+b & 0 \\ 0 & a-b \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \quad (6)$$

This seems to require the exchange symmetry $\nu_\mu \leftrightarrow \nu_\tau$, but since (ν_μ, μ) and (ν_τ, τ) are $SU(2)_{\rm L}$ doublets, we must also have $\mu \leftrightarrow \tau$ exchange. Nevertheless, we still have the option of assigning μ^c and τ^c. If $\mu^c \leftrightarrow \tau^c$ exchange is also assumed, then

$$\mathcal{M}_l = \begin{pmatrix} A & B \\ B & A \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} A+B & 0 \\ 0 & A-B \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}. \quad (7)$$

Hence $U_{l\nu} = (U_{\rm L}^l)^\dagger U_{\rm L}^\nu = 1$ and there is no mixing. If μ^c and τ^c do not transform under this exchange, then

$$\mathcal{M}_l = \begin{pmatrix} A & B \\ A & B \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2(A^2+B^2)} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix}, \quad (8)$$

where $c = A/\sqrt{A^2+B^2}$, $s = B/\sqrt{A^2+B^2}$. Again $U_{l\nu} = (U_{\rm L}^l)^\dagger U_{\rm L}^\nu = 1$.

Obviously a more sophisticated approach is needed. To that end, I will list some non-Abelian discrete symmetries based on geometric solids, in anticipation that some of them will be useful for realizing the HPS *ansatz*. I then focus on the tetrahedral group A_4 and show how the charged-lepton and neutrino mass matrices may be constrained, followed by a catalog of recent models, with one detailed example. I will also discuss the symmetry S_4 with another example and mention briefly the symmetry B_4. These examples show how exact and approximate tribimaximal mixing may be obtained.

2. Some discrete symmetries

The five perfect geometric solids were known to the ancient Greeks. In order to match them to the four elements (fire, air, earth, and water) already known, Plato invented a fifth (quintessence) as that which pervades the cosmos and presumably holds it together. Since a cube (hexahedron) may be embedded inside an octahedron and vice versa, the two must have the same group structure and are thus dual to each other. The same holds for the icosahedron and dodecahedron. The tetrahedron is self-dual. Compare this first theory of everything to today's contender, i.e. string theory. (A) There are five consistent string theories in 10 dimensions. (B) Type I is dual to heterotic $SO(32)$, Type IIA is dual to heterotic $E_8 \times E_8$, and Type IIB is self-dual.

Plato inferred the existence of the fifth element (quintessence) from the mismatch of the four known elements with the five perfect geometric solids. In much the same way, Glashow, Iliopoulos, and Maiani inferred the existence of the fourth quark (charm) from the mismatch of the three known quarks (up, down, strange) with the two charged-current doublets $(u, d\cos\theta_{\rm C} + s\sin\theta_{\rm C})$ and $(?, -d\sin\theta_{\rm C} + s\cos\theta_{\rm C})$.

Question: What sequence has ∞, 5, 6, 3, 3, 3,...?

Answer: Perfect geometric solids in 2, 3, 4, 5, 6, 7,... dimensions. In two dimensions, they are the regular polygons. In three dimensions (table 1) and four dimensions (table 2), they are perfect geometric solids. In five dimensions and above, only the first three types of solids continue.

3. Tetrahedral symmetry A_4

For three families, we should look for a group with a $\underline{3}$ representation, the simplest of which is A_4 (table 3), the group of the even permutation of four objects, which is also the symmetry group of the tetrahedron.

Here $\omega = \exp(2\pi i/3) = -(1/2) + i(\sqrt{3}/2)$, and the fundamental multiplication rule of A_4 is

$$\underline{3} \times \underline{3} = \underline{1}(11 + 22 + 33) + \underline{1}'(11 + \omega^2 22 + \omega 33) + \underline{1}''(11 + \omega 22 + \omega^2 33) + \underline{3}(23, 31, 12) + \underline{3}(32, 13, 21). \tag{9}$$

Table 1. Perfect geometric solids in three dimensions.

Solid	Faces	Vertices	Plato	Hindu	Group
Tetrahedron	4	4	Fire	Agni	A_4
Octahedron	8	6	Air	Vayu	S_4
Cube	6	8	Earth	Prithvi	S_4
Icosahedron	20	12	Water	Jal	A_5
Dodecahedron	12	20	Quintessence	Akasha	A_5

Table 2. Perfect geometric solids in four dimensions.

Solid	Composition	Faces	Vertices
4-Simplex	Tetrahedron	5	5
4-Crosspolytope	Tetrahedron	16	8
4-Cube	Cube	8	16
600-Cell	Tetrahedron	600	120
120-Cell	Dodecahedron	120	600
24-Cell	Octahedron	24	24

Table 3. Character table of A_4.

Class	n	h	χ_1	$\chi_{1'}$	$\chi_{1''}$	χ_3
C_1	1	1	1	1	1	3
C_2	4	3	1	ω	ω^2	0
C_3	4	3	1	ω^2	ω	0
C_4	3	2	1	1	1	-1

Note that $\underline{3}\times\underline{3}\times\underline{3} = \underline{1}$ is possible in A_4, i.e. $a_1b_2c_3$ + permutations, and $\underline{2}\times\underline{2}\times\underline{2} = \underline{1}$ is possible in S_3, i.e. $a_1b_1c_1 + a_2b_2c_2$.

Consider $(\nu_i, l_i) \sim \underline{3}$ under A_4, then $\mathcal{M}_\nu$ is of the form

$$\mathcal{M}_\nu = \begin{pmatrix} a+b+c & f & e \\ f & a+b\omega+c\omega^2 & d \\ e & d & a+b\omega^2+c\omega \end{pmatrix}, \tag{10}$$

where a comes from $\underline{1}$, b from $\underline{1}'$, c from $\underline{1}''$, and (d,e,f) from $\underline{3}$. In this basis, $\mathcal{M}_l$ is generally not diagonal, but under A_4, there are two interesting cases:

(I) Let $l_i^c \sim \underline{1}, \underline{1}', \underline{1}''$, then with $(\phi_i^0, \phi_i^-) \sim \underline{3}$,

$$\mathcal{M}_l = \begin{pmatrix} h_1v_1 & h_2v_1 & h_3v_1 \\ h_1v_2 & h_2v_2\omega & h_3v_2\omega^2 \\ h_1v_3 & h_2v_3\omega^2 & h_3v_3\omega \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \sqrt{3}v, \tag{11}$$

for $v_1 = v_2 = v_3 = v$.

(II) Let $l_i^c \sim \underline{3}$, but $(\phi_i^0, \phi_i^-) \sim \underline{1}, \underline{1}', \underline{1}''$, then $\mathcal{M}_l$ is diagonal with

$$\begin{pmatrix} m_e \\ m_\mu \\ m_\tau \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} h_1v_1 \\ h_2v_2 \\ h_3v_3 \end{pmatrix}. \tag{12}$$

In either case, it solves the fundamental theoretical problem of having a symmetry for the neutrino mass matrix even though the charged-lepton mass matrix has three totally different eigenvalues.

To proceed further, the six parameters of $\mathcal{M}_\nu$ must be restricted, from which $U_{l\nu}$ may be obtained:

$$U_{\rm L}^\dagger \mathcal{M}_\nu U_{\rm L}^* = \mathcal{M}_\nu^{(e,\mu,\tau)} = U_{l\nu} \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} (U_{l\nu})^{\rm T}, \tag{13}$$

where

$$\text{(I)} \qquad U_{\rm L} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},$$

$$\text{(II)} \qquad U_{\rm L} = 1. \tag{14}$$

4. Neutrino mass models

Using (I), the first two proposed A_4 models start with only $a \neq 0$, yielding 3 degenerate neutrino masses. In Ma and Rajasekaran [3], the degeneracy is broken softly by $N_i N_j$ terms, allowing b, c, d, e, f to be nonzero. In Babu, Ma and Valle [4], the degeneracy is broken radiatively through flavor-changing supersymmetric scalar lepton mass terms. In both cases, $\theta_{23} \simeq \pi/4$ is predicted. In BMV03, maximal CP-violation in $U_{l\nu}$ is also predicted. Consider the case $b = c$ and $e = f = 0$ [5], then

$$\begin{aligned} \mathcal{M}_\nu^{(e,\mu,\tau)} &= \begin{pmatrix} a + 2d/3 & b - d/3 & b - d/3 \\ b - d/3 & b + 2d/3 & a - d/3 \\ b - d/3 & a - d/3 & b + 2d/3 \end{pmatrix} \\ &= a \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \frac{d}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \\ &= U_{l\nu} \begin{pmatrix} a - b + d & 0 & 0 \\ 0 & a + 2b & 0 \\ 0 & 0 & -a + b + d \end{pmatrix} (U_{l\nu})^{\rm T}, \end{aligned} \tag{15}$$

where

$$U_{l\nu} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}, \tag{16}$$

i.e. tribimaximal mixing would be achieved. However, although $b \neq c$ would allow $U_{e3} \neq 0$, the assumption $e = f = 0$ and the bound $|U_{e3}| < 0.16$ together imply $0.5 < \tan^2\theta_{12} < 0.52$, whereas experimentally, $\tan^2\theta_{12} = 0.45 \pm 0.05$.

Other models based on (I) with $d \neq 0$ and $e = f = 0$ include AF05-1/2 [6,7] with $b = c = 0$; M05-1 [8] with $a = 0$ and assuming $b = c$; BH05 [9] with $b = c$ and $d^2 = 3b(b - a)$; Z05 [10] assuming $b = c$, and M05-5 [11] with $b = c$. They are summarized in table 4.

Table 4. Particle content of models based on (I).

A_4(I)	ϕ^+, ϕ^0	N	ξ^{++}, ξ^+, ξ^0	χ^0	SUSY
MR01	1,3	3	–	–	No
BMV03	1,1	3	–	3	Yes
M04	3	–	$1, 1', 1'', 3$	–	No
AF05-1	1,1	–	–	1	1,3,3
M05-1	3	–	$1', 1'', 3$	–	No
BH05	1,1	3	–	1,1,3,3	Yes
Z05	3	–	–	–	No
M05-5	1,1	3	–	3	Yes
AF05-2	1,1	–	–	1,1,1,3,3,3,3	Yes

Consider as an example M05-5 [11]. Here

$$\mathcal{M}_\nu^{\rm D} = U_{\rm L}^\dagger \begin{pmatrix} m_{\rm D} & 0 & 0 \\ 0 & m_{\rm D} & 0 \\ 0 & 0 & m_{\rm D} \end{pmatrix}, \quad \mathcal{M}_N = \begin{pmatrix} A & 0 & 0 \\ 0 & B & C \\ 0 & C & B \end{pmatrix} \tag{17}$$

imply $e = f = 0$ and $b = c$. To obtain this $\mathcal{M}_N$, consider the superpotential

$$W = \frac{1}{2} m_N (N_1^2 + N_2^2 + N_3^2) + f N_1 N_2 N_3 + \frac{\lambda_1}{4M_{\rm Pl}} (N_1^4 + N_2^4 + N_3^4) + \frac{\lambda_2}{2M_{{\rm P}l}} (N_2^2 N_3^2 + N_3^2 N_1^2 + N_1^2 N_2^2), \tag{18}$$

and its resulting scalar potential

$$V = \left| m_N N_1 + f N_2 N_3 + \frac{\lambda_1}{M_{\rm Pl}} N_1^3 + \frac{\lambda_2}{M_{\rm Pl}} N_1 (N_2^2 + N_3^2) \right|^2 + \left| m_N N_2 + f N_3 N_1 + \frac{\lambda_1}{M_{\rm Pl}} N_2^3 + \frac{\lambda_2}{M_{\rm Pl}} N_2 (N_3^2 + N_1^2) \right|^2 + \left| m_N N_3 + f N_1 N_2 + \frac{\lambda_1}{M_{\rm Pl}} N_3^3 + \frac{\lambda_2}{M_{\rm Pl}} N_3 (N_1^2 + N_2^2) \right|^2 . \tag{19}$$

The usual solution of $V = 0$ is $\langle N_{1,2,3} \rangle = 0$, but the following is also possible:

$$\langle N_{2,3} \rangle = 0, \quad \langle N_1 \rangle^2 = \frac{-m_N M_{\rm Pl}}{\lambda_1}, \tag{20}$$

yielding the above form of $\mathcal{M}_N$ with

$$A = -2m_N, \quad B = (1 - \lambda_2/\lambda_1) m_N, \quad C = f \langle N_1 \rangle. \tag{21}$$

The soft term $-h\langle N_1 \rangle (\nu_1 \phi^0 - l_1 \phi^+)$ must also be added to allow (ν_1, l_1) and (ϕ^+, ϕ^0) to remain massless at the see-saw scale. The resulting theory is then protected below the see-saw scale by the usual R-parity of a supersymmetric theory. Thus A_4

allows tribimaximal neutrino mixing to be generated automatically from the $N_{1,2,3}$ superfields themselves. However, the neutrino mass eigenvalues are not predicted.

Consider next the assignments of case (II). Here $\mathcal{M}_\nu^{(e,\mu,\tau)} = \mathcal{M}_\nu$ already. Let $d = e = f$, then

$$\mathcal{M}_\nu = \begin{pmatrix} a+b+c & d & d \\ d & a+b\omega+c\omega^2 & d \\ d & d & a+b\omega^2+c\omega \end{pmatrix}. \tag{22}$$

Assume $b = c$ and rotate to the basis $[\nu_e, (\nu_\mu+\nu_\tau)/\sqrt{2}, (-\nu_\mu+\nu_\tau)/\sqrt{2}]$, then

$$\mathcal{M}_\nu = \begin{pmatrix} a+2b & \sqrt{2}d & 0 \\ \sqrt{2}d & a-b+d & 0 \\ 0 & 0 & a-b-d \end{pmatrix}, \tag{23}$$

i.e. maximal $\nu_\mu - \nu_\tau$ mixing occurs and $U_{e3} = 0$. The solar mixing angle is now given by $\tan 2\theta_{12} = 2\sqrt{2}d/(d-3b)$. For $b \ll d$, $\tan 2\theta_{12} \to 2\sqrt{2}$, i.e. $\tan^2\theta_{12} \to 1/2$, but $\Delta m^2_{\rm sol} \ll \Delta m^2_{\rm atm}$ implies $2a+b+d \to 0$, so that $\Delta m^2_{\rm atm} \to 6bd \to 0$ as well. Therefore, $b \neq 0$ is required, and $\tan^2\theta_{12} \neq 1/2$, but should be close to it, because $b = 0$ enhances the symmetry of $\mathcal{M}_\nu$ from Z_2 to S_3. Here $\tan^2\theta_{12} < 1/2$ implies inverted ordering and $\tan^2\theta_{12} > 1/2$ implies normal ordering.

Models based on (II) include CFM05 [12] with $3b = -ef/d - \omega^2 fd/e - \omega de/f$ and $3c = -ef/d - \omega fd/e - \omega^2 de/f$; M05-2 [13] with two complicated equalities; HMVV05 [14] with $d = e = f$ and assuming $b = c$; and M05-3 [15] with $b = c$, $e = f$, and $(a+2b)d^2 = (a-b)e^2$. They are summarized in table 5.

5. Permutation symmetry S_4

In the above application of A_4, approximate tribimaximal mixing involves the *ad hoc* assumption $b = c$. This problem is overcome by using S_4 in a supersymmetric seesaw model proposed recently [16], yielding the result

$$\mathcal{M}_\nu = \begin{pmatrix} a+2b & e & e \\ e & a-b & d \\ e & d & a-b \end{pmatrix}. \tag{24}$$

Here $b = 0$ and $d = e$ are related limits. The permutation group of four objects is S_4 (table 6). It contains both S_3 and A_4. It is also the symmetry group of the hexahedron (cube) and the octahedron.

The fundamental multiplication rules are:

$$\begin{aligned} \underline{3} \times \underline{3} = {} & \underline{1}(11+22+33) + \underline{2}(11+\omega^2 22+\omega 33, 11+\omega 22+\omega^2 33) \\ & +\underline{3}(23+32, 31+13, 12+21) \\ & +\underline{3}'(23-32, 31-13, 12-21), \end{aligned} \tag{25}$$

$$\underline{3}' \times \underline{3}' = \underline{1} + \underline{2} + \underline{3}_S + \underline{3}'_A, \tag{26}$$

$$\underline{3} \times \underline{3}' = \underline{1}' + \underline{2} + \underline{3}'_S + \underline{3}_A. \tag{27}$$

Table 5. Particle content of models based on (II).

A_4 (II)	ϕ^+, ϕ^0	N	ξ^{++}, ξ^+, ξ^0	χ^0
CFM05	$1, 1', 1''$	3	1	3
M05-2	1	3	3	$1, 1', 1'', 3$
HMVV05	$1, 1', 1''$	–	$1, 1', 1'', 3$	–
M05-3	$1, 1', 1'', 3$	$1, 1', 1''$	–	–

Table 6. Character table of S_4.

Class	n	h	χ_1	$\chi_{1'}$	χ_2	χ_3	$\chi_{3'}$
C_1	1	1	1	1	2	3	3
C_2	3	2	1	1	2	−1	−1
C_3	8	3	1	1	−1	0	0
C_4	6	4	1	−1	0	−1	1
C_5	6	2	1	−1	0	1	−1

Note that both $\underline{3}\times\underline{3}\times\underline{3} = \underline{1}$ and $\underline{2}\times\underline{2}\times\underline{2} = \underline{1}$ are possible in S_4. Let $(\nu_i, l_i), l_i^c, N_i \sim \underline{3}$ under S_4. Assume singlet superfields $\sigma_{1,2,3} \sim \underline{3}$ and $\zeta_{1,2} \sim \underline{2}$, then

$$\mathcal{M}_N = \begin{pmatrix} M_1 & h\langle\sigma_3\rangle & h\langle\sigma_2\rangle \\ h\langle\sigma_3\rangle & M_2 & h\langle\sigma_1\rangle \\ h\langle\sigma_2\rangle & h\langle\sigma_1\rangle & M_3 \end{pmatrix}, \tag{28}$$

where $M_1 = A + f(\langle\zeta_2\rangle + \langle\zeta_1\rangle)$, $M_2 = A + f(\langle\zeta_2\rangle\omega + \langle\zeta_1\rangle\omega^2)$, and $M_3 = A + f(\langle\zeta_2\rangle\omega^2 + \langle\zeta_1\rangle\omega)$. The most general S_4-invariant superpotential of σ and ζ is given by

$$\begin{aligned} W = {} & M(\sigma_1\sigma_1 + \sigma_2\sigma_2 + \sigma_3\sigma_3) + \lambda\sigma_1\sigma_2\sigma_3 + m\zeta_1\zeta_2 + \rho(\zeta_1\zeta_1\zeta_1 + \zeta_2\zeta_2\zeta_2) \\ & +\kappa[(\sigma_1\sigma_1 + \sigma_2\sigma_2\omega + \sigma_3\sigma_3\omega^2)\zeta_2 + (\sigma_1\sigma_1 + \sigma_2\sigma_2\omega^2 + \sigma_3\sigma_3\omega)\zeta_1]. \end{aligned} \tag{29}$$

The resulting scalar potential has a minimum at $V = 0$ (thus preserving supersymmetry) only if $\langle\zeta_1\rangle = \langle\zeta_2\rangle$ and $\langle\sigma_2\rangle = \langle\sigma_3\rangle$, so that

$$\mathcal{M}_N = \begin{pmatrix} A+2B & E & E \\ E & A-B & D \\ E & D & A-B \end{pmatrix}. \tag{30}$$

To obtain a diagonal $\mathcal{M}_l$, choose $\phi^l_{1,2,3} \sim \underline{1}+\underline{2}$. To obtain a Dirac $\mathcal{M}_{\nu N}$ proportional to the identity, choose $\phi^N_{1,2,3} \sim \underline{1}+\underline{2}$ as well, but with zero vacuum expectation value for $\phi^N_{2,3}$. This allows $\mathcal{M}_\nu$ to have the form of Eq. (24), and thus approximate tribimaximal mixing.

6. B_4

Exact tribimaximal mixing has also been obtained by Grimus and Lavoura [17] using the Coxeter group B_4, which is the symmetry group of the hyperoctahedron

(4-crosspolytope) with 384 elements. Here $\mathcal{M}_l$ is diagonal with (ν_i, l_i), l_i^c, and (ϕ_i^+, ϕ_i^0) belonging to three different 3-dimensional representations of B_4 with the property $a_1b_1c_1 + a_2b_2c_2 + a_3b_3c_3 = 1$. The $\mathcal{M}_N$ of eq. (30) is then reduced by $D = E + 3B$.

7. Some remarks

With the application of the non-Abelian discrete symmetry A_4, a plausible theoretical understanding of the HPS form of the neutrino mixing matrix has been achieved, i.e. $\tan^2\theta_{23} = 1$, $\tan^2\theta_{12} = 1/2$, $\tan^2\theta_{13} = 0$.

Another possibility is that $\tan^2\theta_{12}$ is not 1/2, but close to it. This has theoretical support in an alternative version of A_4, but is much more natural in S_4.

In the future, this approach to lepton family symmetry should be extended to include quarks, perhaps together in a consistent overall theory.

Acknowledgement

I thank Pankaj Agrawal and the other organizers of WHEPP-9 for their great hospitality and a stimulating workshop. This work was supported in part by the US Department of Energy under Grant No. DE-FG03-94ER40837.

References

[1] See for example S Goswami, in *Lepton Photon Symposium* (Uppsala, Sweden, 2005) (to be published)
[2] P F Harrison, D H Perkins and W G Scott, *Phys. Lett.* **B530**, 167 (2002)
See also X-G He and A Zee, *Phys. Lett.* **B560**, 87 (2003)
[3] E Ma and G Rajasekaran, *Phys. Rev.* **D64**, 113012 (2001)
[4] K S Babu, E Ma and J W F Valle, *Phys. Lett.* **B552**, 207 (2003)
[5] E Ma, *Phys. Rev.* **D70**, 031901(R) (2004)
[6] G Altarelli and F Feruglio, *Nucl. Phys.* **B720**, 64 (2005)
[7] G Altarelli and F Feruglio, *Nucl. Phys.* **B741**, 215 (2006)
[8] E Ma, *Phys. Rev.* **D72**, 037301 (2005)
[9] K S Babu and X-G He, hep-ph/0507217
[10] A Zee, *Phys. Lett.* **B630**, 58 (2005)
[11] E Ma, *Phys. Rev.* **D73**, 057304 (2006)
[12] S-L Chen, M Frigerio and E Ma, *Nucl. Phys.* **B724**, 423 (2005)
[13] E Ma, *Mod. Phys. Lett.* **A20**, 2767 (2005)
[14] M Hirsch, E Ma, A Villanova del Moral and J W F Valle, *Phys. Rev.* **D72**, 091301(R) (2005); Erratum: *Phys. Rev.* **D72**, 119904 (2005)
[15] E Ma, *Mod. Phys. Lett.* **A20**, 2601 (2005)
[16] E Ma, *Phys. Lett.* **B632**, 352 (2006)
[17] W Grimus and L Lavoura, *J. High Energy Phys.* **0601**, 018 (2006)