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Breathers in ferrimagnetic systems

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Abstract. Breathers in discrete nonlinear ferrimagnetic spin lattices are investigated for both easy-axis and easy-plane configurations. The region in frequency space of the formation of breathers is determined and the anticontinuum limit discussed. The monochromatic and the coloured breathers are found out numerically for different parameters and different conditions of excitations.

Keywords. Breathers; ferrimagnetic spin lattice; anticontinuum limit; bifurcation; localisation; easy-axis; easy-plane.

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1. Introduction

Breathers, time-periodic localised excitations in discrete nonlinear lattices, are objects of widespread interest. They are ubiquitous in almost every area of condensed matter physics extending from lattice dynamics to magnetically-ordered systems, in nonlinear optics and other areas of physics [1–4] as well. The experimental realization of these breathers has motivated further search into these intrinsic localised modes [2–4]. The existence of breathers has been demonstrated both analytically and numerically for a system of interacting nonlinear oscillators by starting from anticontinuum limit [1,5–7]. Exact breather solutions have been obtained for piecewise linear models by identifying breathers with homoclinic orbits and studying the intersections of the stable and unstable manifold of the hyperbolic fixed points [8].

Breathers are also believed to be interesting entities with regard to localisation in magnetic systems. There prevail in the literature excellent reviews about localised excitations including solitary waves in magnetic systems (also called magnetic solitons) with both ferromagnetic and antiferromagnetic orderings [9,10]. Intrinsic self-localised modes have been obtained at the classical as well as quantum mechanical levels for both ferromagnetic and antiferromagnetic orderings [11,12]. Zolotaryuk *et al* [13] investigated the problem of discrete breathers in ferromagnetic systems for both easy-axis and easy-plane configurations. Starting from the anticontinuum (AC) limit they deduced the conditions for the existence of breathers and studied both analytically and numerically the profiles of breathers, their stability and the effect of dimension on the breathers. Lai and Sievers [9] also studied in detail the existence and stability of intrinsic localised modes of both even and odd parities for ferromagnetic and antiferromagnetic systems in the presence or absence of an external magnetic field.

Though the most natural extension of ferromagnetic and antiferromagnetic systems would be ferrimagnets, an analysis of breathers in ferrimagnets is lacking to the best of our knowledge. A systematic study of the existence of localised modes, their stability [16] and mobility in ferrimagnets is important in view of the energy localisation and transport of spin waves in ferrimagnetically ordered systems. To this end we have addressed ourselves to a study of discrete breathers in classical two-sublattice ferromagnetic systems, the spins being alternately 1 and S in magnitude (in suitable units) and oppositely directed in two sublattices respectively. That breathers exist in the system discussed here, points to the localisation of the energy of nonlinear spin waves. Though the question of movability of these breathers has not been addressed in this work, it would be an interesting exercise to extend this analysis to include the possibility of the mobile breathers.

The paper is organized as follows. Section 2 discusses the dispersion relation of spin waves in an easy-axis ferrimagnetic system and the region in frequency space relevant to the formation of breathers. Section 2.2 deals with the profiles of breathers under different conditions of excitation. In §3 easy-plane breathers are discussed. Section 4 deals with the energetics involved in the discrete breathers of ferrimagnetic systems. The paper ends with a discussion of the essential results and the possible extension.

2. Dispersion relation for easy-axis ferrimagnet

The Hamiltonian of a system of interacting classical spins with nearest-neighbour coupling is given by $H = -\frac{1}{2}\sum_{n\neq n'}\sum_{\alpha(x,y,z)} J_\alpha^{nn'} S_n^\alpha S_{n'}^\alpha - D\sum_n S_n^{z2}$, where J_x, J_y, $J_z < 0$ for ferrimagnets, $D > 0$ for easy-axis configuration and in the ground state the spins are aligned alternately parallel and antiparallel to the Z-axis. The components of J are assumed to be site-independent. Assuming nearest-neighbour interaction, the Landau–Lifshitz equations governing the evolution of classical spins are given by [13]

$$\frac{\mathrm{d}S_n^x}{\mathrm{d}t} = J_y S_n^z(S_{n-1}^y + S_{n+1}^y) - J_z S_n^y(S_{n-1}^z + S_{n+1}^z) - 2DS_n^y S_n^z, \tag{1}$$

$$\frac{\mathrm{d}S_n^y}{\mathrm{d}t} = J_z S_n^x(S_{n-1}^z + S_{n+1}^z) - J_x S_n^z(S_{n-1}^x + S_{n+1}^x) - 2DS_n^x S_n^z, \tag{2}$$

$$\frac{\mathrm{d}S_n^z}{\mathrm{d}t} = J_x S_n^y(S_{n-1}^x + S_{n+1}^x) - J_y S_n^x(S_{n-1}^y + S_{n+1}^y). \tag{3}$$

(The spins are normalized in terms of a characteristic spin.) For planar isotropy (i.e., $J_x = J_y$), we obtain assuming a two-sublattice model with plane-wave perturbations about the ground state,

$$\omega^2 + 2\omega(J_z + D)(S^z_{2p} + S^z_{2p+1}) + 4(J_z S^z_{2p} + D S^z_{2p+1})(J_z S^z_{2p+1} + D S^z_{2p}) \\ -4J^2 S^z_{2p} S^z_{2p+1} \cos^2 q = 0, \quad (4)$$

where S^z_{2p} and S^z_{2p+1} refer to the z-components of spin at the even and odd sites respectively. The dispersion relation gives

$$\omega = \omega_1 \pm [\omega_2 - \omega_3]^{1/2}, \quad (5)$$

where $\omega_1 = (|J_z| - D)(S^z_{2p} + S^z_{2p+1})$, $\omega_2 = (|J_z| - D)^2(S^z_{2p} + S^z_{2p+1})^2$ and $\omega_3 = 4[|J_z|S^z_{2p} - DS^z_{2p+1})(|J_z|S^z_{2p+1} - DS^z_{2p}) - J^2 S^z_{2p} S^z_{2p+1} \cos^2 q]$. The linear dispersion relation of the spin waves is obtained by putting S^z_{2p} and S^z_{2p+1} equal to 1 and $-S$ respectively or vice versa. The nonlinear dispersion relation is obtained for nonlinear plane-wave excitations by replacing S^z_{2p} and S^z_{2p+1} with $\sqrt{1 - S^2}$ and $-\sqrt{S^2 - B^2}$ or vice versa, where A and B represent the amplitudes of spin waves excited at the adjacent sites. For either branch of the dispersion relation, the magnitudes of the frequencies of nonlinear plane waves lie lower than the minima of the linear dispersion relation obtained at $q = 0$ and $q = \pi$ for positive branch and $q = \pi/2$ for negative branch respectively. Furthermore, the difference increases with the increase of the amplitude of excitation showing that the nonlinear spectrum gets continuously repelled from the linear one with increasing nonlinearity (figures 1 and 2). Thus the nonlinear excitations can bifurcate from the lower band edge to form localised excitations [13,14]. For the present system there is no possibility of the formation of nonlinear localised excitations by bifurcation from the upper band edge because of their becoming resonant with the linear modes in that case.

2.1 *Anticontinuum limit (AC limit) and its significance in respect of finding out breather solutions*

The anticontinuum limit [1,5–7,13] (or sometimes called the antiintegrable limit) is found out by putting $J_x = J_y = 0$ or $J_x = J_y = J_z = 0$ in eqs (1)–(3), which then become integrable, and afterwards exciting only a limited number of spins about the ground state. The solution in this limit is, $S^z_n =$ constant and $S^x_n + iS^y_n = S^+_n = A_n \exp i(\omega_n t + \phi_n)$, where ω_n and ϕ_n are the frequency and phase respectively of the spin excited at the nth site. The magnitude of ω_n can be seen to be equal to $|J_z(S^z_{n-1} + S^z_{n+1}) + 2DS^z_n|$, where $|S^z_n| = \sqrt{1 - A^2}$ or $= \sqrt{S^2 - B^2}$ depending on whether the spin of the excited state is 1 or S, A and B being the respective amplitudes of excitation at the sites. Provided, certain nonresonance and anharmonicity conditions are satisfied (explained below), the solutions can be continued to the range of small but finite values of J_x, J_y and J_z leading to breather solutions with appropriate frequencies.

Let us write the equations as (following Zolotaryuk *et al* [13]),

$$J_y S^z_n(S^y_{n-1} + S^y_{n+1}) - J_z S^y_n(S^z_{n-1} + S^z_{n+1}) - 2DS^y_n S^z_n - \dot{S}^x_n = 0, \quad (6)$$

$$J_z S^x_n(S^z_{n-1} + S^z_{n+1}) - J_x S^z_n(S^x_{n-1} + S^x_{n+1}) + 2DS^x_n S^z_n - \dot{S}^y_n = 0, \quad (7)$$

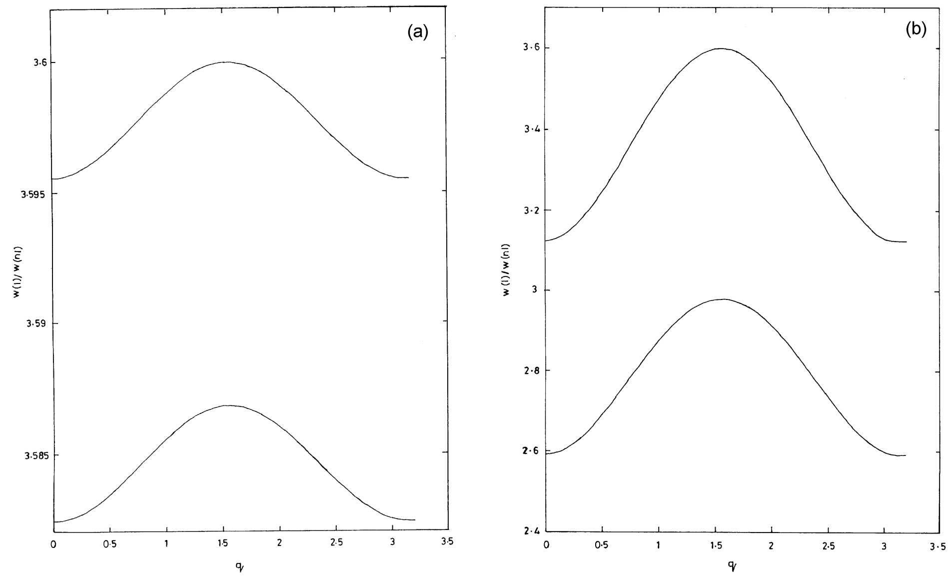

Figure 1. (**a**) Linear and nonlinear dispersion curves for various values of q for an easy-axis ferrimagnetic system ($J_z = -1.0$, $J_x = J_y = -0.1$, $D = 1.0$, $S = 0.8$). The nonlinear dispersion curve has been drawn for $\delta_1 = 0.1$ and $\delta_2 = 0.05$, where δ_1 and δ_2 refer to the amplitudes of nonlinear excitations at the even and the odd sites respectively. Here δ_1 and δ_2 refer to 'Del 1' and 'Del 2' respectively in the graphs shown in figures 2 and 3. The plot shows that the nonlinear dispersion curve (the lower curve) lies below the linear one (upper one). $w(\mathrm{l})$ and $w(\mathrm{nl})$ refer to the linear and nonlinear dispersions respectively. Here w refers to ω in the text. (**b**) Linear and nonlinear dispersion curves similar to figure 1a for isotropic case ($J_z = J_x = J_y = -1.0, D = 1.0, S = 0.8, \delta_1 = 0.5, \delta_2 = 0.5$). The plot shows that the nonlinear dispersion curve (the lower curve) lies below the linear one (the upper one).

$$J_x S_n^y (S_{n-1}^x + S_{n+1}^x) - J_y S_n^x (S_{n-1}^y + S_{n+1}^y) - \dot{S}_n^z = 0. \tag{8}$$

The solution admits of a locally unique continuation by implicit function theorem for sufficiently small J_x, J_y and J_z if the derivatives of the left-hand side expressions with respect to the above parameters are invertible at $J_x = J_y = J_z = 0$. Following Zolotaryuk *et al* [13] the calculations give $k\omega \neq \omega_0$ and $k\omega \neq \omega_1$ for $n \neq 0, \pm 1$ and $n = \pm 1$ respectively, where k is a positive integer, n denotes the lattice site, ω, the frequency at the central site and ω_1, ω_0 denote the frequencies at adjacent sites respectively. For $n = 0$, a similar analysis gives $\omega \neq 0$ for $k \neq 0$ and $D \neq 0$ for $k = 0$. These are the nonresonance conditions required to be satisfied in addition to the condition $k\omega \neq \omega_\mathrm{l}$ where ω_l is the linear frequency and also the condition of anharmonicity of the periodic orbit of the Hamiltonian of the spin system in order that one may extend the solution in the anticontinuum limit to the region of small but finite interaction between the spins. The restrictions on the magnitudes of the parameters that result from these conditions are given in the following for

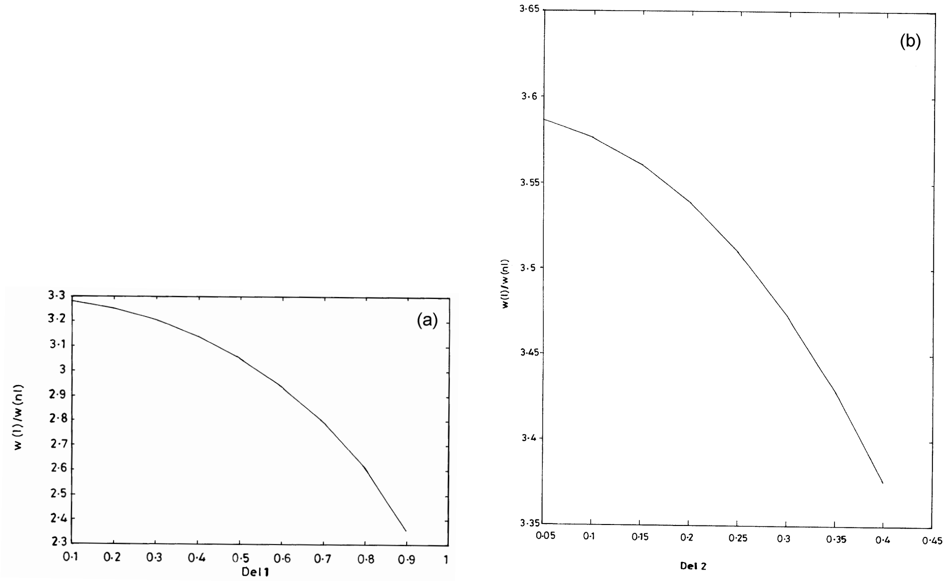

Figure 2. (**a**) Plot of the linear and nonlinear dispersion relations for various magnitudes of δ_1 ($\delta_2 = 0.05$ at $q = 1.57$ of an easy-axis ferrimagnetic system. The nonlinear dispersion curve is seen to get repelled from the linear one with the incrase of the amplitude of excitation and thus to create the possibility of formation of breathers. The parameters chosen are: $J_x = J_y = J_z = -1.0, D = 1.0, S = 0.9$. (**b**) A plot showing the linear and nonlinear dispersion relations for various magnitudes of δ_2. The parameters are $J_z = -1.0, J_x = J_y = -1.0, D = 1.0, \delta_1 = 0.1$. The nonlinear dispersion curve is seen to get repelled from the linear one with the increase of δ_2 and thus to give rise to breathers.

two possible spin configurations in an easy-axis system. A similar set of restriction also follows straightforwardly for an easy-plane system and are not given here for the sake of brevity.

For the configuration $(\cdots -S, 1, S_0, 1, -S, \ldots)$ we obtain $\omega = 2(J_z + DS_0), \omega_1 = J_z(S_0 - S) + 2D$, $\omega_0 = 2(J_z - DS)$. The condition $k|\omega| = |\omega_1|$ gives $|J_z|(2k - |S_0 - S|) = 2D(1 - k|S_0|)$ and hence for this resonance condition we get $|J_z| = 2D(1 - k|S_0|)/[2k - |S_0 - S)|]$. The resonance condition $k|\omega| = |\omega_0|$ gives $k|(J_z + DS_0)| = |J_z| + DS$, or, $|J_z| = D(S - k|S_0|)/[(k-1)]$ for $k \neq 1$ and $|S_0| = S$ for $k = 1$.

For the configuration $(\cdots 1, -S, S_0, -S, 1, \ldots)$ we similarly obtain, $\omega = 2(DS_0 - J_zS)$, $\omega_1 = J_z(1+S_0) - 2DS$, $\omega_0 = 2(D - J_zS)$. The resonance condition $k|\omega| = |\omega_1|$ gives as above, $|J_z| = 2D(S - k|S_0|)/(2kS - 1 - S_0)$. Similarly the resonance condition $k|\omega| = |\omega_0|$ gives $|J_z| = D(1 - k|S_0|)/[S(k-1)]$ for $k \neq 1$ and $|S_0| = 1$ for $k = 1$. These conditions were taken care of in the choice of the parameters J_x, J_y, J_z, D etc. so that the periodic orbit of the original Hamiltonian remains non resonant allowing us to extend the solution in the anticontinuum limit for small but finite values of the coupling parameters. The calculations for the easy-plane

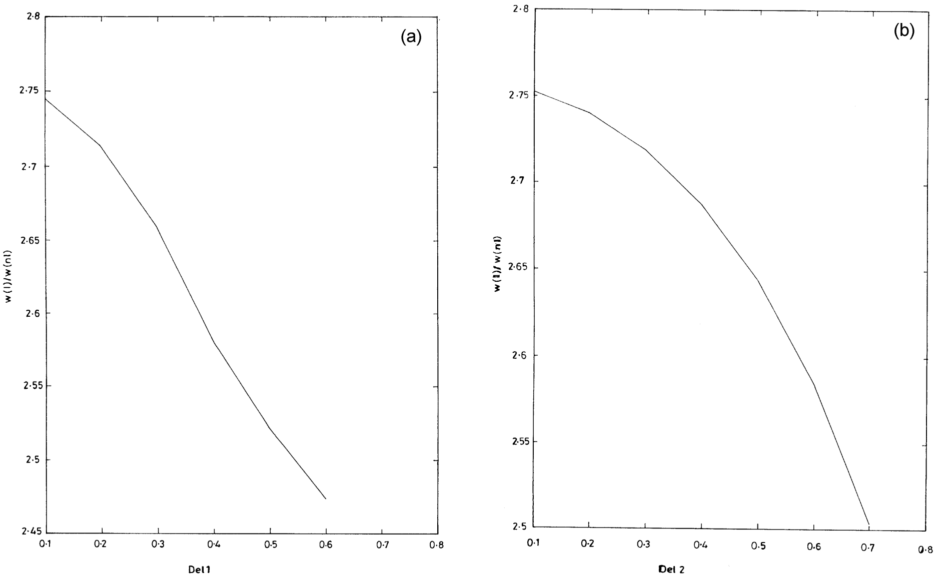

Figure 3. (**a**) The linear and nonlinear dispersion relations for various magnitudes of δ_1 in an easy-plane ferrimagnetic system. The parameters are: $q = 1.57, D = -1.0, J_x = J_y = J_z = -1.0, S = 0.9, \delta_2 = 0.05$. Here also the nonlinear dispersion curve (the lower curve) is seen to get repelled from the linear one (the upper line) with the increasing value of δ_1 and thus to give rise to breathers. (**b**) A similar plot of linear and nonlinear dispersion relations for various magnitudes of δ_2 ($\delta_1 = 0.05$), the other parameters remaining the same as before.

configuration follow similarly leading to the relevant choice of parameters in that case also.

2.2 *Breather solutions in easy-axis ferrimagnets*

For a completely isotropic ($J_x = J_y = J_z$) or planar isotropic ($J_x = J_y \neq J_z$) Heisenberg interaction, eqs (1)–(3) then yield the solution $S_n^z =$ constant and $S_n^x + iS_n^y = A_n \exp(i\omega t)$, where ω is the site-independent frequency of the excited normal mode. We then obtain, assuming symmetry with respect to the nth site, the center of the breather, the difference equation

$$A_n(\omega - 2D\sqrt{1 - A_n^2}) + 2JA_{n+1}\sqrt{1 - A_n^2} + 2J_z A_n\sqrt{S^2 - A_{n+1}^2} = 0, \tag{9}$$

where A_n and A_{n+1} are the amplitudes corresponding to the nth and the $(n+1)$th sites respectively, and the total spin at the nth site is taken to be unity. Knowing A_n we can find out A_{n+1} and continue the scheme to determine the amplitudes for

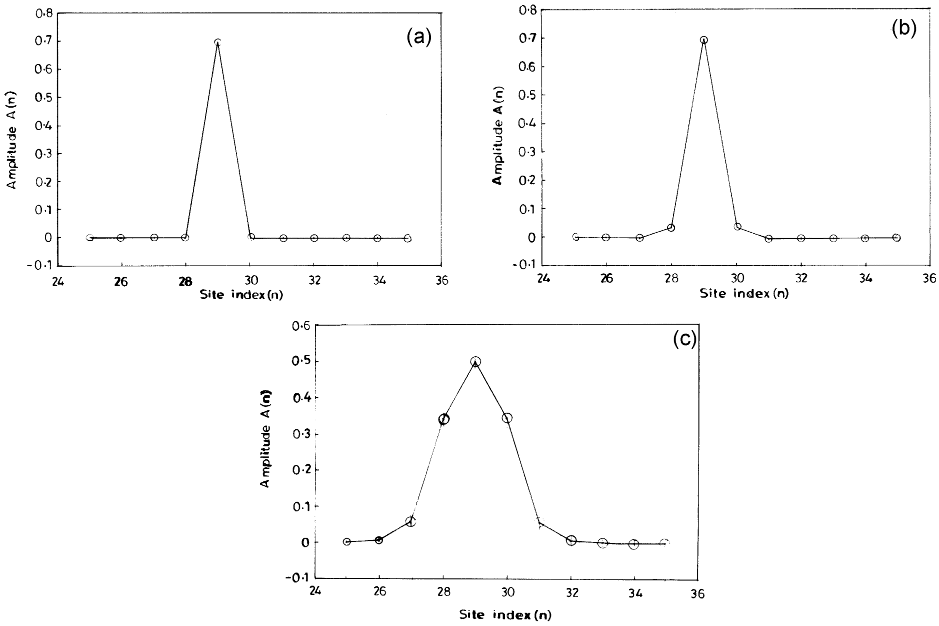

Figure 4. (**a**) Plot of the amplitude of excitation $A(n)$ of monochromatic breathers for various sites n ($J_x = J_y = J_z = 1.0 \times 10^{-04}$, $D = 1.0, S = 0.9$) in an easy-axis ferrimagnetic system. (Initially only one spin has been excited.) (**b**) A similar plot for $J_x = J_y = J_z = 1.0, D = 1.0, S = 0.9$. Localisation is clearly seen to decrease compared to the previous case due to decrease of D/J ratio. (**c**) A similar plot of the amplitude for various sites, where initially three spins have been excited at the three sites, the other parameters remaining the same as before.

the other sites as well. Taking ω to be equal to the value at the anticontinuum limit we have numerically solved eq. (9) (figures 4a, 4b). The three-spin excited monochromatic breather solution is similarly obtained by exciting the nth spin and two adjacent spins lying on either side of the nth one (figure 4c). Assuming symmetry about nth site we can proceed as before to find out the breather profile. From the condition that the frequencies corresponding to the sites of the excited spins must be identical, we get in the anticontinuum limit

$$A_n^2 > \left(1 - \left[\frac{J_z(2S-1) + 2SD}{2D + J_z}\right]^2\right). \tag{10}$$

The coloured breather solutions for general J_x, J_y, J_z are obtained by numerically solving the equations by fourth-order Runge–Kutta method. The breather profile is obtained numerically for initially one-spin excited ferrimagnetic system (figures 5 and 6).

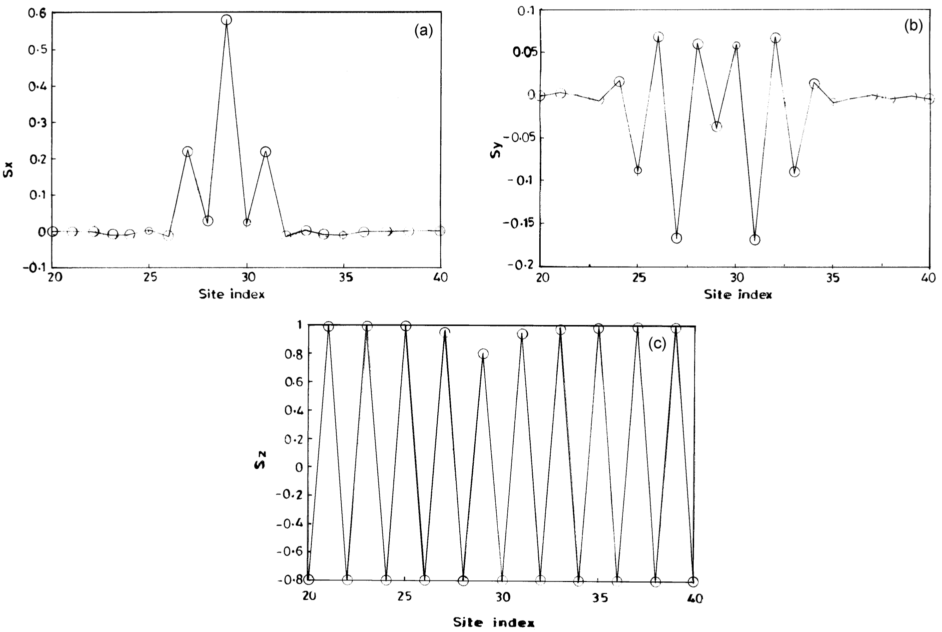

Figure 5. (**a**) Plot of S_x vs. site index n for coloured breathers in an easy-axis ferrimagnetic system obtained by solving the time evolution equation by Runge–Kutta method. The parameters are: $J_x = J_y = J_z = -1.0$, $D = 1.0, S = 0.8$. (**b**) Plot of S_y vs. n for the same choice of parameters. (**c**) Plot of S_z vs. n for the same choice of parameters.

3. Easy-plane ferrimagnetic configuration

For easy-plane ferrimagnets $D < 0$ and the spins are aligned in the XY plane in the ground state with the spins pointing alternately along the positive and the negative X-axis with magnitudes 1 and S respectively. The linear dispersion relation for spin-wave excitation in easy-plane configuration gives in the completely isotropic case

$$\omega^2 = 2(A - B\cos^2 q) \pm 2\sqrt{(F + E\cos^2 q)}, \tag{11}$$

where $A = J^2(1 + S^2) + 2JDS, B = 2J^2S, F = J^4(1 - S^2)^2, E = 4J^2D^2S^2 + 4J^3S(1 - S^2)(D - J)$ and the plane-wave excitation is taken to be of the form $\mathrm{e}^{i(q_n - \omega t)}$. Here unlike in the case of ferromagnets or antiferromagnets [9,13] a gap exists below the minimum of the frequency spectrum. The nonlinear dispersion relation is obtained as before by putting $|S_n^z| = \sqrt{(1 - S^2)}$ or $\sqrt{S^2 - B^2}$, depending on whether the spin of the excited state is 1 or S. For the positive branch (given by the +ve sign of eq. (5)), the frequencies of the non-linearly excited spin waves lie below the minimum of the linear frequency band (as has been seen numerically for various choices of parameters (figures 3a and 3b)) and hence breathers can exist below the lower band edge [14]. For the negative branch, however, the frequencies of nonlinear excitations lie below the maximum of the linear spectrum

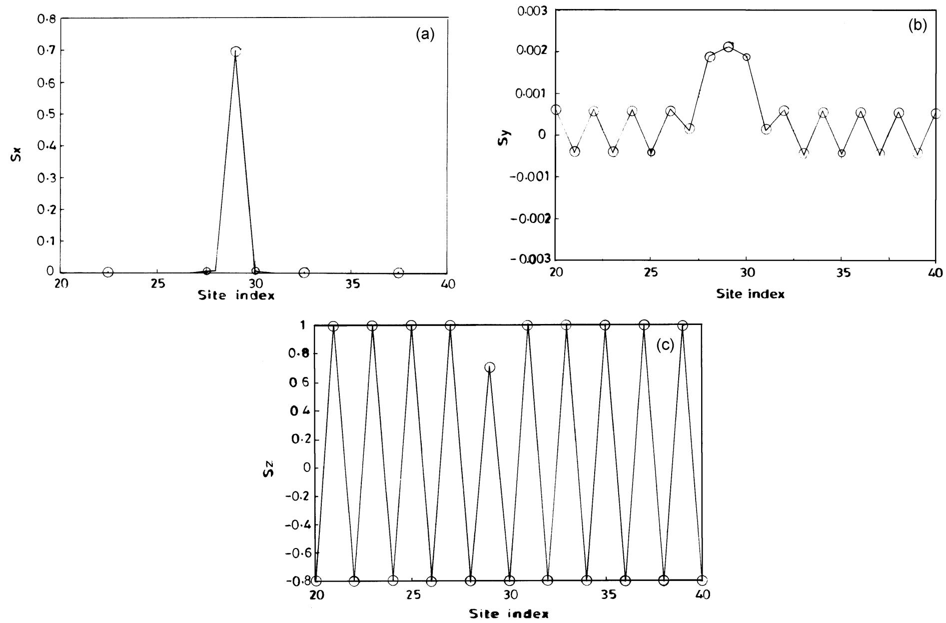

Figure 6. (**a**) Plot of S_x vs. site index n for the parameters: $J_x = J_y = -0.1, J_z = -1.0, D = 1.0, S = 0.8$. (**b**) Plot of S_y vs. n for the same choice of parameters. (**c**) Plot of S_z vs. n for the same choice of parameters.

and hence overlap with it. Thus breathers can exist for the positive branch of the dispersion relation only. Breather profiles are obtained by exciting only a limited number of spins out of the easy plane [9,13], which is the XY-plane in this case and then solving Landau–Lifshitz equations (1)–(3). The solutions are continued from the AC limit which is given by $J_x = J_y = J_z = 0$ in the isotropic case. Provided the frequency of the nonlinearly excited wave lies below the lower band edge and satisfies the nonresonance and the anharmonicity conditions, the solutions go over to spatially localised excitations, i.e., breathers for small J_x, J_y, J_z. The solution is numerically determined and the breather profile shown graphically (figures 7a–c).

4. Energetics

For small intersite coupling the Hamiltonian of the entire lattice can be represented as the sum of the Hamiltonians corresponding to individual lattice points. The Hamiltonian (or energy) contribution from the nth lattice point is written as, $H_n = -[J(S_n^x(S_{n-1}^x+S_{n+1}^x)+S_n^y(S_{n-1}^y+S_{n+1}^y))+J_zS_n^z(S_{n-1}^z+S_{n+1}^z)]-DS_n^{z2}$. The average energy corresponding to the central spin of the excited breather mode (average being calculated over one time period of the breather mode) is seen to be fairly constant when calculated over an interval equal to many time periods (figure 8). This clearly shows the stable nondecaying structure of the breather.

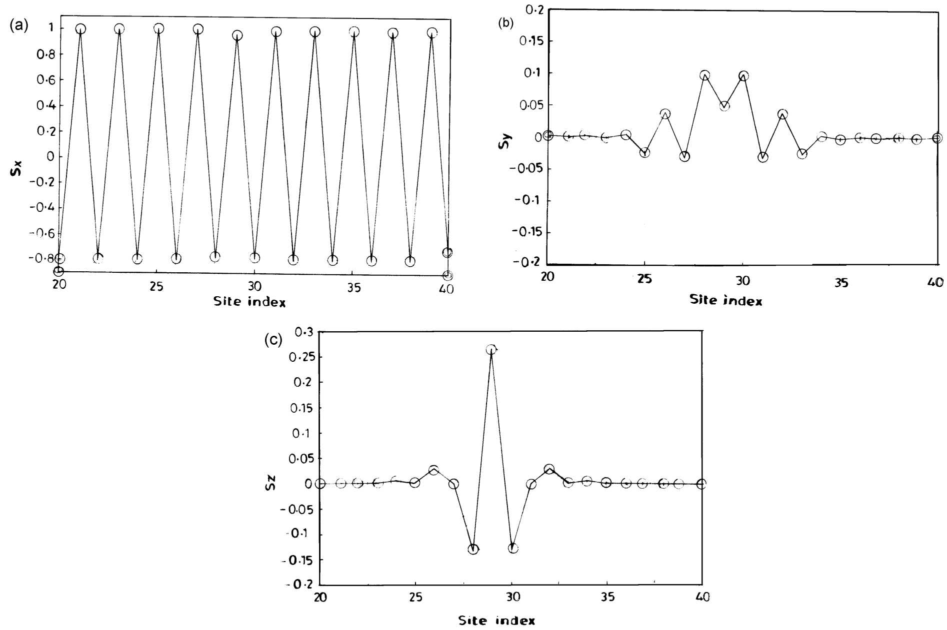

Figure 7. (**a**) Plot of S_x vs. n in an easy-plane ferrimagnetic system. The parameters are: $J_x = J_y = J_z = -0.1, D = -1.0, S = 0.8$. (**b**) Plot of S_y vs. n for the same choice of parameters. (**c**) Plot of S_z vs. n for the same choice of parameters.

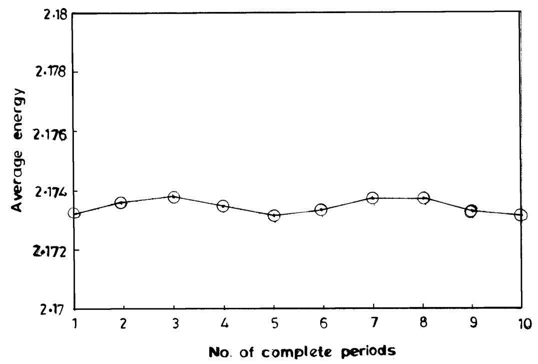

Figure 8. Plot of the average energy vs. number of complete periods of the breather.

It is clear from the expression for the Hamiltonian of the ferrimagnetic system that the energy required for exciting the breather mode in an easy-axis configuration, which is the difference between the energy in the excited state and that in the ground state, is nearly equal to $D[(S_n^{z2})_{\text{ground}} - (S_n^{z2})_{\text{excited}}]$ for small J ($J_x = J_y = J_z = J$), where n corresponds to the site of the central spin. For an easy-axis configuration, where the easy axis is parallel to the Z-axis, the difference is nearly equal to DA_n^2, where A_n is the amplitude of excitation of the nth site. Thus the energy of excitation decreases as the amplitude of excitation decreases. Since the amplitude of the nonlinearly excited breather mode is vanishingly small when it is just formed by bifurcation from the linear band edge, the easy-axis configuration requires no threshold for excitation of the breather mode.

For the easy-plane configuration the frequency corresponding to the lower band edge is given by $\omega = 2\sqrt{JDS}$, while that corresponding to the AC limit is $\omega_n = 2DS_n^z$. In order that breathers may form by bifurcation from the lower band edge, the frequency of the breather mode, which is approximately the same as that corresponding to the AC limit, must be less than that of the lower band edge. This gives us the condition that $S_n^{z2} < JS/D$. Thus, since S_n^z for the ground state is zero here, the energy of excitation of the easy-plane ferrimagnetic breather mode clearly goes to zero at the threshold of excitation. It can be remarked in passing that for dimension ≥ 2 a finite energy threshold for the formation of discrete breathers has been seen to exist [15].

4.1 *Discussions*

The work deals with the problem of excitation of breathers in a ferrimagnetic system. The features worth-mentioning in this connection consist of the following:

(1) For the three-spin excited monochromatic breathers the amplitude of the centre of the breather A_n should satisfy the condition $A_n^2 > (1 - [2S(J_z + D) - J_z/2D + J_z]^2)$ as given by (7). Thus different ferrimagnetic systems modelled by different values of S will require different values of the minimum amplitudes of the central spin for generating three-spin excited monochromatic breathers. For antiferromagnetic systems ($S = 1$) the inequality (7) shows that there is no lower bound for the amplitude. For one-spin excited monochromatic breathers we can similarly deduce from (6), assuming symmetry about the centre of the breather, that $A_n < S_0$, where S_0 is the magnitude of the spin at the site of excitation ($S_0 = 1$ here), and the frequency of the breather is taken to be that in the AC limit.

(2) The easy-plane linear dispersion relation (8) for spin-wave excitation admits of waves with frequencies below the lower linear spin-wave band edge. Thus while the corresponding cases for the ferromagnetic and antiferromagnetic counterparts have acoustic-like spectra and hence do not allow any nonlinear gap mode below the lower band edge [9,13], the ferrimagnetic systems can give rise to breathers below the linear band edge. Also, contrary to the ferromagnetic or antiferromagnetic systems, where breather modes can exist above the top of the linear band, no such mode exists (explained in §3) in the ferrimagnetic case.

(3) The ferrimagnetic breathers we have investigated, have fairly steady structures. This is borne out by the fact that the average energy of the breather (calculated over one time period of the breather) remains constant over a time interval equal to many time periods (figure 8). Localisation property of the breathers was verified by working with spin systems of different sizes. The greater extent of localisation for higher value of D/J was also checked (figures 4a, 5b) for coloured breathers and figures 3a, 3b for monochromatic breathers). Also the energy re-

quired for exciting the easy-axis as well as the easy-plane ferrimagnetic system goes to zero at the threshold of the conventional breather mode. This is in keeping with the results of Zolotaryuk *et al* [13] for ferromagnetic ordering.

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