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On the integral representations of the Jost function and Coulomb off-shell Jost solution

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Abstract. The integral representations of the Jost function (on- and off-shell) are rederived by the judicious use of the transposed operator relation on the particular integrals for Jost solution and using one of these particular integrals an analytical expression for the Coulomb off-shell Jost solution is presented in the maximal reduced form.

Keywords. Integral representations of the Jost function; off-shell Jost solution.

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It is well-known that in charged-particle scattering the long range of Coulomb interaction is a source of special difficulties due to $1/r$ behaviour of the potential at large distances. These difficulties are observed both in classical and quantum scattering. The scattering wave functions for charged-particle scattering exhibit an essentially different (logarithmic) behaviour at large distances compared to scattering wave functions for short-range potentials. This leads to a discontinuity in the behaviour of half- and off-shell functions for the Coulomb and Coulomb-like interactions at the energy shell. The characteristic Coulomb discontinuity arises from the fact that a long-range potential distorts not only the scattering wave but also the incident plane wave [1]. However, in such a situation, one can extract physical information by introducing suitable Coulombian asymptotic states [2].

The behaviour of the irregular solution $f_\ell(k,r)$ of the radial Schrödinger equation near the origin determines the Jost function [3] $f_\ell(k)$ which has an important role in examining the analytic properties of partial wave scattering amplitude. The Jost function has two integral representations [4]: one in terms of irregular solution $f_\ell(k,r)$ and the other in terms of the regular solution $\phi_\ell(k,r)$. The integral representation associated with irregular solution $f_\ell(k,r)$ follows directly from the integral equation for $f_\ell(k,r)$. Contrary to this, the other integral representation is derived with particular emphasis on the asymptotic behaviour of regular solution $\phi_\ell(k,r)$.

The off-shell Jost function [5] $f_\ell(k,q)$, with q, an off-shell momentum, is also determined from the irregular solution $f_\ell(k,q,r)$ of an inhomogeneous Schrödinger equation in the same way as $f_\ell(k)$ is obtained from $f_\ell(k,r)$. The off-shell Jost function $f_\ell(k,q)$ has also two integral representations: one in terms of off-shell Jost solution $f_\ell(k,q,r)$ and the other involves a free particle off-shell solution and the on-shell function $\phi_\ell(k,r)$. The former result follows directly from the integral equation for $f_\ell(k,q,r)$. The latter result was obtained by Fuda [6] by using a momentum space formulation of the off-shell Jost function and Kowalski's generalization of Sasakawa method [7]. In the recent past, we [8] have derived these integral representations by exploiting the particular solution of the inhomogeneous differential equation for $f_\ell(k,q,r)$ together with the boundary conditions [4] on $f_\ell(k,r)$ and $\phi_\ell(k,r)$.

The objective of the present work is to look for a straightforward method to arrive at these results. This will be achieved by exploiting the particular solution of the inhomogeneous differential equation for $f_\ell(k,q,r)$ with judicious use of the transposed operator relation. In this course of study it will be observed that the merit of the present approach is its simplicity. Further, a closed form expression for off-shell Coulomb Jost solution [9] will be derived by using a representation for $f_\ell(k,q,r)$. Only s-wave case will be dealt here and the subscript $\ell=0$ omitted. The higher partial wave treatment to get an expression for $f_\ell(k,q,r)$ involves mathematical complication.

The on-shell Jost solution [3] $f(k,r)$ for a short-range potential obeys the differential equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+k^2-V(r)\right]f(k,r)=0 \tag{1}$$

with the asymptotic boundary condition

$$\lim_{r\to\infty} f(k,r)\mathrm{e}^{-ikr}=1 \tag{2a}$$

and Jost function is defined as

$$f(k)=\lim_{r\to 0} f(k,r). \tag{2b}$$

The off-shell Jost solution $f(k,q,r)$ for a spherically symmetric potential $V(r)$ satisfies the inhomogeneous differential equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+k^2-V(r)\right]f(k,q,r)=(k^2-q^2)\mathrm{e}^{iqr}. \tag{3}$$

The off-shell Jost function is expressed as

$$f(k,q)=\lim_{r\to 0} f(k,q,r). \tag{4a}$$

The function $f(k,q,r)$ has the asymptotic normalization [2,5]

$$f(kq,r)\xrightarrow[r\to\infty]{}\mathrm{e}^{iqr}. \tag{4b}$$

Now eq. (3) is rewritten as

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+k^2\right]\{f(k,q,r)-\mathrm{e}^{iqr}\}=V(r)f(k,q,r). \tag{5}$$

or

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+k^2 f(k,q,r)\right]\{f(k,q,r)-\mathrm{e}^{iqr}\}=V(r)\mathrm{e}^{iqr} \tag{6}$$

When $q=\pm k$ the above equations lead to

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+k^2\right]f(\pm k,r)=V(r)f(\pm k,r), \tag{7}$$

where $f(\pm k,r)$ are the two irregular solutions of the Schrödinger equation which enter into the theory of ordinary Jost function.

According to Fuda and Whiting [5], it is assumed that the particular integrals of eqs (3), (5) and (6) represent the off-shell Jost solution. These are

$$f(k,q,r)=\int_r^\infty G^{(I)}(r,r')(k^2-q^2)\mathrm{e}^{iqr'}\mathrm{d}r', \tag{8}$$

$$f(k,q,r)=\mathrm{e}^{iqr}+\int_r^\infty G_0^{(I)}(r,r')V(r')f(k,q,r')\mathrm{d}r' \tag{9}$$

and

$$f(k,q,r)=\mathrm{e}^{iqr}+\int_r^\infty G^{(I)}(r,r')V(r')\mathrm{e}^{iqr'}\mathrm{d}r'. \tag{10}$$

Here $G^{(I)}(r,r')$, the irregular Green's function for motion in $V(r)$, reads as [4]

$$G^{(I)}(r,r')=\left\{\begin{array}{ll}\frac{1}{f(k)}[\phi(k,r')f(k,r)-\phi(k,r)f(k,r')], & r'>r\\ 0, & r'<r\end{array}\right\} \tag{11}$$

and $G_0^{(I)}(r,r')$ stands for the free particle one. Equation (9) represents an integral equation for $f(k,q,r)$ and eq. (10) is the formal solution of it.

Equation (8) is rewritten as

$$f(k,q,r)=\int_r^\infty G^{(I)}(r,r')\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+k^2\right)\mathrm{e}^{iqr'}\mathrm{d}r'. \tag{12}$$

Using the transposed operator relation $\langle\varphi|\hat{o}|\psi\rangle=\langle\psi|\tilde{o}|\varphi\rangle$, $\tilde{o}=\hat{o}$ and the well-known differential equation for Green's function, eq. (12) takes the form

$$f(k,q,r)=\mathrm{e}^{iqr}+\int_r^\infty G^{(I)}(r,r')V(r')\mathrm{e}^{iqr'}\mathrm{d}r'. \tag{13}$$

From eqs (10), (11) and (13) the integral representations of $f(k,q)$ and $f(k)$ are obtained as

$$f(k,q) = f(k,q,0) = 1 + \int_0^\infty \mathrm{e}^{iqr} V(r)\phi(k,r)\mathrm{d}r \tag{14}$$

and

$$f(k) = f(k,0) = 1 + \int_0^\infty \mathrm{e}^{ikr} V(r)\phi(k,r). \tag{15}$$

The other integral representations follow directly from eq. (9) and its on-shell version. Equations (14) and (15) are derived only by rewriting the differential equations and their particular integrals for $f(k,q,r)$ together with transposed operator relation. Note that eqs (14) and (15) are applicable both for short-range and Coulomb potentials.

Combination of eqs (8) and (11) with Coulomb potential, rearrangement of terms and certain algebraic manipulations yields the Coulomb off-shell Jost solution [8,9] as

$$\begin{aligned} f(k,q,r) = 2ik\Gamma(1+i\eta)r\mathrm{e}^{ikr} & \left[\frac{(q-k)}{2k\Gamma(2+i\eta)} F\left(1, i\eta; 2+i\eta; \frac{(q-k)}{q+k)}\right)\right. \\ & \left. \times\Phi(1+I\eta, 2; -2ikr) - \left(\frac{(q+k)}{(q-k)}\right)^{i\eta} \Psi(1+i\eta, 2; -2ikr)\right] \\ & -\frac{(k^2-q^2)}{2ik} r\mathrm{e}^{ikr} \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{k-q}{2k}\right)^n \theta_{n+1}(1+i\eta, 2; -2ikr). \end{aligned} \tag{16}$$

In deriving the above expression the following relations [10–12] are used:

$$\begin{aligned} \int_0^\infty \mathrm{e}^{-az} z^{s-1} \Psi(b,d;\mu z) = & \frac{\Gamma(1+s-d)\Gamma(s)}{\Gamma(1+b+s-d)a^s} \\ & \times F\left(b, s; 1+b+s-d; 1-\frac{\mu}{a}\right), \end{aligned} \tag{17}$$

$$\int_0^\infty \mathrm{e}^{-\lambda z} z^{\nu} \Phi(a,c;pz) = \frac{\Gamma(\nu+1)}{\lambda^{\nu+1}} F\left(a, \nu+1; c; \frac{p}{\lambda}\right), \tag{18}$$

$$\begin{aligned} \theta_\sigma(a,c;z) = \frac{1}{(c-1)} & \left[\Phi(a,c;z)\int_0^z \mathrm{e}^{-z'} z'^{(\sigma+c-2)} \bar{\Phi}(a,c;z')\mathrm{d}z'\right. \\ & \left. -\bar{\Phi}(a,c;z)\int_0^z \mathrm{e}^{-z'} z'^{(\sigma+c-2)} \Phi(a,c;z')\mathrm{d}z'\right] \end{aligned} \tag{19}$$

and

$$F(a,b;c;z) = (1-z)^{c-a-b} F(c-a, c-b; c; z). \tag{20}$$

A couple of useful checks is made on the expression for $f(k,q,r)$ with particular emphasis on their limiting behaviour and on-shell discontinuity.

In the limit of no Coulomb field, $\eta = 0$, $f(k,q,r) = e^{iqr}$. Making use of the fact that $\Psi(a,c;z)_{z\to 0} \to z^{1-c}\Gamma(c-1)/\Gamma(a)$ and $\Phi(a,c;z)_{z\to 0} \to 1$ in eq. (16) the off-shell Jost function $f(k,q)$ reads as

$$f(k,q,r)_{r\to 0} \to f(k,q) = \left(\frac{(q+k)}{(q-k)}\right)^{i\eta}. \tag{21}$$

For scattering on a short-range potential, $f(k,r)$ is a continuous function of the off-shell momentum so that $f(k,q,r)$ becomes the ordinary Jost solution $f(k,r)$. This is, however, not true for Coulomb potential and Coulomb plus short-range potentials [13,14] and the off-shell Jost solution and Jost function exhibit a discontinuity at the energy shell. The on-shell limiting behaviour of $f(k,q,r)$ and $f(k,q)$ is given by the singular factor $(q-k)^{-i\eta}$. The function $f(k,r)$ can be obtained from the corresponding off-shell quantity by using the relation [13,14]

$$f(k,r) = \lim_{q\to k} \omega f(k,q,r), \quad k > 0, \tag{22}$$

with

$$\omega = \left[\frac{(q-k)}{(q+k)}\right]^{i\eta} \frac{\mathrm{e}^{\pi\eta/2}}{\Gamma(1+i\eta)}. \tag{23}$$

Thus, other useful checks on eq. (16) consist in showing that

$$f(k,r) = Lt_{q\to k} \frac{\mathrm{e}^{\pi\eta/2}}{\Gamma(1+i\eta)} \left(\frac{(q-k)}{(q+k)}\right)^{i\eta} f(k,q,r) \tag{24}$$

and

$$f(k,q,r) \xrightarrow[r\to\infty]{} \mathrm{e}^{iqr}. \tag{25}$$

Equations (24) and (25) can easily be verified from the result given in eq. (16).

In contrast to the derivation of Fuda [6] and Newton [4] the present approach is much more simpler and provide a common basis for deriving all the integral representations for the Jost function. Also it is different and straightforward from our earlier approach [8] to the problem. By exploiting the result of $f(k,q,r)$ an expression for off-shell T-matrix for Coulomb potential can be computed.

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