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Effect of various periodic forces on Duffing oscillator

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Abstract. Bifurcations and chaos in the ubiquitous Duffing oscillator equation with different external periodic forces are studied numerically. The external periodic forces considered are sine wave, square wave, rectified sine wave, symmetric saw-tooth wave, asymmetric saw-tooth wave, rectangular wave with amplitude-dependent width and modulus of sine wave. Period doubling bifurcations, chaos, intermittency, periodic windows and reverse period doubling bifurcations are found to occur due to the applied forces. A comparative study of the effect of various forces is performed.

Keywords. Duffing oscillator; various periodic forces; bifurcations; chaos.

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In the past the dynamics of many nonlinear oscillators are studied with external force being of the form $f \sin \omega t$ [1–3]. In recent years, there are few studies with different periodic forces [4–9]. The study of the influence of different types of periodic forces is very important. This is because certain periodic forces can be generated and easily applied to real physical systems and electronic circuits. Further, a detailed comparative study of the effect of different forces will be helpful to know the role of shape of the forces on nonlinear dynamics and to select an appropriate force in creating and controlling various nonlinear behaviours.

Motivated by the above in the present paper we numerically study the effect of certain periodic forces in the Duffing oscillator equation

$$\ddot{x} + \alpha \dot{x} + \omega_0^2 x + \beta x^3 = F(t), \quad (1)$$

where α is the damping constant, ω_0 is the natural frequency and β is the stiffness constant which plays the role of nonlinear parameter. The period T of all the forces considered in our study is fixed as $2\pi/\omega$. Mathematical forms of periodic forces such as sine wave ($F_{\rm sin}(t)$), square wave ($F_{\rm sq}(t)$), rectified sine wave ($F_{\rm rec}(t)$), symmetric saw-tooth wave ($F_{\rm sst}(t)$), asymmetric saw-tooth wave ($F_{\rm ast}(t)$), modulus of sine wave ($F_{\rm msi}(t)$) and rectangular wave with amplitude-dependent width ($F_{\rm rw}(t)$) are the following:

$$F_{\rm sin}(t) = F_{\rm sin}(t + 2\pi/\omega) = f \sin\omega t, \tag{2}$$

$$F_{\rm sq}(t) = F_{\rm sq}(t + 2\pi/\omega) = f {\rm sgn}(\sin\omega t), \tag{3}$$

$$F_{\rm rec}(t) = F_{\rm rec}(t + 2\pi/\omega) = \begin{cases} f \sin\omega t, & 0 < t < \pi/\omega \\ 0, & \pi/\omega < t < 2\pi/\omega, \end{cases} \tag{4}$$

$$F_{\rm sst}(t) = F_{\rm sst}\,(t + 2\pi/\omega) = \begin{cases} 4ft/T, & 0 < t < \pi/2\omega \\ -4ft/T + 2f, & \pi/2\omega < t < 3\pi/2\omega \\ 4ft/T - 4f, & 3\pi/2\omega < t < 2\pi/\omega, \end{cases} \tag{5}$$

$$F_{\rm ast}(t) = F_{\rm ast}\,(t + 2\pi/\omega) = \begin{cases} 2ft/T, & 0 < t < \pi/\omega \\ 2ft/T - 2f, & \pi/\omega < t < 2\pi/\omega, \end{cases} \tag{6}$$

$$F_{\rm msi}(t) = F_{\rm msi}\,(t + 2\pi/\omega) = f \sin\omega t/2, \tag{7}$$

$$F_{\rm rw}(t) = F_{\rm rw}\,(t + 2\pi/\omega) = \begin{cases} f, & \pi/\omega - 1/2f < t < \pi/\omega + 1/2f \\ 0, & \text{otherwise,} \end{cases} \tag{8}$$

where sgn(y) is sign of y, T=$2\pi/\omega$ and t is taken as mod(T).

For our numerical study we fix $\alpha = 0.5$, $\beta = 1.0$, $\omega_0^2 = -1.0$ and $\omega = 1.0$. The response of the system to various periodic forces is analyzed by varying the amplitude f.

Figure 1 shows the bifurcation diagrams for various forces. In figure 1 we can clearly notice many similarities and differences in the bifurcation pattern when the parameter f is varied. Consider the effect of the force $f \sin\omega t$. For small values of f two period-T ($= 2\pi/\omega$) orbits coexist, one in each potential well of the system. At $f = f_{\rm t} = 0.321102$, a transcritical bifurcation occurs at which maximal Lyapunov exponent ($\lambda_{\rm m}$) ≈ 0. When the force $f \sin\omega t$ is replaced by other forces similar behaviour is found to occur. The values of f of the different forces at which transcritical bifurcation is observed are given in table 1.

We have numerically computed the length (L) of the periodic orbit during one-drive cycle of the applied force near the transcritical bifurcation. Figure 2 shows the variation of L with $f - f_{\rm t}$. When the amplitude of the forces such as rectified sine wave, rectangular wave with amplitude-dependent width and modulus of sine wave is varied from the value just below $f_{\rm t}$ a smooth variation of L is found. For other forces, a sudden jump in the values of L is observed at $f_{\rm t}$.

The repetition period and frequency of all the forcing terms are $T = 2\pi/\omega$ and ω respectively. However, the Fourier series of all the forces except the force $\sin\omega t$ considered in our study have various frequencies. The frequencies present in the forces and in the periodic solution confined to the left well alone corresponding to the amplitude $f = 0.2$ are studied by constructing the Fourier series

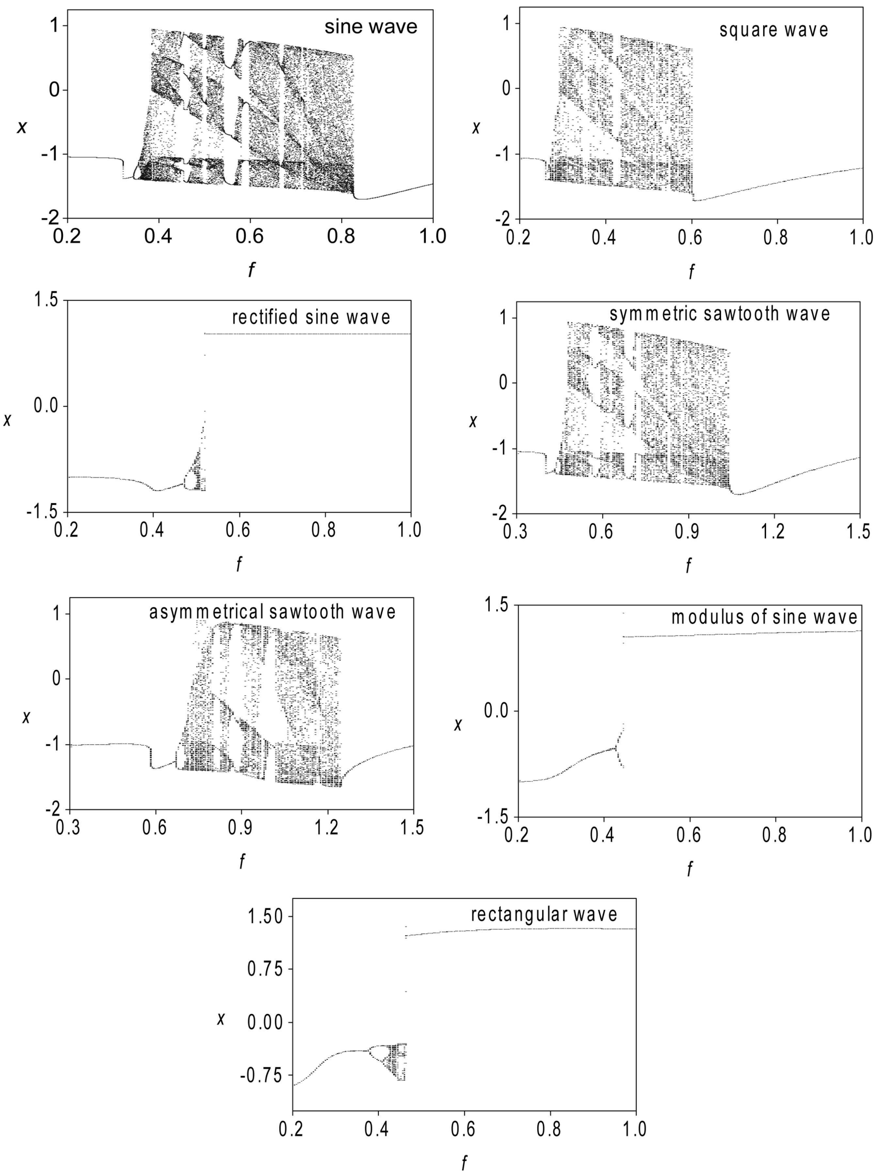

Figure 1. Bifurcation diagrams of system (1) driven by various periodic forces. The parameters of the system (1) are fixed at $\alpha = 0.5$, $\beta = 1.0$, $\omega_0^2 = -1.0$ and $\omega = 1.0$. Bifurcation structures due to sine, square, symmetric saw-tooth and asymmetric saw-tooth waves are similar. The bifurcation structures of the other three forces are different from these forces.

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t) \tag{9}$$

Table 1. Summary of bifurcation phenomena of the double-well Duffing oscillator eq. (1) driven by various periodic forces with $\alpha = 0.5, \omega_0^2 = -1.0, \beta = 1.0$ and $\omega = 1.0$. For modulus of sine wave transcritical and period-$2T$ bifurcations are found to occur at 0.29250 and 0.45625 respectively.

	Critical values of amplitude of the various forces					
Bifurcations	Sine wave	Square wave	Rectified sine wave	Symmetric saw-tooth wave	Asymmetric saw-tooth wave	Rectangular wave
Transcritical	0.321102	0.257281	0.394500	0.397080	0.579302	0.250664
Period-$2T$	0.343798	0.258208	0.469201	0.431410	0.666365	0.343777
Period-$4T$	0.355197	0.265795	0.488993	0.449260	0.690449	0.365679
Period-$8T$	0.357900	0.267736	0.494002	0.450024	0.695820	0.370712
Period-$16T$	0.358480	0.268149	0.495206	0.450173	0.696910	0.371653
Onset of chaos	0.358655	0.268265	0.496405	0.450225	0.697262	0.372128
Band-merging	0.361718	0.270403	0.501019	0.454036	0.703285	0.376768
Sudden widening	0.383340	0.290639	0.516592	0.459148	0.727939	0.382301
Intermittency	0.360320	0.270142	0.498730	0.453550	0.700421	0.372611

and fast Fourier transform (FFT). a_0 is nonzero for rectified sine wave, modulus of sine wave and rectangular wave while it is zero for other wave forms. But a_0 is nonzero for the periodic solution induced by the forces. This is because the solution is oscillatory about the equilibrium point $x^* = -|\omega_0^2|/\beta = -1$. a_n's alone are nonzero for modulus of sine and rectangular waves. For rectified sine wave, only even a_n's are nonzero. b_n's alone are nonzero for asymmetric saw-tooth wave. For square and symmetric saw-tooth waves only odd b_n's are present. Large number of frequencies are present in the square, asymmetric and rectangular waves. The Fourier series of the solution of the system contained both sine and cosine components including even and odd integral multiples of frequency ω. However, for the solution induced by sine, rectified sine, symmetric saw-tooth and modulus of sine waves only the first few a_n's and b_n's are found to be nonzero. For $n > 5$ the magnitude of a_n's and b_n's are less than 10^{-5}. But for the solution due to square, asymmetric saw-tooth and rectangular waves, large number of frequencies are present.

The critical values of f at which various bifurcations occur for different forms of forces are summarized in table 1. From table 1 and figure 1, we note that when f is increased from a small value, period doubling bifurcation phenomenon is realized much earlier for square wave while it is realized relatively at a higher value of f for the asymmetric saw-tooth wave. For modulus of sine wave, bifurcation from period-T orbit to period-$2T$ orbit and then to period-T orbit is found.

The bifurcation structures for sine, square, symmetric saw-tooth and asymmetric saw-tooth waves are similar. The bifurcation pattern for the modulus of sine, rectangular and rectified sine waves are different from others. To understand the observed difference we studied the phase portrait and the corresponding basin of attraction by varying the forcing amplitude. Since the external modulus of sine wave force is always positive, the left well alone moves up and down from the

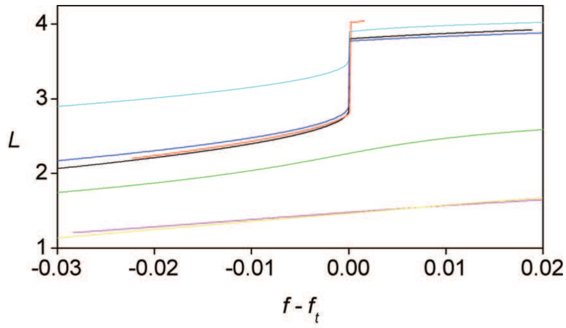

Figure 2. $f - f_t$ vs. the length of the orbit in the x–$\dot{x}$ plane during the one drive cycle of various forces. The curves from top to bottom correspond to sine wave, square wave, rectified sine wave, symmetric saw-tooth wave, asymmetric saw-tooth wave, modulus of sine wave and rectangular wave.

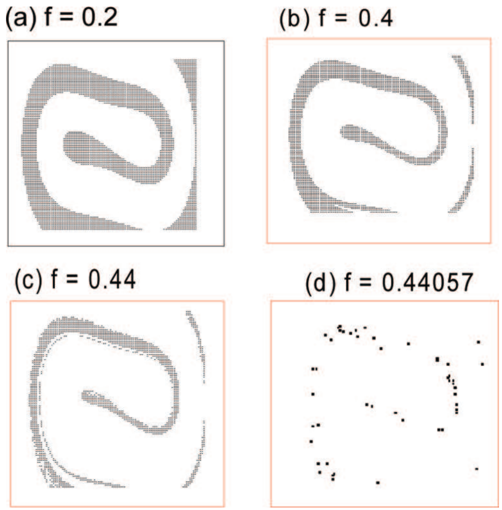

Figure 3. Basin of attraction of left well orbit (black region) and right well orbit (white region) in the x–$\dot{x}$ plane for the system driven by modulus of sine wave. Here $\alpha = 0.5$, $\beta = 1.0$, $\omega = 1.0$ and $\omega_0^2 = 1.0$. Each basin of attraction map consists of 100×100 initial conditions chosen in the domain $(x, \dot{x}) = (-3, -5) - (3, 5)$.

equilibrium position. For small values of f two period-T orbits, one in each well exist. As f increases the size of the orbit increases. Figure 3 shows the numerically computed basin of attraction for few values of f. As f increases the size of the basin of attraction of the orbit confined to the left well decreases while that of the right well orbit increases. At $f = 0.4406$ for all initial conditions in the interval $x \in [-3, 3]$ and $\dot{x} \in [-5, 5]$ after transient evolution the trajectories end up on the right well orbit only. This is because for $f \geq 0.4406$ the external force is sufficiently large that it is able to push the system from the left well to the right well. However, since the right well is stationary there is no cross well motion. Similar results are found for the system driven by the rectified sine wave.

For $f < 2\pi/\omega$ the force is nonzero during the time intervals

$$nT + \frac{\pi}{\omega} - \frac{f}{2} < t < nT + \frac{\pi}{\omega} + \frac{f}{2}, \quad n = 0, 1, ... \tag{10}$$

and zero otherwise. The intervals of time during which the force is nonzero increase with increase in f. For $f \geq 2\pi/\omega$ the rectangular force becomes a constant force. In this case the equation of motion (1) is a damped oscillator.

In summary, we have noticed several similarities and differences in the bifurcation structure in the presence of different periodic forces. It is important to study the effect of other types of forces such as amplitude-modulated and frequency-modulated waves. Analytical methods such as multiple scale perturbation theory and Melnikov technique can be employed to eq. (1) to investigate certain nonlinear phenomena with different periodic forces. These will be studied in future.

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