PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 67, No. 2
August 2006
pp. 249–256

On the exact solutions of nonlinear diffusion-reaction equations with quadratic and cubic nonlinearities

R S KAUSHAL[1], RANJIT KUMAR[2] and AWADHESH PRASAD[2]
[1]Department of Physics, Ramjas College, University of Delhi, Delhi 110 007, India
[2]Department of Physics and Astrophysics, University of Delhi, Delhi 110 007, India
E-mail: awadhesh@physics.du.ac.in

MS received 23 March 2006; accepted 23 June 2006

Abstract. Attempts have been made to look for the soliton content in the solutions of the recently studied nonlinear diffusion-reaction equations [R S Kaushal, *J. Phys.* **38**, 3897 (2005)] involving quadratic or cubic nonlinearities in addition to the convective flux term which renders the system nonconservative and the corresponding Hamiltonian non-Hermitian.

Keywords. Soliton; elliptic functions; diffusion equations; integrable systems; partial differential equations.

PACS Nos 05.45.Yv; 02.30.Ik; 02.30.Jr

1. Introduction

The diffusion-reaction (D-R) equation and its variants have been the subject of study in various branches of physical, chemical and biological sciences [1–4]. Besides its study in one dimension and higher dimensions, this all depends on the nature of the phenomenon under study as to what form of this equation (whether linear or nonlinear version) needs to be used. While a variety of simplified versions of nonlinear D-R equations are studied [3,5–7] in the literature, a D-R equation with cubic nonlinearity has been found recently in several interesting applications [5,8,9]. In fact, the presence of velocity-term, which is also attributed to the turbulent or anomalous diffusion in this later version is found to play an important role in several applications, particularly in biological studies [1,4,8,9] and the same is also responsible for making the corresponding Hamiltonian non-Hermitian. Earlier, one of us had studied [5] the possibility of real eigenvalues of the non-Hermitian Hamiltonian operator

$$H = -D\frac{\partial^2}{\partial x^2} + v\frac{\partial}{\partial x} + V(x), \tag{1}$$

associated with the linear version of the D-R equation, $C_t + vC_x = DC_{xx} - V(x)C$, and also the problem of integrability associated with the classical analogue of this operator. Solitary wave solutions of the D-R equations with quadratic and cubic nonlinearities, namely

$$C_t + vC_x = DC_{xx} + [a + U(x)]C - b|C|C, \tag{2}$$

$$C_t + vC_x = DC_{xx} + [a + U(x)]C - b|C|^2C, \tag{3}$$

are obtained under some simplifying assumptions. This was done mainly to obtain the closed form solutions of eqs (2) and (3) for complex concentration function $C(x,t)$. Here, we relax this assumption of complex $C(x,t)$ and demonstrate that eqs (2) and (3) for real $C(x,t)$ do admit a variety of exact solutions including the solitonic ones in terms of elliptic functions more on the lines of the KdV equation. In particular, the existence of soliton solutions of eq. (3) is explicitly demonstrated in a certain parametric domain. We shall, however, restrict ourselves to the case of constant value of the random potential $U(x)$, say U_0, in (2) and (3) in this work. In some sense eqs (2) and (3) can be considered as the generalization of the Fisher and Nagumo equations respectively, discussed in the literature [2,3].

In the next section we obtain the quasi-exact solutions of both eqs (2) and (3) using the so-called eigenfunction ansatz method [10]. In §3, we discuss the solitonic and solitary wave solutions of eq. (3) in a very general manner in terms of Jacobian elliptic function. Finally, concluding remarks are made in §4.

2. Exact solutions of eqs (2) and (3)

For the solution of eq. (3), we introduce the variable $\xi = x - wt$ and recast eq. (2) for the real $C(x,t)$ in the form

$$DC'' - (v - w)C' + \alpha C - bC^2 = 0, \tag{4}$$

where $U(x)$ is assumed to be constant (say U_0) and $\alpha = a + U_0$. Primes on C denote derivatives with respect to ξ. Further, for the solutions of eq. (4) we make an ansatz

$$C(\xi) = a_0 + a_1 \tanh(\mu\xi) + a_2 \tanh^2(\mu\xi), \tag{5}$$

where a_0, a_1, a_2, μ and w are constants to be fixed later. The use of (5) in (4) and the rationalization of the resultant expression with respect to the powers of $\tanh(\mu\xi)$ yields the following set of equations:

$$2\mu^2 Da_2 - (v - w)\mu a_1 + \alpha a_0 - ba_0^2 = 0, \tag{6}$$

$$-2\mu^2 Da_1 - 2(v - w)\mu a_2 + \alpha a_1 - 2ba_0a_1 = 0, \tag{7}$$

$$-8\mu^2 Da_2 + (v - w)\mu a_1 + \alpha a_2 - b(a_1^2 + 2a_0a_2) = 0, \tag{8}$$

$$2\mu^2 Da_1 + 2(v-w)\mu a_2 - 2ba_1a_2 = 0, \tag{9}$$

$$6\mu^2 Da_2 - ba_2^2 = 0. \tag{10}$$

These equations can be easily solved for the constants a_0, a_1, a_2, μ and w in terms of v, α and b appearing in eq. (2) or (4). From eqs (8), (9) and (10), one immediately derives the results for a_0, a_1 and a_2 as

$$a_0 = \frac{1}{2Db}\left[\alpha D - 8\mu^2 D^2 - \frac{1}{25}(v-w)^2\right], \tag{11}$$

$$a_1 = \frac{6(v-w)\mu}{5b}, \tag{12}$$

$$a_2 = \frac{6\mu^2 D}{b}. \tag{13}$$

Note that the parameter w (which has the dimension of velocity) is of somewhat different nature compared to other constants appearing in the ansatz (5). Using eqs (11)–(13) in (7) one obtains μ as $\mu = \pm(v-w)/(10D)$. On the other hand, the use of the values of a_0, a_1, a_2, and μ in eq. (6) yield a constraining relation among various quantities, viz.,

$$25\alpha D = \pm 6(v-w)^2. \tag{14}$$

Relation (14) can be used to fix w from $v - w = \pm 5\sqrt{\alpha D/6}$. Finally a_0, a_1, a_2 and μ turn out to be $a_0 = a_2 = \alpha/4b, a_1 = \pm\alpha/2b, \mu = \pm\sqrt{\alpha/24D}$. The results for $D = b = 1$ and $\alpha = 0.2$ are worth comparable to those available in [2,3]. Thus, the exact solution (5) of (4) takes very simple form, viz.,

$$C(\xi) = (\alpha/4b)[1 \pm \tanh(\mu\xi)]^2. \tag{15}$$

Next, we obtain the exact solution for the case of cubic nonlinear equation (3) which is recast in the form

$$DC'' - (v-w)C' + \alpha C - bC^3 = 0. \tag{16}$$

As before, for the solution of eq. (16) we make an ansatz

$$C(\xi) = a_0 + a_1 \tanh(\mu\xi), \tag{17}$$

and substitute the same in (16). The rationalization of the resultant expression with respect to the powers of $\tanh(\mu\xi)$ now yields the following set of equations involving the unknowns a_0, a_1, μ and w:

$$-\alpha a_0 + (v-w)\mu a_1 + ba_0^3 = 0, \tag{18}$$

$$2\mu^2 Da_1 - \alpha a_1 + 3ba_0^2 a_1 = 0, \tag{19}$$

$$-(v-w)\mu a_1 + 3ba_0a_1^2 = 0, \tag{20}$$

$$-2\mu^2 Da_1 + ba_1^3 = 0. \tag{21}$$

These equations can be solved to give the values of a_0, a_1, μ and w. In fact, the pair of eqs (18) and (20), and (19) and (21) separately provide the expression for a_1^2 and a comparison yields $a_0 = \pm\sqrt{\alpha/4b}$. Thus, the solution (17) of (16) becomes

$$C(\xi) = \pm\sqrt{\alpha/4b}\ \ [1 + \tanh(\mu\xi)], \tag{22}$$

with $\mu = \pm\sqrt{\alpha/8D}$ and w in $\xi = x - wt$ is given by $v - w = \pm 3\sqrt{\alpha D/2}$.

A few remarks about the solutions (15) and (22), respectively, of eqs (4) and (16) (or of eqs (2) and (3)) are in order: (i) While the coefficients in ansatz (5) and (17) depend on the strength of the nonlinear coupling b in general, the quantities μ and w, on the other hand, depend only on the diffusion coefficient D. (ii) Corresponding to the two values of μ, each of eqs (15) and (22) represents four traveling wave solutions of eqs (2) and (3). However, two of them will turn out to be linearly independent, for each case. (iii) Note that for positive α, the solutions are real as long as the nonlinear coupling $b > 0$. For negative b however the solutions become complex. If we label the four solutions of (4) or (16) as $C_{++}(\xi), C_{+-}(\xi), C_{-+}(\xi)$ and $C_{--}(\xi)$ corresponding to the (upper,upper), (upper,lower), (lower,upper) and (lower,lower) signs on the right-hand side of (15) and (22), then a number of properties of the function $C_{ij}(\xi)(i, j = +, -)$ can be listed. For example, for the solution (15) written as

$$C(\xi) = \left(\frac{\alpha}{4b}\right)\left[1 \pm \tanh\left(\pm\sqrt{\frac{\alpha}{24D}}\xi\right)\right]^2, \tag{23}$$

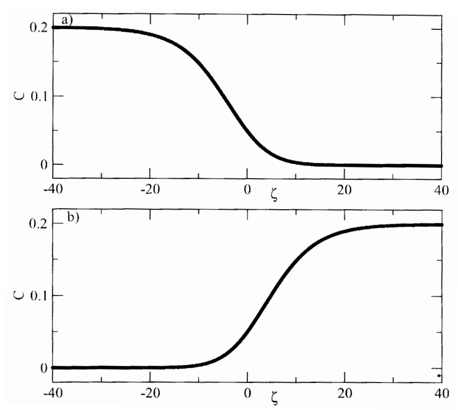

Figure 1. Solutions of eq. (15). (**a**) Kink-shaped soliton (C_{+-}) and (**b**) antikink-shaped soliton (C_{++}), for $v = 2.0$, $D = 1.0$, $\alpha = 0.2$, $b = 1.0$.

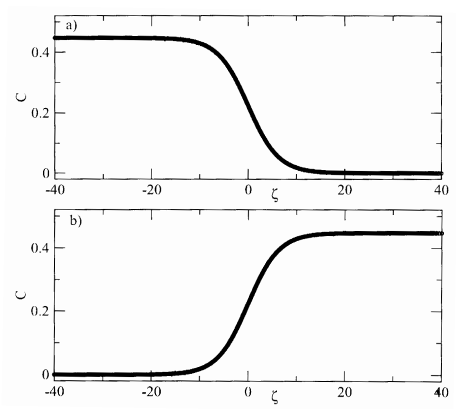

Figure 2. Solutions of eq. (22). (**a**) Kink-shaped soliton (C_{+-}) and (**b**) antikink-shaped soliton (C_{++}), for $v = 2.0$, $D = 1.0$, $\alpha = 0.2$, $b = 1.0$.

note that $C_{++}(\xi) = C_{+-}(-\xi) = C_{-+}(-\xi) = C_{--}(\xi)$. Similarly, for (22) written as

$$C(\xi) = \pm\sqrt{\frac{\alpha}{4b}}\left[1 + \tanh\left(\pm\sqrt{\frac{\alpha}{4D}}\xi\right)\right], \tag{24}$$

the following symmetries are obvious: $C_{++}(\xi) = -C_{-+}(\xi) = C_{+-}(-\xi) = -C_{--}(-\xi)$. However, in both the cases, the pair (C_{++}, C_{+-}) or (C_{--}, C_{-+}) turns out to be linearly independent.

It is interesting to note that the linearly independent solutions C_{++} and C_{+-} of eq. (16) respectively represent the kink and antikink-shaped soliton solutions in analogy with other versions of the nonlinear D-R equations studied in the literature (see p. 289 of [4]). The kink and antikink-shaped solitons resulting from these solutions of eqs (15) and (22) are shown in figures 1a,b and 2a,b, respectively for $v = 2.0, D = 1.0, \alpha = 0.2, b = 1.0$. For the same value of the nonlinear coupling b, the distinct features for the quadratic and cubic nonlinearities can be noticed from these figures.

3. Other general solutions of eq.(3)

Using the so-called polar reduction method, we investigate other general soliton solutions of eq. (3) for the complex $C(x,t)$. In particular, we look for the solution of

$$DC'' - (v-w)C' + \alpha C - b|C|^2C = 0 \tag{25}$$

in the form

$$C(\xi) = r(\xi)\exp[i\Theta(\xi)]. \tag{26}$$

The use of (26) in (25) and subsequent separation of the resultant expression into real and imaginary parts respectively yields [5]

$$Dr'' - Dr\Theta'^2 - (v-w)r' + \alpha r - br^3 = 0, \tag{27}$$

$$2Dr'\Theta' + Dr\Theta'' - (v-w)r\Theta' = 0. \tag{28}$$

As before, the first integral of eq. (28) can be written as [5]

$$\Theta' = \frac{k}{r^2}\exp\left[\frac{(v-w)\xi}{D}\right], \tag{29}$$

in which, again, we resort to the $v = w$ case for simplicity leading to $\Theta' = k/r^2$. Using this form of Θ', the first integral of eq. (27) turns out to be

$$\frac{r'^2}{2} = -\frac{k^2}{2r^2} - \frac{\alpha r^2}{2D} + \frac{br^4}{4D} + k_1, \tag{30}$$

which, after introducing the variable $S = r^2$, can be expressed as

$$S'^2 = AS^3 + BS^2 + k_1 S - 4k^2 \quad \equiv \quad AF(S) \tag{31}$$

or its integral version as

$$\xi = \pm \int^S \frac{\mathrm{d}S}{\sqrt{AF(S)}}, \tag{32}$$

where $F(S) = (S - \beta_1)(S - \beta_2)(S - \beta_3)$. Here $A = (2b/D), B = -(4\alpha/D), k_1$ is the constant of integration and the roots $\beta_i^{'s}$ of $F(S)$ satisfy

$$\beta_1 + \beta_2 + \beta_3 = -B/A, \tag{33}$$

$$\beta_1\beta_2 + \beta_2\beta_3 + \beta_3\beta_1 = k_1/A, \tag{34}$$

$$\beta_1\beta_2\beta_3 = +4k^2/A. \tag{35}$$

Now we make the substitution $z^2 = (S - \beta_3)/(\beta_2 - \beta_3)$ and write the right-hand side of eq. (32) in the form of elliptic integral [2,3], viz.,

$$\xi(z, m) = \int_o^z \frac{\mathrm{d}z'}{\lambda\sqrt{1 - m^2 z'^2}\sqrt{1 - z'^2}}, \tag{36}$$

where $\lambda = -A(\beta_3 - \beta_1)/4$, $m^2 = (\beta_3 - \beta_2)/(\beta_3 - \beta_1)$. It can be readily seen that for the case $z = \sin\phi$, the integral in (36) defines the elliptic function sn and the same allows one to write $z = sn(\lambda\xi, m)$ or the general solution $r(\xi)$ as [2]

$$r(\xi) = \pm\sqrt{\beta_3 - (\beta_3 - \beta_2)sn^2(\lambda\xi, m)}. \tag{37}$$

For $k = 0$, the polar angle Θ becomes constant, say Θ_0, and for this choice we investigate the following cases differing in terms of nature of zeroes of $F(S)$.

Case I: When $\beta_3 \neq 0$ and $\beta_1 = \beta_2 = 0$, eq. (37) renders the solution of (25) as $(sn^2 + cn^2 = 1)$,

$$C(x, t) = \sqrt{\beta_3}cn(\lambda\xi, m)\exp(i\Theta_0). \tag{38}$$

Case II: When β_2 and β_3 are not zero but $\beta_1 = 0$, the solution of (25) can be expressed as

$$C(x, t) = \pm\sqrt{\beta_3 - (\beta_3 - \beta_2)sn^2(\lambda\xi, m)}\exp(i\Theta_0). \tag{39}$$

Case III: When $\beta_3 = 0$ but β_1 and β_2 are not zero, then solution of (25) becomes

$$C(x, t) = \sqrt{\beta_2}sn(\lambda\xi, m)\exp(i\Theta_0). \tag{40}$$

Solutions corresponding to these three cases are respectively shown in figures 3–5, for $D = 1.0, \alpha = 0.2, b = 1.0$. Note that Case I is interesting as it clearly displays the soliton content in the solution (see figure 3). Cases II and III however represent periodic solutions of (3) which are respectively displayed in figures 4 and 5.

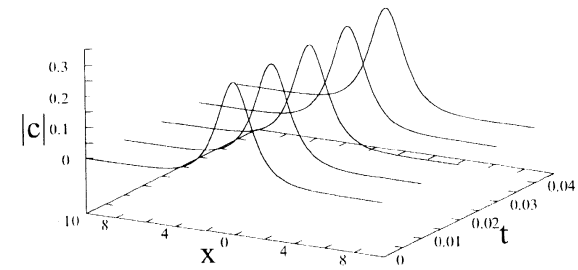

Figure 3. Solitonic solution of eq. (3) (see Case I) for $D = 1.0$, $\alpha = 0.2$, $b = 1.0$, $\beta_1 = \beta_2 = 0$, $\beta_3 = 0.4$.

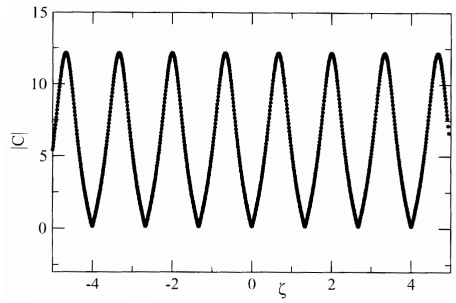

Figure 4. Periodic wave solution of eq. (3) (see Case II) for $D = 1.0$, $\alpha = 0.2$, $b = 1.0$, $\beta_1 = 0$, $\beta_2 = 0.4$, $\beta_3 = 0.02$.

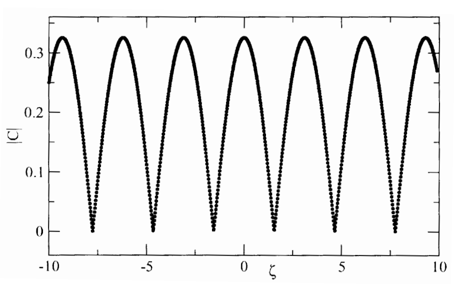

Figure 5. Periodic wave solution of eq. (3) (see Case III) for $D = 1.0$, $\alpha = 0.2$, $b = 1.0$, $\beta_3 = 0$, $\beta_1 = 0.4$, $\beta_2 = 0.02$.

4. Concluding discussion

Certain aspects ignored earlier [5] in the studies of the D-R equations involving quadratic and cubic nonlinearities, are now investigated in this work. In particular, the existence of soliton content in the solutions of eqs (2) and (3) is demonstrated explicitly.

In view of the fact that nonlinear D-R equations are in use in explaining a variety of physical phenomenon and the choice of nonlinearity in these equations is a part of the modeling process of the phenomenon under study, it is expected that the results obtained in this paper can offer some clue in making such choices.

Acknowledgments

The authors would like to thank Dr D Parashar for a critical reading of the manuscript. One of the authors (AP) would like to thank the Department of Science and Technology, India, for financial support. RK thanks CSIR, Gov. of India for Junior Research Fellowship.

References

[1] J D Murray, *Mathematical biology* (Springer-Verlag, New York, 1993)
[2] M Lakshmanan and S Rajasekhar, *Nonlinear dynamics: Integrability, chaos, and patterns* (Springer-Verlag, Indian reprint, 2003)
[3] R S Kaushal and D Parashar, *Advanced methods of mathematical physics* (Narosa Pub. House, New Delhi, CRC Press, USA, Alpha Science Int. London, 2000)
[4] M Remoissent, *Waves called solitons: Concepts and experiments* (Springer-Verlag, Berlin-Heidelberg, 1999)
[5] R S Kaushal, *J. Phys.* **A38**, 3897 (2005)
[6] D R King, *J. Phys.* **A24**, 3213 (1991)
[7] R S Benarjee, *Int. J. Theo. Phys.* **32**, 879 (1993)
[8] N Hatano and D R Nelson, *Phys. Rev. Lett.* **77**, 570 (1996)
D R Nelson and N M Shnerb, *Phys. Rev.* **E58**, 1383 (1998); and references therein
[9] J T Chalker and Z Jane Wang, *Phys. Rev. Lett.* **79**, 1797 (1997)
N Moislyeb and M Gluck, *Phys. Rev.* **E63**, 041103 (2001)
[10] R S Kaushal, *Classical and quantum mechanics of noncenteral potentials: A survey of two-dimensional systems* (Narosa Publishing House, New Delhi, 1998; Springer-Verlag, Heidelberg, 1998)