PRAMANA — journal of physics

Vol. 67, No. 2
August 2006
pp. 227–237

Magnetized cosmological models in bimetric theory of gravitation

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MS received 8 December 2005; revised 19 May 2006; accepted 23 June 2006

Abstract. Bianchi type-III magnetized cosmological model when the field of gravitation is governed by either a perfect fluid or cosmic string is investigated in Rosen's [1] bimetric theory of gravitation. To complete determinate solution, the condition, viz., $A = (BC)^n$, where n is a constant, between the metric potentials is used. We have assumed different equations of state for cosmic string [2] for the complete solution of the model. Some physical and geometrical properties of the exhibited model are discussed and studied.

Keywords. Bimetric theory; perfect fluid; cosmic string; magnetic field; Bianchi type-III.

PACS Nos 04.20.-q; 04.20.Cv; 04.20.Ex; 98.90

1. Introduction

A new theory of gravitation, called the bimetric theory of gravitation, was proposed by Rosen [1] to modify the Einstein's general theory of relativity by assuming two metric tensors, viz., a Riemannian metric tensor g_{ij} and a background metric tensor f_{ij}. The metric tensor g_{ij} determines the Riemannian geometry of the curved space–time which plays the same role as given in the Einstein's general relativity and it interacts with matter. The background metric tensor f_{ij} referes to the geometry of the empty (free from matter and radiation) universe and describes the inertial forces. This metric tensor f_{ij} has no direct physical significance but appears in the field equations. Therefore it interacts with g_{ij} but not directly with matter. One can regard f_{ij} as giving the geometry that would exist if there were no matter. In the absence of matter one would have $g_{ij} = f_{ij}$. Moreover, the bimetric theory also satisfied the covariance and equivalence principles: the formation of general relativity. The theory agrees with the present observational facts pertaining to general relativity. Thus at every point of space–time, there are two line elements:

$$ds^2 = g_{ij}dx^i dx^j \tag{1}$$

and

$$d\sigma^2 = f_{ij} dx^i dx^j. \tag{2}$$

The field equations of Rosen's bimetric theory of gravitation are

$$N_j^i - \frac{1}{2} N \delta_j^i = -8\pi k T_j^i, \tag{3}$$

where $N_j^i = \frac{1}{2} f^{ab} (g^{hi} g_{hj|a})_{|b}$ and $k = \sqrt{g/f}$, together with $g = \det(g_{ij})$ and $f = \det(f_{ij})$. Here the vertical bar (|) stands for f-covariant differentiation and T_j^i is the energy–momentum tensor of matter fields.

Rosen [1,3–6], Yilmaz [7], Karade and Dhoble [8], Karade [9], Israelit [10–12], Adhav *et al* [13] are some of the eminent authors who have studied several aspects of bimetric theory of gravitation. In particular, Reddy and Venkateswarlu [14], Reddy and Venkateswara Rao [15], Mohanti *et al* [16] and Reddy [17] have established the non-existence of spatially homogeneous and isotropic cosmological models of Bianchi types and Kantowski–Sachs in bimetric theory of gravitation when the field of gravitation is governed by either perfect fluid or string dust. Recently, Adhav *et al* [18] and Katore *et al* [19] have shown that non-static plane symmetric metric as well as n-dimensional static plane symmetric space–time do not occur in bimetric theory of gravitation when the field of gravitation is governed by either perfect fluid or mesonic massive scalar field.

Recently, there has been a lot of interest in cosmological model on the basis of Rosen's bimetric theory of gravitation. The purpose of Rosen's bimetric theory is to get rid of the singularities that occur in general relativity that was appearing in the big-bang in cosmological models. In bimetric theory, according to Rosen [5], the background metric tensor f_{ij} should not be taken as describing an empty universe but it should rather be chosen on the basis of cosmological consideration. Hence Rosen proposed that the metric f_{ij} be taken as the metric tensor of a universe in which perfect cosmological principle holds. In accordance with this principle, the large scale structure of the universe presents the same aspect from everywhere in space and at all times. The fact, however, is that while taking the matter actually present in the universe, this principle is not valid on small scale structure due to irregularities in the matter distribution and also not valid on large scale structure due to the evolution of the matter. Therefore, we adopt the perfect cosmological principle as the guiding principle. It does not apply to g_{ij} and the matter in the universe but to the metric f_{ij}. Hence f_{ij} describes a space–time of constant curvature.

It is interesting to note that magnetic field plays a significant role at cosmological model. Melvin [20] suggested in the cosmological solution for dust and electromagnetic field that during the evolution of the universe, the matter was in highly ionized state and smoothly coupled with magnetic field and consequently form a neutral matter as a result of universe expansion. Hence in string dust universe the presence of magnetic field is not unrealistic.

In the present work, keeping in view of the importance of Maxwell's electromagnetic field interactions with a perfect fluid or a cosmic string, we have discussed this problem in the context of Bianchi type-III cosmological model in Rosen's bimetric theory of gravitation by paying special attention to different equations of state for cosmic string. It is shown that the model does not exist in bimetric relativity when the field of gravitation is governed by perfect fluid with electromagnetic field and hence a vacuum model is obtained. In this paper we have presented a class of exact solution of the field equations for Bianchi type-III magnetized cosmological model which represents a barotropic equation of state for string model but in the other cases of string model, i.e. both the geometric string and p-string, the model admits vacuum solutions. The metric of four-dimensional space–time is g_{ij} of signature -2. Some physical and geometrical aspects of the exhibited model are discussed and studied at the end.

2. Perfect fluid coupled with an electromagnetic source

The energy–momentum tensor for perfect fluid distribution with an electromagnetic field is given by

$$T_{ij} = (\rho + p)u_i u_j - pg_{ij} + E_{ij} \tag{4}$$

together with $g_{ij}u^i u^j = 1$, where

$$E_{ij} = \frac{1}{4\pi}\left[-F_{is}F^s_j + \frac{1}{4}g_{ij}F_{sp}F^{sp}\right]. \tag{5}$$

Here p and ρ are proper pressure and energy density of the fluid respectively, u^i is the four-velocity vector, E_{ij} is the electromagnetic field and F_{ij} is the Maxwell's electromagnetic tensor.

We consider the spatially homogeneous and anisotropic Bianchi type-III metric in the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - A^2\mathrm{d}x^2 - B^2\mathrm{e}^{2x}\mathrm{d}y^2 - C^2\mathrm{d}z^2. \tag{6}$$

The flat metric corresponding to (6) is

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{e}^{2x}\mathrm{d}y^2 - \mathrm{d}z^2. \tag{7}$$

We assume the coordinates to be co-moving so that

$$u^1 = 0 = u^2 = u^3 \quad \text{and} \quad u^4 = 1.$$

In co-moving coordinates, the magnetic field is taken along x-direction so that the only non-vanishing components of electromagnetic field tensor F_{ij} is F_{23}. Hence due to assumption of infinite electrical conductivity $F_{14} = 0 = F_{24} = F_{34}$.

The first set of Maxwell's equation

$$F_{[ij,k]} = 0$$

leads to the result

$$F_{23} = \text{ constant } = H \text{ (say)}.$$

From eq. (5) we obtain

$$E_1^1 = -E_2^2 = -E_3^3 = E_4^4 = \frac{H^2}{8\pi B^2 C^2 \mathrm{e}^{2x}} = \eta \text{ (say)}. \tag{8}$$

Equation (4) leads to

$$T_1^1 = -p + \eta; \quad T_2^2 = -p - \eta = T_3^3 \quad \text{and} \quad T_4^4 = \rho + \eta. \tag{9}$$

The Rosen's field equation (3) for the metric (6) and (7) with the help of (9) becomes

$$\left(\frac{A_4}{A}\right)_4 - \left(\frac{B_4}{B}\right)_4 - \left(\frac{C_4}{C}\right)_4 = -16\pi k(-p + \eta), \tag{10}$$

$$\left(\frac{A_4}{A}\right)_4 - \left(\frac{B_4}{B}\right)_4 + \left(\frac{C_4}{C}\right)_4 = -16\pi k(p + \eta), \tag{11}$$

$$\left(\frac{A_4}{A}\right)_4 + \left(\frac{B_4}{B}\right)_4 - \left(\frac{C_4}{C}\right)_4 = -16\pi k(p + \eta) \tag{12}$$

and

$$\left(\frac{A_4}{A}\right)_4 + \left(\frac{B_4}{B}\right)_4 + \left(\frac{C_4}{C}\right)_4 = 16\pi k(\rho + \eta), \tag{13}$$

where suffix 4 following an unknown function denotes an ordinary differentiation with respect to time. Equations (10)–(13) are four equations connecting five unknowns A, B, C, λ and ρ. For the complete determination of these field equations, we assume the relation between metric potentials as

$$A = (BC)^n, \tag{14}$$

where n is a constant.

Equations (10)–(13) with the help of eq. (14) for $n = 1/2$ yields

$$\left(\frac{A_4}{A}\right)_4 = \left(\frac{B_4}{B}\right)_4 = \left(\frac{C_4}{C}\right)_4. \tag{15}$$

Using eq. (15) in eqs (12) and (13) we obtain

$$\left(\frac{A_4}{A}\right)_4 = -16\pi k(p + \eta) \tag{16}$$

$$3\left(\frac{A_4}{A}\right)_4 = 16\pi k(\rho + \eta). \tag{17}$$

Solving eqs (16) and (17) we obtain

$$\rho + 3p + 4\eta = 0. \tag{18}$$

In view of reality conditions $\rho > 0$, $p > 0$ and $\eta > 0$.

Equation (18) immediately implies that

$$\rho = 0, \quad p = 0 \quad \text{and} \quad \eta = 0.$$

This means that the perfect fluid with an electromagnetic field Bianchi type-III cosmological model do not exist in bimetric theory of gravitation.

When $\rho = 0, p = 0$ and $\eta = 0$ (vacuum), the field eqs (10)–(13) admit the solution

$$A = \mathrm{e}^{k_1 t}, \quad B = \mathrm{e}^{k_2 t} \quad \text{and} \quad C = \mathrm{e}^{k_3 t},$$

where k_1, k_2 and k_3 are constants of integration.

Hence Bianchi type-III vacuum model in bimetric theory of gravitation becomes

$$\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{e}^{2k_1 t}\mathrm{d}x^2 - \mathrm{e}^{2(k_2 t + x)}\mathrm{d}y^2 - \mathrm{e}^{2k_3 t}\mathrm{d}z^2. \tag{19}$$

This model can be transformed through a proper choice of coordinates and constants to

$$\mathrm{d}s^2 = \mathrm{d}T^2 - \mathrm{e}^{2T}[\mathrm{d}X^2 + \mathrm{e}^{2X}\mathrm{d}Y^2 + \mathrm{d}Z^2]. \tag{20}$$

It is interesting to note that the vacuum model (20) is spatially homogeneous, isotropic and has no initial singularity. The energy–momentum tensor T_{ij} has no contribution of Maxwell electromagnetic field to perfect fluid. The model reduces to flat space–time when $T = 0$. Hence a vacuum Bianchi type-III homogeneous, isotropic and singularity free magnetized cosmological model is obtained. Hence one can conclude that in Rosen's bimetric relativity, the only possible solutions of Bianchi type magnetized cosmological models sourced by perfect fluid is a vacuum solution.

3. Cosmic string coupled with an electromagnetic field

The energy–momentum tensor for a cosmic string [2,21] in the presence of an electromagnetic field is written as

$$T_{ij} = \rho u_i u_j - \lambda x_i x_j + E_{ij}, \tag{21}$$

where ρ is the rest energy density of the system of the string with massive particle attached to them (p-string), λ is the tension density of the string and E_{ij} is that for magnetic field. As pointed out by Letelier [2], λ may be positive or negative, u^i represents the four-velocity vector and x^i represents an anisotropic direction, i.e. direction of the string.

We have

$$u_i u^i = -x_i x^i = 1 \quad \text{and} \quad u^i x_i = 0. \tag{22}$$

We consider

$$\rho = \rho_{\mathrm{p}} + \lambda,$$

where ρ_{p} is the particle energy density attached to the string.

From eqs (6) and (22) we write

$$u_i = u^i = (0,\ 0,\ 0,\ 1)$$

and x^i can be taken parallel to any of the directions $\partial/\partial x, \partial/\partial y, \partial/\partial z$. We choose x^i parallel to $\partial/\partial x$ so that

$$x^i = (A^{-1},\ 0,\ 0,\ 0).$$

The components of energy tensor for an electromagnetic field are given by eq. (8). Using eq. (8), the non-zero components of T^i_j yields

$$T^1_1 = \lambda + \eta; \quad T^2_2 = -\eta = T^3_3 \quad \text{and} \quad T^4_4 = \rho + \lambda. \tag{23}$$

The Rosen's field equations (3) of bimetric theory of gravitation for Bianchi type-III metric (6) with the help of (7) and (23) can be written as

$$\left(\frac{A_4}{A}\right)_4 - \left(\frac{B_4}{B}\right)_4 - \left(\frac{C_4}{C}\right)_4 = -16\pi k(\lambda + \eta), \tag{24}$$

$$\left(\frac{A_4}{A}\right)_4 - \left(\frac{B_4}{B}\right)_4 + \left(\frac{C_4}{C}\right)_4 = -16\pi k\eta, \tag{25}$$

$$\left(\frac{A_4}{A}\right)_4 + \left(\frac{B_4}{B}\right)_4 - \left(\frac{C_4}{C}\right)_4 = -16\pi k\eta \tag{26}$$

and

$$\left(\frac{A_4}{A}\right)_4 + \left(\frac{B_4}{B}\right)_4 + \left(\frac{C_4}{C}\right)_4 = 16\pi k(\rho + \eta), \tag{27}$$

where suffix 4 following an unknown function denotes an ordinary differentiation with respect to time t. We are at liberty to make some assumption as we have more unknowns, i.e. two physical parameters ρ, λ and three metric potentials A, B and C, with lesser number of field equations (24)–(27) to determine them. For the complete determination of these field equations, we assume the following different equations of state for string model [2]:

(I) $\rho = \rho(\lambda)$ (barotropic equation of state),

(II) $\rho = \lambda$ (geometric string),

(III) $\rho = (1 + W)\lambda$ (p-string or Takabayasi string).

Case I: *Barotropic equation of state* ($\rho = \rho(\lambda)$)

To solve the field equations (24)–(27), we have four equations connecting five unknowns. So one more relation between metric potentials is needed. Here we assume the relation

$$A = (BC)^n, \tag{28}$$

where n is a constant. Raj Bali [22] obtained a magnetized cosmological model in general relativity with the above condition where $n = 1$. In this paper we shall choose $n = 1/2$.

From eqs (25) and (26) we have

$$\left(\frac{A_4}{A}\right)_4 - \left(\frac{B_4}{B}\right)_4 + \left(\frac{C_4}{C}\right)_4 = -16\pi k\eta \tag{29}$$

and

$$\left(\frac{B_4}{B}\right)_4 - \left(\frac{C_4}{C}\right)_4 = 0. \tag{30}$$

Using condition (28) in (29) we obtain

$$\left(\frac{B_4}{B}\right)_4 + m\left(\frac{C_4}{C}\right)_4 = -b\frac{A}{BC}, \tag{31}$$

where

$$m = \frac{n+1}{n-1} \quad \text{and} \quad b = \frac{2H^2}{(n-1)\mathrm{e}^{2x}}.$$

Setting $BC = \alpha$ and $B/C = \beta$ in eqs (30) and (31) we obtain

$$\left(\frac{a_4}{\alpha}\right)_4 + F\left(\frac{\beta_4}{\beta}\right)_4 = a\alpha^{(n-1)}, \tag{32}$$

where

$$F = \frac{1-m}{1+m} \quad \text{and} \quad a = -\frac{2b}{1+m}$$

and

$$\left(\frac{\beta_4}{\beta}\right)_4 = 0. \tag{33}$$

Using (33) in eq. (32), we obtain

$$\left(\frac{\alpha_4}{\alpha}\right)_4 = a\alpha^{(n-1)}. \tag{34}$$

Integrating eq. (34) we obtain

$$\left(\frac{\alpha_4}{\alpha}\right)^2 = 2a\int \alpha^{(n-2)}\mathrm{d}\alpha + a_1, \tag{35}$$

where a_1 is the constant of integration. We take $a_1 = 0$ for closed form solution.

In this case, eqs (33) and (35) yield the solutions on integration

$$\alpha = \left[\frac{(1-n)}{2}(k_1 t + k_2)\right]^{2/(1-n)}$$

and

$$\beta = k_4 \mathrm{e}^{k_3 t},$$

where k_1, k_2, k_3 and k_4 are constants of integration (not all zeros).

The metric (6) can be transformed through a proper choice of coordinates and constants to the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - t^4(\mathrm{d}x^2 + \mathrm{e}^{t+2x}\mathrm{d}y^2 + \mathrm{e}^{-t}\mathrm{d}z^2). \tag{36}$$

Case II: *Geometric string* ($\rho = \lambda$)

Here we assume a relation between metric coefficients, i.e.

$$A = (BC)^n, \tag{37}$$

where n is a constant. We choose $n = 1/2$.

Using $\rho = \lambda$, adding eqs (24) and (27) we obtain

$$\left(\frac{A_4}{A}\right)_4 = 0. \tag{38}$$

Using (37) and adding eqs (25) and (26) we obtain

$$\eta = 0. \tag{39}$$

With eqs (37) and (38), eq. (25) becomes

$$\left(\frac{B_4}{B}\right)_4 = \left(\frac{C_4}{C}\right)_4. \tag{40}$$

The relation (37) with the help of eqs (38) and (40) becomes

$$\left(\frac{B_4}{B}\right)_4 = 0. \tag{41}$$

Using eqs (37)–(41), we obtain

$$\rho = 0 = \lambda$$

which shows that geometric string do not exist in bimetric theory of gravitation for Bianchi type-III space–time with an electromagnetic field. Hence in this case we, again, get the vacuum model as obtained in model (20). The model (20) has no singularity at $T = 0$. The energy density, tension density as well as electromagnetic field being vanished, the energy–momentum tensor T_{ij} turns out to be zero and consequently gives an empty space–time.

Case III: *p-string or Takabayasi string* ($\rho = (1 + W)\lambda$)

Using eq. (25) in eqs (24), (26) and (27) we obtain

$$\left(\frac{C_4}{C}\right)_4 = 8\pi k\lambda, \tag{42}$$

$$\left(\frac{A_4}{A}\right)_4 + \left(\frac{C_4}{C}\right)_4 = 8\pi k\rho, \tag{43}$$

and

$$\left(\frac{B_4}{B}\right)_4 = \left(\frac{C_4}{C}\right)_4. \tag{44}$$

Here the equation of state $\rho = (1+W)\lambda$, where $W > 0$, a constant and it is small for string dominated era and large for particle dominated era.

Further, using the polynomial relation $A = (BC)^n$ with $n = 1/2$ and eq. (44) we obtain

$$\left(\frac{A_4}{A}\right)_4 = \left(\frac{B_4}{B}\right)_4 = \left(\frac{C_4}{C}\right)_4. \tag{45}$$

Using p-string, i.e. $\rho = (1+W)\lambda$, the field equations (42) and (43) yield

$$\left(\frac{A_4}{A}\right)_4 - W\left(\frac{C_4}{C}\right)_4 = 0. \tag{46}$$

From eqs (45) and (46), we get

$$\left(\frac{A_4}{A}\right)_4 = 0. \tag{47}$$

Equations (42) and (43) with the help of eqs (45) and (47) become

$$\rho = 0 = \lambda$$

which shows that p-string do not occur in bimetric theory of gravitation for Bianchi type-III when the field is governed by cosmic string with an electromagnetic source. So the vacuum model is obtained as the model (20).

4. Some physical and geometrical aspects of the model

Case I: The model (36) has no initial singularity at $t = 0$. It does not seem to be a flat model.

The physical parameters λ (tension density) and ρ (energy density) of the string are given by eqs (24) and (27) respectively as

$$\lambda = -\frac{1}{4\pi t^8} \tag{48}$$

and

$$\rho = -\frac{1}{2\pi t^8}. \tag{49}$$

In order to gain a further insight into the behaviour of the universe (36), we write the kinematical parameters for the model (36) as follows:

The spatial volume $V^3 = (-g)^{1/2} = t^6 \mathrm{e}^x$.
Expansion scalar θ is given by

$$\theta = u^4_{;4} = \frac{6}{t}.$$

and shear scalar

$$\sigma^2 = \frac{1}{2}\sigma^{ij}\sigma_{ij} = \frac{1}{6}\theta^2.$$

The deceleration parameters

$$q = -3\theta^{-2}\left[\theta_{;i}u^i + \frac{1}{3}\theta^2\right] = -\frac{1}{2} < 0.$$

From the above parameters, we observed that ρ, λ, expansion scalar (θ), as well as shear scalar (σ) all decrease as time increases. At the singularity stage $t \to 0, V^3 \to 0$ and ρ, λ, θ and σ all infinitely large but, at $t \to \infty, V^3 \to \infty$ and ρ, λ, θ and σ all vanish which represent an empty universe. But all these parameters remain finite and physically significant for all $t > 0$. Hence we see that space–time admits a big-bang singularity but the rate of expansion of the universe decreases with increase of time. We also see that there is no effect of magnetic field in the expansion of the universe. But the physical parameters ρ and λ of the strings vanish which remains a dust filled universe. Moreover, since $\lim_{t\to\infty}(\sigma/\theta) \neq 0$, the model does not approach isotropy for large values of t. The role of the deceleration parameter q seems to specify the expansion of the universe. In the last decades, the standard cosmological model favoured a presently matter dominated universe expanding in a decelerated fashion. The positive value of the deceleration parameter q indicates that the model decelerates in the standard way. But the recent observation seems to be a negative value of this parameter which shows that the model inflates.

Cases II and III: When the energy density and tension density of the string vanish, the field equations (24)–(27) admit an exact vacuum solution in Rosen's bimetric relativity. Hence in these cases we, again, get empty universe as the model (20). The model (20) has no singularity and it reduces to an empty (i.e. no matter and radiation) universe when $T = 0$.

5. Conclusion

The present work is an extension of the work of Reddy and Venkateswara Rao [14]. In addition, keeping in view of recent interest in electromagnetic field in free space, we have developed the idea of perfect fluid and cosmic string by considering the Bianchi type-III cosmological model in Rosen's bimetric theory of gravitation. Our model will be useful for a better understanding of magnetized cosmological model in Rosen's bimetric relativity.

Acknowledgement

The authors wish to acknowledge the UGC, New Delhi, for sanctioning research project and financial support. The authors are very much thankful to the referee for his valuable comments to improve the paper in a standard way. They are also thankful to IUCAA for sending preprints and papers.

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