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On a revisit to the Painlevé test for integrability and exact solutions for Yang's self-dual equations for $SU(2)$ gauge fields

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Abstract. Painlevé test (Jimbo *et al* [1]) for integrability for the Yang's self-dual equations for $SU(2)$ gauge fields has been revisited. Jimbo *et al* analysed the complex form of the equations with a rather restricted form of singularity manifold. They did not discuss exact solutions in that context. Here the analysis has been done starting from the real form of the same equations and keeping the singularity manifold completely general in nature. It has been found that the equations, in real form, pass the Painlevé test for integrability. The truncation procedure of the same analysis leads to non-trivial exact solutions obtained previously and auto-Backlund transformation between two pairs of those solutions.

Keywords. Painlevé analysis; integrability; auto-Backlund transformation; exact solutions; $SU(2)$ gauge field; self-duality.

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1. Introduction

This communication revisits some observations regarding a set of well-known equations, namely Yang's self-dual equations for $SU(2)$ gauge fields [1–3]. Jimbo *et al* [1] adopted the algorithm of Weiss *et al* [4] and showed that the equations [2] pass the Painlevé test for integrability in the sense of Weiss *et al* [4]. Ward [5] used a completely different approach, complicated indeed, and arrived at the same conclusion. Both the investigations [4,5] used the complex form of the equations [2] and neither of them reported any solution from the analysis.

Recently, Chakraborty and Chanda [6] have shown that the equations admit spreading wave with solitary profile, which tends to zero as time tends to infinity and spreading wave packets. Chakraborty and Chanda [6] found another interesting thing about the equations. They observed that the self-dual equations reported by Yang [2] and the equations reported by Charap [7] for the chiral invariant model of pion dynamics under tangential parametrization have some common characteristics, which are mathematically interesting. This has been elaborated in their publication [6]. With this observation they have combined the two sets of equations and have obtained a new set of equations wherefrom the previous two sets can be obtained as particular cases. It has been found [6] that the solutions of the combined equations do not deviate much from those of Yang's equations, whereas the solutions of the combined equations deviate considerably from those of Charap's equations. A very recent investigation by Chakraborty and Chanda [8] shows that the Charap's equations [7] pass the Painlevé test for integrability in the sense of Weiss *et al* [4] and admit truncation of series [4,9,10].

Here it has been found that the real form of Yang's self-dual equations [2,3] passes the Painlevé test for integrability in the sense of Weiss *et al* [4] and admit truncation of series leading to non-trivial exact solutions obtained previously and auto-Backlund transformation between two pairs of these solutions (see for example the work of Larsen [9] and Roychowdhury [10]). The important aspect of our analysis is that we have analysed the equation keeping the singularity manifold completely general, whereas Jimbo *et al* [1] analysed the same equation with a restricted nature of singularity manifold.

2. The equations

These were obtained by Yang [2,3] while discussing the condition of self-duality of $SU(2)$ gauge fields on Euclidean four-dimensional space. The equations are given by

$$\phi(\phi_{yy^*} + \phi_{zz^*}) - \phi_y\phi_{y^*} - \phi_z\phi_{z^*} + \rho_y\rho_{y^*} + \rho_z\rho_{z^*} = 0, \tag{2.1a}$$

$$\phi(\rho_{yy^*} + \rho_{zz^*}) - 2\rho_y\phi_{y^*} - 2\rho_z\phi_{z^*} = 0, \tag{2.1b}$$

where the $*$ mark denotes complex conjugate, ϕ and ρ are functions of y, y^*, z and z^*, ϕ is real, ρ is complex and

$$\sqrt{2}y = x^1 + ix^2, \quad \sqrt{2}z = x^3 - ix^4, \tag{2.1c}$$

where x^1, x^2, x^3, x^4 are real.

Once ρ and ϕ are known, the corresponding R-gauge potentials are given by Yang [2]

$$\phi\vec{b}_y = (i\rho_y, \rho_y, -i\phi_y), \quad \phi\vec{b}_{y^*} = (-i\rho_{y^*}, \rho^*_{y^*}, i\phi_{y^*}) \tag{2.2a}$$

$$\phi\vec{b}_z = (i\rho_z, \rho_z, -i\phi_z), \quad \phi\vec{b}_{z^*} = (-i\rho_{z^*}, \rho^*_{z^*}, i\phi_{z^*}) \tag{2.2b}$$

and R-gauge field strengths $F_{\mu\nu}$ are given by

$$F_{\mu\nu} = B_{\mu,\nu} - B_{\nu,\mu} - B_\mu B_\nu + B_\nu B_\mu, \tag{2.3a}$$

$$B_\mu = b^i_\mu X_i \tag{2.3b}$$

$$X_i = -(1/2)i\sigma_i, \tag{2.3c}$$

where σ_i are the 2×2 Pauli matrices.

All such solutions represent the condition of self-duality except when ϕ is zero because when ϕ is zero $F_{\mu\nu}$ become singular and the solutions obtained can only be treated as solutions of Yang's R-gauge and not self-dual solutions unless a transformation like $F_{\mu\nu} \to U^{-1}F_{\mu\nu}U$ removes the singularities. In terms of real variables eqs (2.1) read

$$\begin{aligned}\phi(\phi_{11} + \phi_{22} + \phi_{33} + \phi_{44}) = {} & (\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2) - (\psi_1^2 + \psi_2^2 + \psi_3^2 + \psi_4^2) \\ & -(\chi_1^2 + \chi_2^2 + \chi_3^2 + \chi_4^2) \\ & -2(\psi_1\chi_2 - \psi_2\chi_1 + \psi_4\chi_3 - \psi_3\chi_4),\end{aligned} \tag{2.4a}$$

$$\begin{aligned}\phi(\psi_{11} + \psi_{22} + \psi_{33} + \psi_{44}) = {} & 2(\phi_1\psi_1 + \phi_2\psi_2 + \phi_3\psi_3 + \phi_4\psi_4) \\ & +2(\phi_1\chi_2 - \phi_2\chi_1 + \phi_4\chi_3 - \phi_3\chi_4),\end{aligned} \tag{2.4b}$$

$$\begin{aligned}\phi(\chi_{11} + \chi_{22} + \chi_{33} + \chi_{44}) = {} & 2(\phi_1\chi_1 + \phi_2\chi_2 + \phi_3\chi_3 + \phi_4\chi_4) \\ & +2(\phi_2\psi_1 - \phi_1\psi_2 + \phi_3\psi_4 - \phi_4\psi_3),\end{aligned} \tag{2.4c}$$

where $\rho = \psi + i\chi, \phi_1 \equiv \partial\phi/\partial x^1, \phi_{11} = \partial^2\phi/\partial x^2$ etc.

3. Painlevé test for integrability of eq. (2.4) in the sense of Weiss *et al*

Jimbo *et al* [1] analysed eqs (2.1) for the Painlevé test. They did not attempt to obtain any explicit solution from that analysis. Furthermore, they analysed the situation with special form of the singularity manifold. Here we analyse eqs (2.4) with arbitrary singularity manifold and obtain exact solutions from the analysis itself. One important feature about the solutions presented here is that they are free from critical singularities. For the theoretical background of Painlevé analysis one can consult refs [11–18]. Here we proceed with the basic algorithm.

For eqs (2.4) we define the singularity manifold given by

$$u = u(x^1, x^2, x^3, x^4) = 0 \tag{3.1}$$

and set

$$\phi = u^\alpha \sum_{j=0}^{\infty} \phi_j u^j, \quad \psi = u^\beta \sum_{j=0}^{\infty} \psi_j u^j, \quad \chi = u^\gamma \sum_{j=0}^{\infty} \chi_j u^j, \tag{3.2}$$

where $\phi(x^1, x^2, x^3, x^4), \psi(x^1, x^2, x^3, x^4), \chi(x^1, x^2, x^3, x^4)$ are a set of solutions of (2.4); ϕ_j, ψ_j, χ_j are all analytic functions of (x^1, x^2, x^3, x^4) in the neighborhood of the manifold (3.1); $\phi_0 \neq 0, \psi_0 \neq 0, \chi_0 \neq 0$.

The test may be divided into three main steps after the substitution of (3.2) in the differential equations concerned, i.e. (2.4):

(I) Make the leading order analysis (where one gets all possible $\alpha, \beta, \gamma, \phi_0, \psi_0$ and χ_0 in (3.2)).
(II) Define the recursion relations for u_j for leading orders obtained in Step I and determine the resonance positions (those values of j for which some or all of the relations are not defined).
(III) Check whether the expansions allow requisite number of arbitrary functions at the resonance positions.

3.1 *Leading order analysis*

We assume

$$\phi \sim \phi_0 u^{\alpha}, \quad \psi \sim \psi_0 u^{\beta}, \quad \chi \sim \chi_0 u^{\gamma}. \tag{3.3}$$

We substitute (3.3) in (2.4) and equate the coefficients of the negative powers of u (considering that all α, β and γ are negative). This leads to

$$\alpha = -1, \quad \beta = -1, \quad \gamma = -1 \tag{3.4a}$$

and

$$\phi_0^2 + \psi_0^2 + \chi_0^2 = 0 \tag{3.4b}$$

so that

$$\phi = u^{-1} \sum_{j=0}^{\infty} \phi_j u^j, \quad \psi = u^{-1} \sum_{j=0}^{\infty} \psi_j u^j, \quad \chi = u^{-1} \sum_{j=0}^{\infty} \chi_j u^j, \tag{3.5a}$$

where

$$\phi_0^2 + \psi_0^2 + \chi_0^2 = 0. \tag{3.5b}$$

One can notice the similarity between eq. (3.5b) obtained here and eq. (3) of Jimbo *et al* [1]. Again, all the terms in their case equivalent to α, β and γ were equal to -1 as has been obtained here.

3.2 *Resonance positions*

We directly substitute (3.5) in (2.4). We have not written explicitly the recursion relations because of their involved structure. In order to have an idea of the style of writing one can consult the recursion relations of the work of Chanda and Roy-chowdhury [18].

Here the resonance positions are

$$R = -1, 0, 0, 1, 1, 2. \tag{3.6}$$

For Jimbo *et al* [1] the resonance positions were exactly the same as those given in (3.6).

(i) $R = -1$ indicates that the singularity manifold defined in (3.1) is required to be arbitrary.
(ii) $R = 0, 0$ indicate that any two of the coefficients ϕ_0, ψ_0 and χ_0 are required to be arbitrary.
(iii) $R = 1, 1$ indicate that any two of the coefficients ϕ_1, ψ_1 and χ_1 are required to be arbitrary.
(iv) $R = 2$ indicates that any one of the coefficients ϕ_2, ψ_2 and χ_2 are required to be arbitrary.

3.3 *To check whether the expansions allow requisite number of arbitrary functions at the resonance positions*

(i) The singularity manifold $u = 0$, by definition, is arbitrary.
(ii) There is only one equation involving ϕ_0, ψ_0, χ_0 which is given by (3.5b). Hence any two of the coefficients ϕ_0, ψ_0, χ_0 are arbitrary.
(iii) We get only one equation involving $\phi_1, \psi_1, \chi_1, \phi_0, \psi_0, \chi_0$. Hence two of the coefficients ϕ_1, ψ_1 and χ_1 can be kept arbitrary when the third is determined in terms of those two arbitrary functions and ϕ_0, ψ_0, χ_0.
(iv) We get only two equations involving $\phi_2, \psi_2, \chi_2, \phi_1, \psi_1, \chi_1, \phi_0, \psi_0, \chi_0$. Hence only one of the coefficients ϕ_2, ψ_2, χ_2 can be kept arbitrary. The second and the third are determined in terms of that arbitrary function and $\phi_1, \psi_1, \chi_1, \phi_0, \psi_0, \chi_0$.

With these observations one can conclude that eqs (2.4) pass the Painlevé test for integrability in the sense of Weiss *et al.*

4. Truncation of the series in eq. (3.2), auto-Backlund transformation and exact solutions

Here we forcefully make the coefficients ϕ_j, ψ_j, χ_j of u^{j-1} in the expansion (3.5a) zero for $j > 1$.

The coefficients ϕ_1, ψ_1 and ψ_1 in (3.5a) are rewritten as p, q, r respectively in order to differentiate them from $\partial\phi/\partial x^1$, $\partial\psi/\partial x^1$, $\partial\chi/\partial x^1$ etc.

Then from (3.5a) one gets

$$\phi = \phi_0 u^{-1} + p, \quad \psi = \psi_0 u^{-1} + q, \quad \chi = \chi_0 u^{-1} + r. \tag{4.1}$$

Putting these in (2.4) one gets

$$Au^{-4} + Cu^{-3} + Eu^{-2} + Fu^{-1} + G = 0, \tag{4.2a}$$

$$A'u^{-4} + C'u^{-3} + E'u^{-2} + F'u^{-1} + G' = 0, \tag{4.2b}$$

$$A''u^{-4} + C''u^{-3} + E''u^{-2} + F''u^{-1} + G'' = 0, \tag{4.2c}$$

where

$$A = (u_1^2 + u_2^2 + u_3^2 + u_4^2)(\phi_0^2 + \psi_0^2 + \chi_0^2), \tag{4.2.1a}$$

$$\begin{aligned} C = [&2\phi_0 p(u_1^2+u_2^2+u_3^2+u_4^2) - \phi_0^2(u_{11}+u_{22}+u_{33}+u_{44}) \\ &-2\psi_0(\psi_{01}u_1+\psi_{02}u_2+\psi_{03}u_3+\psi_{04}u_4) \\ &-2\chi_0(\chi_{01}u_1+\chi_{02}u_2+\chi_{03}u_3+\chi_{04}u_4) \\ &+2\psi_0(\chi_{01}u_2-\chi_{02}u_1-\chi_{03}u_4+\chi_{04}u_3) \\ &-2\chi_0(\psi_{01}u_2-\psi_{02}u_1-\psi_{03}u_4+\psi_{04}u_3)], \end{aligned} \tag{4.2.1b}$$

$$\begin{aligned} E = [&\phi_0(\phi_{011}+\phi_{022}+\phi_{033}+\phi_{044}) \\ &-2p(\phi_{01}u_1+\phi_{02}u_2+\phi_{03}u_3+\phi_{04}u_4) \\ &-(\phi_{01}^2+\phi_{02}^2+\phi_{03}^2+\phi_{04}^2)+(\psi_{01}^2+\psi_{02}^2+\psi_{03}^2+\psi_{04}^2) \\ &+(\chi_{01}^2+\chi_{02}^2+\chi_{03}^2+\chi_{04}^2)-\phi_0 p(u_{11}+u_{22}+u_{33}+u_{44}) \\ &+2\phi_0(u_1p_1+u_2p_2+u_3p_3+u_4p_4) \\ &-2\psi_0(u_1q_1+u_2q_2+u_3q_3+u_4q_4) \\ &-2\chi_0(u_1r_1+u_2r_2+u_3r_3+u_4r_4) \\ &-2\psi_0(u_1r_2-u_2r_1-u_3r_4+u_4r_3) \\ &+2\chi_0(u_1q_2-u_2q_1-u_3q_4+u_4q_3) \\ &+2(\psi_{01}\chi_{02}-\psi_{02}\chi_{01}-\psi_{03}\chi_{04}+\psi_{04}\chi_{03}) \\ &-2p(\phi_{01}u_1+\phi_{02}u_2+\phi_{03}u_3+\phi_{04}u_4)], \end{aligned} \tag{4.2.1c}$$

$$\begin{aligned} F = [&\phi_0(p_{11}+p_{22}+p_{33}+p_{44})+p(\phi_{011}+\phi_{022}+\phi_{033}+\phi_{044}) \\ &-2(\phi_{01}p_1+\phi_{02}p_2+\phi_{03}p_3+\phi_{04}p_4) \\ &+2(\phi_{01}q_1+\psi_{02}q_2+\psi_{03}q_3+\psi_{04}q_4) \\ &+2(\chi_{01}r_1+\chi_{02}r_2+\chi_{03}r_3+\chi_{04}r_4) \\ &+2(\psi_{01}r_2-\psi_{02}r_1-\psi_{03}r_4+\psi_{04}r_3) \\ &-2(\chi_{01}q_2-\chi_{02}q_1-\chi_{03}q_4+\chi_{04}q_3)], \end{aligned} \tag{4.2.1d}$$

$$\begin{aligned} G = &p(p_{11}+p_{22}+p_{33}+p_{44})-(p_1^2+p_2^2+p_3^2+p_4^2) \\ &+(q_1^2+q_2^2+q_3^2+q_4^2)+(r_1^2+r_2^2+r_3^2+r_4^2) \\ &+2(q_1r_2-q_2r_1-q_3r_4+q_4r_3)]. \end{aligned} \tag{4.2.1e}$$

Also,

$$A' = 0, \tag{4.2.2a}$$

$$\begin{aligned} C' = &2\psi_0 p(u_1^2+u_2^2+u_3^2+u_4^2)-\phi_0\psi_0(u_{11}+u_{22}+u_{33}+u_{44}) \\ &+2\psi_0(\phi_{01}u_1+\phi_{02}u_2+\phi_{03}u_3+\phi_{04}u_4) \\ &-2\phi_0(\chi_{01}u_2-\chi_{02}u_1-\chi_{03}u_4+\chi_{04}u_3) \\ &+2\chi_0(\phi_{01}u_2-\phi_{02}u_1-\phi_{03}u_4+\phi_{04}u_3), \end{aligned} \tag{4.2.2b}$$

$$E' = \phi_0(\psi_{011} + \psi_{022} + \psi_{033} + \psi_{044}) \\ -2p(\psi_{01}u_1 + \psi_{02}u_2 + \psi_{03}u_3 + \psi_{04}u_4) \\ -\psi_0 p(u_{11} + u_{22} + u_{33} + u_{44}) \\ +2\phi_0(u_1q_1 + u_2q_2 + u_3q_3 + u_4q_4) \\ -2(\phi_{01}\psi_{01} + \phi_{02}\psi_{02} + \phi_{03}\psi_{03} + \phi_{04}\psi_{04}) \\ +2\psi_0(u_1p_1 + u_2p_2 + u_3p_3 + u_4p_4) \\ +2\phi_0(u_1r_2 - u_2r_1 - u_3r_4 + u_4r_3) \\ -2\chi_0(u_1p_2 - u_2p_1 - u_3p_4 + u_4p_3) \\ -2(\phi_{01}\chi_{02} - \phi_{02}\chi_{01} - \phi_{03}\chi_{04} + \phi_{04}\chi_{03}), \tag{4.2.2c}$$

$$F' = \phi_0(q_{11} + q_{22} + q_{33} + q_{44}) + p(\psi_{011} + \psi_{022} + \psi_{033} + \psi_{044}) \\ -2(\phi_{01}q_1 + \phi_{02}q_2 + \phi_{03}q_3 + \phi_{04}q_4) \\ -2(\psi_{01}p_1 + \psi_{02}p_2 + \psi_{03}p_3 + \psi_{04}p_4) \\ -2(\phi_{01}r_2 - \phi_{02}r_1 - \phi_{03}r_4 + \phi_{04}r_3) \\ +2(\chi_{01}p_2 - \chi_{02}p_1 - \chi_{03}p_4 + \chi_{04}p_3), \tag{4.2.2d}$$

$$G' = p(q_{11} + q_{22} + q_{33} + q_{44}) - 2(p_1r_2 - p_2r_1 - p_3r_4 + p_4r_3) \\ -2(p_1q_1 + p_2q_2 + p_3q_3 + p_4q_4). \tag{4.2.2e}$$

And

$$A'' = 0, \tag{4.2.3a}$$

$$C'' = 2\chi_0 p(u_1^2 + u_2^2 + u_3^2 + u_4^2) - \phi_0\chi_0(u_{11} + u_{22} + u_{33} + u_{44}) \\ +2\chi_0(\phi_{01}u_1 + \phi_{02}u_2 + \phi_{03}u_3 + \phi_{04}u_4) \\ +2\phi_0(\psi_{01}u_2 - \psi_{02}u_1 - \psi_{03}u_4 + \psi_{04}u_3) \\ -2\psi_0(\phi_{01}u_2 - \phi_{02}u_1 - \phi_{03}u_4 + \phi_{04}u_3), \tag{4.2.3b}$$

$$E'' = \phi_0(\chi_{011} + \chi_{022} + \chi_{033} + \chi_{044}) \\ -2p(\chi_{01}u_1 + \chi_{02}u_2 + \chi_{03}u_3 + \chi_{04}u_4) \\ -\chi_0 p(u_{11} + u_{22} + u_{33} + u_{44}) \\ +2\phi_0(u_1r_1 + u_2r_2 + u_3r_3 + u_4r_4) \\ -2(\phi_{01}\chi_{01} + \phi_{02}\chi_{02} + \phi_{03}\chi_{03} + \phi_{04}\chi_{04}) \\ +2\chi_0(u_1p_1 + u_2p_2 + u_3p_3 + u_4p_4) \\ -2\phi_0(u_1q_2 - u_2q_1 - u_3q_4 + u_4q_3) \\ +2\psi_0(u_1p_2 - u_2p_1 - u_3p_4 + u_4p_3) \\ +2(\phi_{01}\psi_{02} - \phi_{02}\psi_{01} - \phi_{03}\psi_{04} + \phi_{04}\psi_{03}), \tag{4.2.3c}$$

$$F'' = \phi_0(r_{11} + r_{22} + r_{33} + r_{44}) + p(\chi_{011} + \chi_{022} + \chi_{033} + \chi_{044}) \\ -2(\phi_{01}r_1 + \phi_{02}r_2 + \phi_{03}r_3 + \phi_{04}r_4) \\ -2(\chi_{01}p_1 + \chi_{02}p_2 + \chi_{03}p_3 + \chi_{04}p_4) \\ +2(\phi_{01}q_2 - \phi_{02}q_1 - \phi_{03}q_4 + \phi_{04}q_3) \\ -2(\psi_{01}p_2 - \psi_{02}p_1 - \psi_{03}p_4 + \psi_{04}p_3), \tag{4.2.3d}$$

$$G'' = p(r_{11} + r_{22} + r_{33} + r_{44}) + 2(p_1q_2 - p_2q_1 - p_3q_4 + p_4q_3) \\ -2(p_1r_1 + p_2r_2 + p_3r_3 + p_4r_4). \tag{4.2.3e}$$

As has been stated in §3.3, the equations $C = 0, C' = 0, C'' = 0$ are equivalent to only one equation and $E = 0, E' = 0, E'' = 0$ are equivalent to two equations subject to the condition given by $\phi_0^2 + \psi_0^2 + \chi_0^2 = 0$.

Normally these conditions are written as overdetermined system by equating the coefficients of u^{-j} separately to zero.

But, here if we do so we get

$$\phi_0^2 + \psi_0^2 + \chi_0^2 = 0 \tag{4.3}$$

which is valid only when ϕ_0, ψ_0 and χ_0 are complex quantities. At the time of checking the existence of Painlevé property we obtained the same result, (i.e. (4.3)), at the leading order stage. The same kind of expression was obtained by Jimbo *et al* (see eq. (3) of [1]). That does not create any problem in the infinite Laurent-like expansion (3.2). However, when we truncate the series we must make sure that ultimately ϕ, ψ and χ are real.

It has been shown in the Appendix that if we start with the coefficients of u equal to zero and all p, q, r equal to zero then we are left with solutions of ϕ, ψ and χ which involve the complex quantity $\sqrt{-1}$. This happens even when all p, q, r are not equal to zero.

In the following we show that two types of solutions obtained previously by Chakraborty and Chanda [6], De and Ray [20] and Chanda and Ray [21] can be rediscovered from (4.1) and (4.2).

4.1 *Solutions reported by Chakraborty and Chanda* [6]

Here the solutions can be obtained from (4.1) and (4.2) with

$$p = 0, \quad q = 0, \quad r = 0 \tag{4.4}$$

and by the assumption that

$$(\phi_0/u) = a(v), \quad (\psi_0/u) = h(v), \quad (\chi_0/u) = c(v), \tag{4.5}$$

where a, h, c are functions of v and $v = v(x^1, x^2, x^3, x^4)$.

If one uses the expressions (4.4) and (4.5) in (4.1) one arrives exactly at

$$\phi = a(v), \quad \psi = h(v), \quad \chi = \chi(v)$$

which is the same as eq. (2.1) of Chakraborty and Chanda [6].

It can be checked that starting from (4.2) with $p = 0, q = 0, r = 0$ and (4.4) one can also arrive at eq. (2.1) of Chakraborty and Chanda [6].

4.2 *Solutions reported by Chanda and Ray* [21]

Here the solutions can be obtained from (4.1) and (4.2) by assuming that $p \neq 0, q \neq 0, r \neq 0$

$$(\phi_0/u) = a'(p), \quad (\psi_0/u) = h'(p), \quad (\chi_0/u) = c'(p), \tag{4.6}$$

where a', h', c' are functions of p. Hence we get from (4.1)

$$\phi = a'(p) + p = a(p), \quad \text{where } a \text{ is another function of } p, \tag{4.7a}$$

$$\psi = h'(p) + q, \tag{4.7b}$$

$$\chi = c'(p) + r, \tag{4.7c}$$

where (p, q, r) and (ϕ, ψ, χ) are two separate sets of solutions of eqs (2.4).

Yang [3] noted that if p, q, r are three functions of x^1, x^2, x^3 and x^4 such that

$$q_1 + r_2 = p_3, \tag{4.8a}$$

$$q_2 - r_1 = p_4, \tag{4.8b}$$

$$q_3 - r_4 = -p_1, \tag{4.8c}$$

$$q_4 + r_3 = -p_2, \tag{4.8d}$$

then a set of solutions (2.4) is obtained by setting

$$\phi = p, \quad \psi = q, \quad \chi = r. \tag{4.9}$$

Yang [3] indicated some particular solutions of (4.8) as well.

Chanda and Ray [21] obtained some new solutions using

$$\phi = a''(p), \tag{4.10a}$$

$$\psi = h''(p) + Mq - Nr, \tag{4.10b}$$

$$\chi = c''(p) + Mr + Nq, \tag{4.10c}$$

where (i) M and N are real arbitrary constants, (ii) p, q and r are functions of (x^1, x^2, x^3, x^4) and satisfy (4.8) and (iii) a'', h'', c'' are functions of p.

It is easy to identify that (4.7) is a particular situation of (4.10) where $M = 1, N = 0, a'' = a', b'' = b', c'' = c'$. It may be checked that with eqs (4.7)–(4.9) one can arrive from eq. (4.2) to eq. (2.4) of Chanda and Ray [21] through straightforward manipulation. Thus, eqs (4.7) with (4.8) can be treated as an auto-Backlund transformation between two sets of solutions where one set of solutions (p, q, r) are given by (4.8) and another set of solutions (ϕ, ψ, χ) are given by eqs (3) of Chanda and Ray [21] with $A = 1, B = 0$ in eq. (2.4) of the same paper.

5. Summary

Yang's self-dual equations for $SU(2)$ gauge fields (expressed in real form; eqs (2.4)) pass the Painlevé test for integrability in the sense of Weiss *et al.* Jimbo *et al* analysed the complex form (2.1) of the same equations and found that the equations pass the Painlevé test for integrability. However, they did not show whether eqs (2.1) allowed truncation of series. Here we have shown that eqs (2.4), i.e. the real form of (2.1), admit truncation of series. The truncation procedure leads to previously obtained two pairs non-trivial exact solutions and the auto-Backlund transformation between those pairs solutions. Another important aspect of our analysis is that Jimbo *et al* analysed the equation with a restricted nature of the singularity manifold. Here we have analysed the equation keeping the singularity manifold completely general.

Appendix

Let us start from the trivial solution $p = 0, q = 0, r = 0$. From (4.1), one gets

$$\phi = \phi_0 u^{-1}, \quad \psi = \psi_0 u^{-1}, \quad \chi = \chi_0 u^{-1}. \tag{A.1}$$

Equating the different coefficients of u^{-j} in (4.2) separately to zero one finally gets

$$\phi_0^2 + \psi_0^2 + \chi_0^2 = 0, \tag{A.2a}$$

$$\begin{aligned}\phi_0(u_{11} + u_{22} + u_{33} + u_{44}) &= 2(\phi_{01}u_1 + \phi_{02}u_2 + \phi_{03}u_3 + \phi_{04}u_4) + (2u_2/\psi_0)(\chi_0\phi_{01} - \phi_0\chi_{01}) \\ &\quad + (2u_1/\psi_0)(\phi_0\chi_{02} - \chi_0\phi_{02}) + (2u_4/\psi_0)(\phi_0\chi_{03} - \chi_0\phi_{03}),\end{aligned} \tag{A.2b}$$

$$\begin{aligned}\phi_0(\psi_{011} + \psi_{022} + \psi_{033} + \psi_{044}) &= 2(\phi_{01}\psi_{01} + \phi_{02}\psi_{02} + \phi_{03}\psi_{03} + \phi_{04}\psi_{04}) \\ &\quad + 2(\phi_{01}\chi_{02} - \phi_{02}\chi_{01} - \phi_{03}\chi_{04} + \phi_{04}\chi_{03}),\end{aligned} \tag{A.2c}$$

$$\begin{aligned}\phi_0(\chi_{011} + \chi_{022} + \chi_{033} + \chi_{044}) &= 2(\phi_{01}\chi_{01} + \phi_{02}\chi_{02} + \phi_{03}\chi_{03} + \phi_{04}\chi_{04}) \\ &\quad + 2(\psi_{01}\phi_{02} - \psi_{02}\phi_{01} - \psi_{03}\phi_{04} - \psi_{04}\phi_{03}).\end{aligned} \tag{A.2d}$$

In (A.2) there are four equations and four unknowns, namely ϕ_0, ψ_0, χ_0 and u. Let us assume, again,

$$\psi_0 = \chi_0 \tag{A.3}$$

when we get (as a result of the requirement of consistency between (A.2c) and (A.2d))

$$\phi_0^2 = -2\psi_0^2, \tag{A.4a}$$

$$\begin{aligned}\phi_0(u_{11} + u_{22} + u_{33} + u_{44}) &= 2(\phi_{01}u_1 + \phi_{02}u_2 + \phi_{03}u_3 + \phi_{04}u_4) \\ &\quad - (2u_2/\psi_0)(\psi_0\phi_{01} - \phi_0\psi_{01}) - (2u_1/\psi_0)(\phi_0\psi_{02} - \psi_0\phi_{02}) \\ &\quad - (2u_3/\psi_0)(\psi_0\phi_{04} - \phi_0\psi_{04}) - (2u_4/\psi_0)(\phi_0\psi_{03} - \psi_0\phi_{03}),\end{aligned} \tag{A.4b}$$

$$\phi_0(\psi_{011} + \psi_{022} + \psi_{033} + \psi_{044})$$
$$= 2(\phi_{01}\psi_{01} + \phi_{02}\psi_{02} + \phi_{03}\psi_{03} + \phi_{04}\psi_{04}), \quad \text{(A.4c)}$$

$$\phi_{01}\psi_{02} - \phi_{02}\psi_{01} + \phi_{04}\psi_{03} - \phi_{03}\psi_{04} = 0. \quad \text{(A.4d)}$$

With a little manipulation, eqs (A.4) lead to

$$\phi_0^2 = -2\psi_0^2, \quad \text{(A.5a)}$$

$$\phi_0(u_{11} + u_{22} + u_{33} + u_{44}) = 2(\phi_{01}u_1 + \phi_{02}u_2 + \phi_{03}u_3 + \phi_{04}u_4) \quad \text{(A.5b)}$$

$$\phi_0(\psi_{011} + \psi_{022} + \psi_{033} + \psi_{044})$$
$$= 2(\phi_{01}\psi_{01} + \phi_{02}\psi_{02} + \phi_{03}\psi_{03} + \phi_{04}\psi_{04}). \quad \text{(A.5c)}$$

Again we have three equations and three unknowns. From (A.5a) and (A.5c) one can write

$$\phi_0 = (\sqrt{2}i)/\zeta, \quad \text{(A.6a)}$$

$$\psi_0 = 1/\zeta, \quad \text{(A.6b)}$$

where ζ satisfies $\zeta_{11} + \zeta_{22} + \zeta_{33} + \zeta_{44} = 0$.

Using (A.6) in (A.5) we get

$$u = \int (\mathrm{d}U/\zeta^2), \quad \text{(A.7)}$$

where U satisfies $U_{11} + U_{22} + U_{33} + U_{44} = 0$.

Hence finally one can conclude that

$$\phi = (2i/\zeta) \bigg/ \left(\int \mathrm{d}U/\zeta^2\right), \quad \text{(A.8a)}$$

$$\psi = (1/\zeta) \bigg/ \left(\int \mathrm{d}U/\zeta^2\right), \quad \text{(A.8b)}$$

$$\chi = (1/\zeta) \bigg/ \left(\int \mathrm{d}U/\zeta^2\right), \quad \text{(A.8c)}$$

where ζ and U satisfy

$$\zeta_{11} + \zeta_{22} + \zeta_{33} + \zeta_{44} = 0 \quad \text{(A.9a)}$$

and

$$U_{11} + U_{22} + U_{33} + U_{44} = 0 \quad \text{(A.9b)}$$

respectively. Equations (A.8) demand that

$$U = W(\zeta), \tag{A.10}$$

where W is a function of ζ.

By using (A.10), eqs (A.9) reduce to

$$\zeta_{11} + \zeta_{22} + \zeta_{33} + \zeta_{44} = 0 \tag{A.11a}$$

$$\zeta_1^2 + \zeta_2^2 + \zeta_3^2 + \zeta_4^2 = 0 \tag{A.11b}$$

which has the solution

$$\zeta = Px^1 + Qx^2 + Rx^3 + Sx^4 + T. \tag{A.12}$$

Here P, Q, R, S, T are all arbitrary constants. Finally, ϕ, ψ and χ are given by (A.8), (A.10) and (A.12).

Obviously another solution of (A.9) can be written as

$$U = H\zeta, \tag{A.13}$$

where H is an arbitrary constant. With (A.13) one can arrive from (A.8) at

$$\phi = (2Hi/\zeta)(\ln \zeta) \tag{A.14a}$$

$$\psi = (H/\zeta)(\ln \zeta) \tag{A.14b}$$

$$\chi = (H/\zeta)(\ln \zeta). \tag{A.14c}$$

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