PRAMANA — journal of physics
© Indian Academy of Sciences
Vol. 66, No. 3
March 2006
pp. 601–607

Construction of exact dynamical invariants of two-dimensional classical system

S C MISHRA and FAKIR CHAND
Department of Physics, Kurukshetra University, Kurukshetra 136 119, India
E-mail: subash_kuk@rediffmail.com; fchand_kuk@yahoo.com

MS received 29 April 2005; revised 2 December 2005; accepted 29 December 2005

Abstract. A general method is used for the construction of second constant of motion of fourth order in momenta using the complex coordinates $(z, \bar{z})$. A fourth-order potential equation is obtained whose solutions directly provide a large class of integrable systems. The potential equation is tested with an interesting example which admits second constants of motion.

Keywords. Classical invariant; integrable systems; second constant of motion.

PACS No. 02.30.IK

1. Introduction

Integrability is considered as a mathematical property that can be successfully used to obtain more predictive power and quantitative information to understand the dynamics of the system globally [1–8]. Whittaker [9] first investigated to construct an invariant (other than total energy), which is called the second constant of motion for a system whose equations of motion are, $\ddot{x}_1 = -(\partial V/\partial x_1)$, $\ddot{x}_2 = -(\partial V/\partial x_2)$, where $V = V(x_1, x_2)$.

A classical Hamiltonian system in n dimensions is said to be classical integrable if there exist $(n-1)$ independent, well-defined global functions whose Poisson brackets with each other and with Hamiltonian vanish. The utility of second- or higher-order invariants, if constructed for a system, may reduce some nonlinear dynamical problems to a quadrature [2,4,5,9]. These invariants are also used to solve several problems of plasma physics [10], hydrodynamics and in the study of classical analogue of Yang–Mills field equations with reference to the generation of potentials for both time-independent and time-dependent systems by choosing suitable gauges [11]. Higher-order invariants provide internal symmetry of a physical system particularly in molecular dynamics [12].

Construction of invariants using Cartesian coordinates are frequently employed in literature [1–3,13]. However, not much work has been done for the construction of invariants using complex coordinates [4–6,14]. Complexification of coordinates can be equally well used to study integrability of dynamical systems and it can make the system more symmetric and appealing in some cases than the original. A lot of simplifications were achieved in the derivation in time-independent classical systems in two dimensions in comparison to time-dependent classical systems [6,14].

In this paper we consider complex coordinates $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$. Some simplifications have been achieved in the derivation and analysis turns out to be more transparent. This complexification method is found not only to produce the known results but also led us to suggest several new integrable systems which perhaps could not be obtained otherwise. We obtain the most general form of fourth-order potential equation (corresponding to fourth-order invariants) whose solutions may directly provide the integrable systems. Here we examine time-independent systems for the potential of the type $V(z,\bar{z}) = lz^4 + mz^2\bar{z}^2 + n\bar{z}^4$, where l, m and n are arbitrary constants. By using rationalisation method, a simple analysis reveals an interesting case of two-coupled oscillators.

In this note we derive the method in §2. Examples and conclusions are given in subsequent sections.

2. Method

We consider a dynamical system described by the Lagrangian

$$L = \frac{1}{2}|\dot{z}|^2 - V(z,\bar{z}), \tag{2.1}$$

where $z = x_1 + ix_2$ and $\bar{z} = x_1 - ix_2$, with the concomitant equations of motion

$$\ddot{z} = -2\frac{\partial V}{\partial \bar{z}}, \quad \ddot{\bar{z}} = -2\frac{\partial V}{\partial z}. \tag{2.2}$$

Let us assume a fourth-order invariant in momenta

$$I = a_0 + \frac{1}{2!}a_{ij}\xi_i\xi_j + \frac{1}{4!}a_{ijkl}\xi_i\xi_j\xi_k\xi_l, \tag{2.3}$$

where $i,j,k,l = 1,2, \xi_1 = \dot{z}, \xi_2 = \dot{\bar{z}}$ and a_0, a_{ij}, a_{ijkl} are functions of $z, \bar{z}$ only. The invariant I implies $(\mathrm{d}I/\mathrm{d}t) = 0$. So using eq. (2.3) we get

$$\begin{aligned} &a_{0,i}\xi_i + \tfrac{1}{2}a_{ij,k}\xi_i\xi_j\xi_k + \tfrac{1}{2}a_{ij}(\dot{\xi}_i\xi_j + \xi_i\dot{\xi}_j) + \tfrac{1}{24}a_{ijkl,m}\xi_i\xi_j\xi_k\xi_l\xi_m \\ &+\tfrac{1}{24}a_{ijkl}(\dot{\xi}_i\xi_j\xi_k\xi_l + \xi_i\dot{\xi}_j\xi_k\xi_l + \xi_i\xi_j\dot{\xi}_k\xi_l + \xi_i\xi_j\xi_k\dot{\xi}_l) = 0. \end{aligned} \tag{2.4}$$

After accounting for the proper symmetrisation of the coefficients and since eq. (2.4) must hold identically in ξ_i's, we get the following equations:

$$a_{ijkl,m} + a_{jklm,i} + a_{klmi,j} + a_{lmij,k} + a_{mijk,l} = 0, \tag{2.5}$$

$$a_{ij,k} + a_{jk,i} + a_{ki,j} + a_{ijkl}\dot{\xi}_l = 0 \quad \text{and} \quad a_{0,i} + a_{ij}\dot{\xi}_j = 0. \tag{2.6}$$

Equations (2.5) and (2.6) correspond to the following set of partial differential equations:

$$\frac{\partial a_{1111}}{\partial z} = 0; \quad \frac{\partial a_{1111}}{\partial \bar{z}} + 4\frac{\partial a_{1112}}{\partial z} = 0; \tag{2.7a,b}$$

$$3\frac{\partial a_{1122}}{\partial z} + 2\frac{\partial a_{1112}}{\partial \bar{z}} = 0; \quad 3\frac{\partial a_{1122}}{\partial \bar{z}} + 2\frac{\partial a_{1222}}{\partial z} = 0; \tag{2.7c,d}$$

$$4\frac{\partial a_{1222}}{\partial \bar{z}} + \frac{\partial a_{2222}}{\partial z} = 0; \quad \frac{\partial a_{2222}}{\partial \bar{z}} = 0. \tag{2.7e,f}$$

$$3\frac{\partial a_{11}}{\partial z} = 2a_{1111}\frac{\partial V}{\partial \bar{z}} + 2a_{1112}\frac{\partial V}{\partial z}; \quad 3\frac{\partial a_{22}}{\partial \bar{z}} = 2a_{2222}\frac{\partial V}{\partial z} + 2a_{1222}\frac{\partial V}{\partial \bar{z}}; \tag{2.8a, b}$$

$$\frac{\partial a_{11}}{\partial \bar{z}} + 2\frac{\partial a_{12}}{\partial z} = 2a_{1112}\frac{\partial V}{\partial \bar{z}} + 2a_{1122}\frac{\partial V}{\partial z}; \frac{\partial a_{22}}{\partial z} + 2\frac{\partial a_{12}}{\partial \bar{z}} = 2a_{1222}\frac{\partial V}{\partial z} + 2a_{1122}\frac{\partial V}{\partial \bar{z}}; \tag{2.9a, b}$$

$$\frac{\partial a_0}{\partial z} = 2a_{11}\frac{\partial V}{\partial \bar{z}} + 2a_{12}\frac{\partial V}{\partial z}; \quad \frac{\partial a_0}{\partial \bar{z}} = 2a_{12}\frac{\partial V}{\partial \bar{z}} + 2a_{22}\frac{\partial V}{\partial z}. \tag{2.10a, b}$$

To solve these coupled equations, consider eqs (2.7a) and (2.7f), which give $a_{1111} = a_{1111}(\bar{z}) = \sigma_1(\bar{z})$ and $a_{2222} = a_{2222}(z) = \chi_1(z)$. Now differentiate eq. (2.7b) with respect to z and using eq. (2.7a), we have

$$a_{1112} = \sigma_2(\bar{z})z + \sigma_3(\bar{z}). \tag{2.11}$$

Exactly, from eqs (2.7e) and (2.7f) we get

$$a_{1222} = \chi_2(z)\bar{z} + \chi_3(z). \tag{2.12}$$

On differentiating eq. (2.7c) with respect to $\bar{z}$, and eq. (2.7d) with respect to z, and on correspondingly subtracting the results and making use of eqs (2.11) and (2.12), we obtain

$$z\frac{\partial^2 \sigma_2}{\partial \bar{z}^2} + \frac{\partial^2 \sigma_3}{\partial \bar{z}^2} = \bar{z}\frac{\partial^2 \chi_2}{\partial z^2} + \frac{\partial^2 \chi_3}{\partial z^2}. \tag{2.13}$$

Now we set $\sigma_3 = D_1$ and $\chi_3 = D_2$ (note that here σ_i's and χ_i's are functions of only $\bar{z}$ and z, respectively, and D_i's are arbitrary integration constants), eq. (2.13) reduces to

$$\frac{1}{\bar{z}}\frac{\partial^2 \sigma_2}{\partial \bar{z}^2} = \frac{1}{z}\frac{\partial^2 \chi_2}{\partial z^2} = D_3 \quad \text{or} \quad \sigma_2 = \frac{1}{6}D_3\bar{z}^3 + D_4\bar{z} + D_5, \tag{2.14}$$

and

$$\chi_2 = \tfrac{1}{6}D_3 z^3 + D_6 z + D_7. \tag{2.15}$$

Similarly, we find σ_1 and χ_1 from eqs (2.7b) and (2.7e) respectively as

$$\sigma_1 = -\tfrac{1}{6}D_3\bar{z}^4 - 2D_4\bar{z}^2 - 4D_5\bar{z} + D_8, \tag{2.16}$$

$$\chi_1 = -\tfrac{1}{6}D_3 z^4 - 2D_6 z^2 - 4D_7 z + D_9. \tag{2.17}$$

Equation (2.7c) yields $\frac{\partial a_{1122}}{\partial z} = -\frac{2}{3}z\frac{\partial \sigma_2}{\partial \bar{z}} = -\frac{1}{3}D_3 z\bar{z}^2 - \frac{2}{3}D_4 z$, which gives on integration

$$a_{1122} = -\tfrac{1}{6}D_3 z^2\bar{z}^2 - \tfrac{1}{3}D_4 z^2 + \sigma_4(\bar{z}). \tag{2.18}$$

Finally the solutions of eqs (2.7a)–(2.7f) turn to be

$$a_{1111} = -\tfrac{1}{6}D_3\bar{z}^4 - 2D_4\bar{z}^2 - 4D_5\bar{z} + D_8, \tag{2.19}$$

$$a_{2222} = -\tfrac{1}{6}D_3 z^4 - 2D_6 z^2 - 4D_7 z + D_9, \tag{2.20}$$

$$a_{1112} = \tfrac{1}{6}D_3 z\bar{z}^3 + D_4 z\bar{z} + D_5 z + D_1, \tag{2.21}$$

$$a_{1222} = \tfrac{1}{6}D_3\bar{z}z^3 + D_6 z\bar{z} + D_7\bar{z} + D_2, \tag{2.22}$$

$$a_{1122} = -\tfrac{1}{6}D_3 z^2\bar{z}^2 - \tfrac{1}{3}D_4 z^2 - \tfrac{1}{3}D_6\bar{z}^2 + D_{10}. \tag{2.23}$$

Differentiate eq. (2.8a) with respect to $\bar{z}$ and eq. (2.9a) with respect to z, and on subtracting these, we get an expression for $\partial^2 a_{12}/\partial z^2$. Similarly, an expression for $\partial^2 a_{12}/\partial \bar{z}^2$ from eqs (2.8b) and (2.9b) is obtained. Now eliminate a_{12} from the resultant expressions by taking appropriate second-order derivatives and subtracting, we finally arrive at the following equation:

$$\begin{aligned}
&\Big(\frac{\partial^3 a_{1122}}{\partial z\partial\bar{z}^2} - \frac{1}{3}\frac{\partial^3 a_{1112}}{\partial\bar{z}^3} - \frac{\partial^3 a_{1222}}{\partial\bar{z}\partial z^2} + \frac{1}{3}\frac{\partial^3 a_{2222}}{\partial z^3}\Big)\frac{\partial V}{\partial z}\\
&+\Big(\frac{\partial^2 a_{1122}}{\partial\bar{z}^2} - \frac{\partial^2 a_{1222}}{\partial z\partial\bar{z}} + \frac{\partial^2 a_{2222}}{\partial z^2}\Big)\frac{\partial^2 V}{\partial z^2}\\
&+\Big(\frac{\partial a_{2222}}{\partial z} - \frac{\partial a_{1222}}{\partial\bar{z}}\Big)\frac{\partial^3 V}{\partial z^3} + \frac{1}{3}a_{2222}\frac{\partial^4 V}{\partial z^4} + \Big(\frac{\partial a_{1122}}{\partial\bar{z}} - \frac{\partial a_{1222}}{\partial z}\Big)\frac{\partial^3 V}{\partial\bar{z}\partial z^2}\\
&-\frac{2}{3}a_{1222}\frac{\partial^4 V}{\partial\bar{z}\partial z^3} + \frac{2}{3}a_{1112}\frac{\partial^4 V}{\partial z\partial\bar{z}^3} + \Big(\frac{\partial a_{1112}}{\partial\bar{z}} - \frac{\partial a_{1122}}{\partial z}\Big)\frac{\partial^3 V}{\partial z\partial\bar{z}^2}\\
&+\Big(\frac{\partial^3 a_{1112}}{\partial z\partial\bar{z}^2} + \frac{1}{3}\frac{\partial^3 a_{1222}}{\partial z^3} - \frac{\partial^3 a_{1122}}{\partial\bar{z}\partial z^2} - \frac{1}{3}\frac{\partial^3 a_{1111}}{\partial\bar{z}^3}\Big)\frac{\partial V}{\partial\bar{z}}\\
&+\Big(\frac{\partial^2 a_{1112}}{\partial z\partial\bar{z}} - \frac{\partial^2 a_{1122}}{\partial z^2} - \frac{\partial^2 a_{1111}}{\partial\bar{z}^2}\Big)\frac{\partial^2 V}{\partial\bar{z}^2}\\
&+\Big(\frac{\partial a_{1112}}{\partial z} - \frac{\partial a_{1111}}{\partial\bar{z}}\Big)\frac{\partial^3 V}{\partial\bar{z}^3} - \frac{1}{3}a_{1111}\frac{\partial^4 V}{\partial\bar{z}^4} = 0.
\end{aligned} \tag{2.24}$$

This is a general potential equation. On substituting the coefficients a_{ijkl} from eqs (2.19)–(2.23), the potential eq. (2.24) reduces to the following form:

$$\begin{aligned}
&-\frac{10}{3}D_3 z\frac{\partial V}{\partial z} - \frac{17}{3}\Big(\frac{1}{2}D_3 z^2 + D_6\Big)\frac{\partial^2 V}{\partial z^2}\\
&-5\Big(\frac{1}{6}D_3 z^3 + D_6 z + D_7\Big)\frac{\partial^3 V}{\partial z^3}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{3}\Big(-\frac{1}{6}D_3z^4-2D_6z^2-4D_7z+D_9\Big)\frac{\partial^4 V}{\partial z^4}\\
&-\frac{5}{3}\Big(\frac{1}{2}D_3\bar{z}z^2+D_6\bar{z}\Big)\frac{\partial^3 V}{\partial\bar{z}\partial z^2}\\
&-\frac{2}{3}\Big(\frac{1}{6}D_3\bar{z}z^3+D_6z\bar{z}+D_7\bar{z}+D_2\Big)\frac{\partial^4 V}{\partial\bar{z}\partial z^3}\\
&+\frac{2}{3}\Big(\frac{1}{6}D_3z\bar{z}^3+D_4z\bar{z}+D_5z+D_1\Big)\frac{\partial^4 V}{\partial z\partial\bar{z}^3}\\
&+\frac{5}{3}\Big(\frac{1}{2}D_3z\bar{z}^2+D_4z\Big)\frac{\partial^3 V}{\partial z\partial\bar{z}^2}+\frac{10}{3}D_3\bar{z}\frac{\partial V}{\partial\bar{z}}\\
&+\frac{17}{3}\Big(\frac{1}{2}D_3\bar{z}^2+D_4\Big)\frac{\partial^2 V}{\partial\bar{z}^2}+5\Big(\frac{1}{6}D_3\bar{z}^3+D_4\bar{z}+D_5\Big)\frac{\partial^3 V}{\partial\bar{z}^3}\\
&+\frac{1}{3}\Big(\frac{1}{6}D_3\bar{z}^4+2D_4\bar{z}^2+4D_5\bar{z}-D_8\Big)\frac{\partial^4 V}{\partial\bar{z}^4}=0.
\end{aligned}
\tag{2.25}
$$

As such, solving eq. (2.25) is a difficult task, but we provide the following recipe for the construction of invariant. For a given form of $V(z,\bar{z})$ the unknown constants D_i's can be determined by rationalising eq. (2.25). Subsequently, determination of other coefficients a_0, a_{ij} from eqs (2.8a,b), (2.9a,b) and (2.10a,b) lead to the final form of second constant of motion from eq. (2.3).

3. Construction of second constant of motion

Examples

Case (*a*): Let us consider the potential of the type

$$V(z,\bar{z})=lz^4+mz^2\bar{z}^2+n\bar{z}^4. \tag{3.1}$$

On substitution of eq. (3.1) in eq. (2.25), and rationalising the result, we get, $l=n=1$ and $m=6$, with $D_3=D_4=D_6=D_7=0$. After using eqs (2.8a) and (2.8b), we get

$$a_{11}=\tfrac{2}{3}D_1\big(z^4+\bar{z}^4+6z^2\bar{z}^2\big)+\tfrac{8}{3}D_8\bar{z}z\big(z^2+\bar{z}^2\big), \tag{3.2}$$

$$a_{22}=\tfrac{2}{3}D_2\big(z^4+\bar{z}^4+6z^2\bar{z}^2\big)+\tfrac{8}{3}D_9\bar{z}z\big(z^2+\bar{z}^2\big). \tag{3.3}$$

Considering eqs (2.9a) and (2.9b), and with $D_1=D_2, D_8=D_9$, we find

$$a_{12}=-\tfrac{1}{3}\big(D_9-3D_{10}\big)\big(z^4+\bar{z}^4+6z^2\bar{z}^2\big)+\tfrac{8}{3}D_2z\bar{z}\big(z^2+\bar{z}^2\big). \tag{3.4}$$

To find the value of a_0, use eqs (2.10a) and (2.10b), and we get

$$
\begin{aligned}
a_0=&\tfrac{1}{3}\big(3D_{10}-D_9\big)\big(z^8+\bar{z}^8\big)+\tfrac{16}{3}D_2z\bar{z}\big(z^6+\bar{z}^6\big)\\
&+\tfrac{4}{3}\big(5D_9+9D_{10}\big)z^2\bar{z}^2\big(z^4+\bar{z}^4\big)\\
&+\tfrac{112}{3}D_2z^3\bar{z}^3\big(z^2+\bar{z}^2\big)+\tfrac{2}{3}\big(57D_{10}+13D_9\big)z^4\bar{z}^4.
\end{aligned}
\tag{3.5}
$$

Substituting these values of a_0, a_{ij} and a_{ijkl} in eq. (2.3), the invariant becomes

$$\begin{aligned}I(z,\bar{z}) = &-\tfrac{1}{3}\big(D_9-3D_{10}\big)\big(z^8+\bar{z}^8\big)+\tfrac{16}{3}D_2 z\bar{z}\big(z^6+\bar{z}^6\big)\\&+\tfrac{4}{3}\big(5D_9+9D_{10}\big)z^2\bar{z}^2\big(z^4+\bar{z}^4\big)\\&+\tfrac{112}{3}D_2 z^3\bar{z}^3\big(z^2+\bar{z}^2\big)+\tfrac{2}{3}\big(57D_{10}+13D_9\big)z^4\bar{z}^4\\&+\big(\tfrac{1}{3}D_2\big(z^4+\bar{z}^4+6z^2\bar{z}^2\big)+\tfrac{4}{3}D_9\bar{z}z\big(z^2+\bar{z}^2\big)\big)\big(\dot{z}^2+\dot{\bar{z}}^2\big)\\&+\big(-\tfrac{1}{3}\big(D_9-3D_{10}\big)\big(z^4+\bar{z}^4+6z^2\bar{z}^2\big)\\&+\tfrac{8}{3}D_2 z\bar{z}\big(z^2+\bar{z}^2\big)\big)\big(\dot{z}\dot{\bar{z}}\big)+\tfrac{1}{24}D_9\big(\dot{z}^4+\dot{\bar{z}}^4\big)\\&+\tfrac{1}{6}D_2\dot{z}\dot{\bar{z}}\big(\dot{z}^2+\dot{\bar{z}}^2\big)+\tfrac{1}{4}D_{10}\dot{z}^2\dot{\bar{z}}^2.\end{aligned}\tag{3.6}$$

Case (*b*): Consider the potential as

$$V(x,y) = x^4+6x^2y^2+y^4,\tag{3.7}$$

in Cartesian coordinates. This potential may be written in complex coordinates using transformations $x=\frac{1}{2}(z+\bar{z})$ and $y=\frac{1}{2\iota}(z-\bar{z})$ as

$$V(z,\bar{z}) = -\tfrac{1}{4}(z^4-6z^2\bar{z}^2+\bar{z}^4).\tag{3.8}$$

By substituting eq. (3.8) in eq. (2.25), and rationalising the result, we find, $D_{10}\neq 0$ and all other D_i's $=0$. After using eqs (2.8a) and (2.8b), we get

$$a_{11}=a_{22}=0.\tag{3.9}$$

Equations (2.9a) and (2.9b) lead us to

$$a_{12} = -\tfrac{1}{4}D_{10}(z^4-6z^2\bar{z}^2+\bar{z}^4),\tag{3.10}$$

the value of a_0, may be obtained by using eqs (2.10a) and (2.10b), i.e.

$$a_0 = \tfrac{1}{16}D_{10}\big(z^8+\bar{z}^8+38z^4\bar{z}^4-12z^2\bar{z}^2(z^4+\bar{z}^4)\big).\tag{3.11}$$

Using these values of a_0, a_{ij} and a_{ijkl} from eqs (3.9), (3.10) in eq. (2.3), and we get

$$\begin{aligned}I(z,\bar{z}) = &\tfrac{1}{16}D_{10}\big(z^8+\bar{z}^8+38z^4\bar{z}^4-12z^2\bar{z}^2(z^4+\bar{z}^4)\big)\\&-D_{10}\big(\tfrac{1}{4}(z^4-6z^2\bar{z}^2+\bar{z}^4)\big)\dot{z}\dot{\bar{z}}+\tfrac{1}{4}D_{10}\dot{z}^2\dot{\bar{z}}^2.\end{aligned}\tag{3.12}$$

4. Conclusions

Basically, this report was intended to derive fourth-order invariants. Fourth-order potential equation was derived which would provide a large class of integrable systems in complex coordinates. Two examples of a two-coupled oscillator system were studied. Case (a) is the two-coupled oscillators in complex plane, whose second constant of motion is obtained (3.6). However, Case (b) is the potential (3.8) obtained using transformations from x–p plane to complex plane of the same system which also admits fourth-order invariant. The invariant constructed in Case (b) seems to be more simpler than that of in x–p plane.

Acknowledgement

The authors are thankful to the referee for several useful suggestions.

References

[1] B Grammaticos, B Dorizzi and A Ramani, *J. Math. Phys.* **24**, 2289 (1983)
L S Hall, *Physica* **D8**, 90 (1983)
[2] J Hietarinta, *Phys. Rep.* **147**, 87 (1987)
[3] C R Holt, *J. Math. Phys.* **23**, 1037 (1982)
[4] R S Kaushal and S C Mishra, *Pramana – J. Phys.* **26**, 109 (1986)
[5] R S Kaushal, S C Mishra and K C Tripathy, *J. Math. Phys.* **26**, 420 (1985)
R L Hiranwal, S C Mishra and Veena Mishra, *Ann. Phys.* **309**, 390 (2004)
Roshan Lal and S C Mishra, *Ind. J. Phys.* **B77**, 567 (2003)
[6] R S Kaushal, S C Mishra and K C Tripathy, *Phys. Lett.* **A102**, 7 (1984)
[7] R S Kaushal, *Classical and quantum mechanics of noncentral potentials* (Narosa Publishing House, 1998)
[8] A Goriely, *Integrability and nonintegrability of dynamical systems* (World Scientific, Singapore, 2001)
W H Steeb and N Euler, *Nonlinear evolution equations and Painlevé test* (World Scientific, Singapore, 1988)
[9] E T Whittaker, *A treatise on the analytical dynamics of particle and rigid bodies* (Cambridge University Press, London, 1960)
[10] H R Lewis and B Abraham Shrauner, *Bull. Am. Phys. Soc.* **34**, 1974 (1989)
[11] G K Savvidy, *Phys. Lett.* **B130**, 303 (1983)
G K Savvidy, *Nucl. Phys.* **B246**, 302 (1984)
S J Chang, *Phys. Rev.* **D29**, 259 (1984)
[12] R Rajaraman and E J Weinberg, *Phys. Rev.* **D11**, 2950 (1975)
R Damburg, R Propin and S Graffi, *Int. J. Quant. Chem.* **21**, 191 (1982)
[13] H J Korsch, *Phys. Lett.* **A90**, 113 (1982)
[14] S C Mishra, R S Kaushal and K C Tripathy, *J. Math. Phys.* **25**, 2217 (1984)