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Gentile statistics and restricted partitions

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Abstract. In a recent paper (Tran *et al*, *Ann. Phys.* **311**, 204 (2004)), some asymptotic number theoretical results on the partitioning of an integer were derived exploiting its connection to the quantum density of states of a many-particle system. We generalise these results to obtain an asymptotic formula for the *restricted or coloured* partitions $p_k^s(n)$, which is the number of partitions of an integer n into the summand of sth powers of integers such that each power of a given integer may occur utmost k times. While the method is not rigorous, it reproduces the well-known asymptotic results for $s = 1$ apart from yielding more general results for arbitrary values of s.

Keywords. Restricted partitions; colour partitions; quantum density of states; Gentile statistics.

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1. Introduction

It is well-known that in one dimension, the density of states of a system of ideal bosons confined in a harmonic trap at a given energy E, in the $N \to \infty$ limit, is the same as the number of ways of partitioning an integer $n = E$ into a sum of integers [1]. The result is the celebrated Hardy–Ramanujan formula [2]. An in-depth analysis of the fundamental additive decomposition of positive integers by sums of other positive integers with various restrictions is given in ref. [3].

In a recent paper, Tran *et al* [4] exploited the connection between quantum many-particle density of states and integer partitions in number theory to obtain more general results for the additive decomposition or partitioning of an integer. It was shown that the asymptotic density of states for ideal bosons in a system with a power-law energy spectrum, is analogous to $p^s(n)$, that is, the number of ways of partitioning an integer n as a sum of sth powers of integers. It was further shown that *distinct* partitions $d^s(n)$ correspond to the asymptotic density of states of a *pseudofermion*-like system in a power-law energy spectrum. The name pseudofermion is coined to indicate that the system's density of states was constructed by applying the exclusion principle only to the particle distribution. The hole distribution was neglected while allowing any number of particles in the ground state, i.e., a collapse of the ground state, where the Fermi energy E_F is set to zero. For the rest of the paper we follow the same procedure as in [4] where only particle distribution, relevant to partitions, is taken into account but not the hole distribution. We note that the method is not rigorous but yields correct asymptotic results in the well-known cases.

In this paper we extend the analysis to obtain an asymptotic formula for restricted or coloured partitions. We define $p_k^s(n)$ as the number of partitions of an integer n into a sum of sth powers of integers, where each power of a given number occurs utmost k times. The method used here is similar to the one used in the previous paper [4]. The leading exponential approximation for the asymptotic partition of an integer into integer summands, $s = 1$, has been discussed recently by Blencowe and Koshnick [5].

We first construct the partition function for a statistical system in which the maximal occupancy of each state is given by k. Obviously $k = 1$ corresponds to a fermionic system whereas there are no restrictions on k for a bosonic system. Indeed the partition function obtained by this minimal restriction is the same as that obtained by Gentile [6] and the corresponding statistics is known as *Gentile statistics*. The partition function of Gentile statistics also has the property that it nicely interpolates between the Fermi and Bose statistics. While the asymptotic formula, obtained using Gentile statistics, reduces to the result for distinct partitions in the fermionic, $k = 1$, limit, the unrestricted or bosonic partitions are obtained by taking the $k \to \infty$ limit.

2. Many-particle density of states

Before we proceed with the main theme of this paper, we recall briefly the results from the previous analysis for the sake of completeness. Consider a system of N particles, where N is very large. In general, the particles may obey any arbitrary statistics. The canonical N-particle partition function is given by

$$Z_N(\beta) \;=\; \int_0^\infty \rho_N(E)\exp(-\beta E)\mathrm{d}E, \tag{1}$$

where β is the inverse temperature, $E_i^{(N)}$ are the eigenenergies of the N-particle system and $\rho_N(E) = \sum_i \delta(E - E_i^{(N)})$ is the N-particle density of states. The density of states $\rho_N(E)$ may therefore be obtained through the inverse Laplace transform of the canonical partition function

$$\rho_N(E) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \exp(\beta E) Z_N(\beta)\mathrm{d}\beta. \tag{2}$$

In general, it is not always possible to do this inversion analytically. As in the case of single-particle density of states, the many-particle density of states may be decomposed into an average (smooth) part and an oscillating part [7]. The smooth part $\bar{\rho}_N(E)$ may be obtained by evaluating eq. (2) using the saddle-point method [8]. However, unlike the one-particle case, where the oscillating part may be obtained using the periodic orbits in a 'trace formula' [7], it remains a challenging task to find an expression for the oscillating part $\delta\rho_N(E)$ [9]. In what follows we shall use the saddle-point method to obtain the smooth part of the density of states, $\bar{\rho}_N(E)$, and identify it with the number of ways the energy E is partitioned, on the average, among N particles.

Defining the entropy, $S(\beta)$, as

$$S(\beta) = \beta E + \log Z_N \tag{3}$$

and expanding the entropy around the stationary point β_0 and retaining only up to the quadratic term in eq. (2) yields the standard result [8]

$$\bar{\rho}_N(E) = \frac{\exp[S(\beta_0)]}{\sqrt{2\pi S''(\beta_0)}}, \tag{4}$$

where the prime denotes differentiation with respect to inverse temperature and

$$E = -\left(\frac{\partial \ln Z_N}{\partial \beta}\right)_{\beta_0}. \tag{5}$$

The above analysis holds for any N-particle system irrespective of the nature of the single-particle spectrum. Hereafter we omit the subscript N since we are only interested in the asymptotic limit $N \to \infty$.

3. Restricted partitions

We consider a statistical system in which a given quantum state has a maximal occupancy of k in each single-particle state. Thus when $k = 1$ the system corresponds to a system of fermions whereas there are no such restrictions on k for a system of bosons except for symmetry considerations. We now construct the partition function for such a system which also incorporates the property of interpolation between Bose and Fermi statistics as k takes on different values.

In order to relate to the problem of integer partitions, we consider a system with a single-particle spectrum given by $\epsilon_m = m^s$, where the integer $m \geq 1$ and $s > 0$. The energy is rendered dimensionless by appropriate choice of units. For example, when $s = 1$ the spectrum can be mapped on to the spectrum of a one-dimensional oscillator, where the energy is measured in units of $\hbar\omega$. For $s = 2$, it is equivalent to setting energy unit as $\hbar^2/2M$, where M is the particle mass in a one-dimensional square well with unit length. These are the only two physically interesting cases. As in [4], we however keep s arbitrary even though for $s > 2$ there are no quadratic Hamiltonian systems.

We define a *restricted or coloured partition* of an integer n as its additive decomposition into integer powers where no integer is repeated more than k times. For example, consider partitions of an integer, say $N = 6$ as a summand of integers:

$$
\begin{aligned}
&6,\ 5+1,\ 4+2,\ 3+2+1, (p_1^1(6)=4)\\
&3+3,\ 4+1+1,\ 2+2+1+1, (p_2^1(6)=7)\\
&3+1+1+1,\ 2+2+2, (p_3^1(6)=9)\\
&2+1+1+1+1, (p_4^1(6)=10)\\
&1+1+1+1+1+1, (p_6^1(6)=11). \qquad (6)
\end{aligned}
$$

We are interested in the relation between the asymptotic expression for such restricted partitions and the density of states of the system which allows utmost k occupancy of the single-particle states.

The relevant physical system for obtaining partitions is thus one in which at zero temperature all the particles are in the ground state whose energy is set to zero. At any excitation energy, particles are excited from the ground state and distributed in the single-particle states with maximal occupancy given by k in each state. The number of ways in which such a distribution can be achieved is the density of states at that energy (as also the number of ways of partitioning the energy).

The *partition function* for such a system in the limit of the number of particles $N \to \infty$ is given by

$$
\begin{aligned}
Z_\infty(\beta) &= \prod_{m=1}^{\infty} [1 + \exp(-\beta m^s) + \cdots + \exp(-k\beta m^s)]\\
&= \prod_{m=1}^{\infty} \sum_{n=0}^{k} \exp(-n\beta m^s), \qquad (7)
\end{aligned}
$$

where the single-particle spectrum is given by a power law. We note that the partition function in eq. (7) is indeed the grand-canonical partition function with the chemical potential $\mu = 0$. In particular $k = 1, \infty$ correspond to the two cases considered in the earlier paper [4]. Note that with k arbitrary this is indeed the partition function for the well-studied Gentile statistics [6].

By setting $x = \exp(-\beta)$, the partition function may be written as

$$
Z_\infty(x) = \sum_{n=1}^{\infty} p_k^s(n) x^n = \prod_{n=1}^{\infty} \frac{[1 - x^{(k+1)n^s}]}{[1 - x^{n^s}]}. \qquad (8)
$$

In the asymptotic limit of a large number of particles, $p_k^s(n)$ is the number of ways of partitioning n.

Using eq. (3), we obtain

$$
\begin{aligned}
S = \beta E &+ \sum_{n=1}^{\infty} \ln[1 - \exp(-(k+1)\beta n^s)]\\
&- \sum_{n=1}^{\infty} \ln[1 - \exp(-\beta n^s)]. \qquad (9)
\end{aligned}
$$

We now evaluate the sums approximately using the Euler–Maclaurin series. We have

$$-\sum_{n=1}^{\infty}\ln[1-\exp(-\beta n^s)] = -\sum_{n=0}^{\infty}\ln[1-\exp(-\beta(n+1)^s)]$$
$$= -\int_0^{\infty}\ln[1-\exp(-\beta(x+1)^s)]\mathrm{d}x$$
$$-\frac{1}{2}\ln[1-\exp(-\beta)] + s\sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)}, \quad (10)$$

where B_{2k} are the Bernoulli numbers. The integral in the Euler–Maclaurin expansion may be evaluated as follows:

$$-\int_0^{\infty}\ln[1-\exp(-\beta(x+1)^s)]\mathrm{d}x$$
$$= -\int_0^{\infty}\ln[1-\exp(-\beta x^s)]\mathrm{d}x + \int_0^1 \ln[1-\exp(-\beta x^s)]\mathrm{d}x$$
$$= \frac{C(s)}{\beta^{1/s}} + \ln[1-\exp(-\beta)] - s\int_0^1 \frac{\beta x^s}{\exp(\beta x^s)-1}\mathrm{d}x, \quad (11)$$

where

$$C(s) = \Gamma\left(1+\frac{1}{s}\right)\zeta\left(1+\frac{1}{s}\right) \quad (12)$$

is given in terms of the Riemann zeta function. In the *high-temperature limit*, that is $\beta \to 0$, we have

$$-\int_0^{\infty}\ln[1-\exp(-\beta(x+1)^s)]\mathrm{d}x \approx \frac{C(s)}{\beta^{1/s}} + \ln[1-\exp(-\beta)] - s + O(\beta). \quad (13)$$

The series involving Bernoulli numbers may be evaluated approximately as follows: The Euler–Maclaurin series applied to logarithm of gamma function gives

$$\ln(\Gamma(n+1)) = n\ln(n) - n + \frac{1}{2}\ln(2\pi) + \sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)n^{2k-1}}. \quad (14)$$

We may consider this as the expansion of zero by putting $n = 1$ and get

$$s\sum_{k=1}^{\infty}\frac{B_{2k}}{2k(2k-1)} = s - \frac{s}{2}\ln(2\pi). \quad (15)$$

Thus keeping terms up to $O(\beta)$, and using the above arguments, the last term in eq. (9) may be written as

$$-\sum_{n=1}^{\infty}[1-\exp(-\beta n^s)] \approx \frac{C(s)}{\beta^{1/s}} + \frac{\ln[1-\exp(-\beta)]}{2} - \frac{s}{2}\ln(2\pi) + O(\beta). \quad (16)$$

One may also get the above result directly using the symbolic math program MAPLE. The infinite sum in the second term of eq. (9) is also evaluated similarly by scaling β appropriately. Note that the constant term does not get scaled.

Thus the entropy to the leading order in β is given by

$$S = \beta E + \frac{\alpha C(s)}{\beta^{1/s}} - \frac{1}{2}\ln[1-\exp(-(k+1)\beta)] + \frac{1}{2}\ln[1-\exp(-\beta)] + O(\beta), \tag{17}$$

where

$$\alpha = 1 - \frac{1}{(k+1)^{1/s}}. \tag{18}$$

We note that the expansion in powers of β is equivalent to taking the high-temperature limit though this limit should be employed cautiously when β is weighted by k which can in principle take large values. The last term of $O(\beta)$ in the RHS of eq. (17) causes a shift in the energy E by a constant (for example 1/24 in the $s=1$ case) in S. In the asymptotic formula for the level density for large E, obtained by the saddle-point method, this may be ignored. Note, however, that such a shift was included by Rosenzweig [10] to improve the Bethe formula [11] for the nuclear level density. We further note that the $k \to \infty$, or the bosonic limit, is special since this limit involves precisely the sum given in eq. (16) and not the difference of two series as in eq. (9). As a result the β independent term, namely $-(s/2)\ln(2\pi)$, appears only in this limit and not for any finite k. Indeed, as we shall see below, this term is crucial in getting the prefactor correctly in the Hardy–Ramanujan Formula.

The saddle point on the real axis [11a] is evaluated by taking the derivative of the entropy,

$$S'(\beta) = E - \frac{1}{s}\,\frac{\alpha C(s)}{\beta^{(1+1/s)}}\ , \tag{19}$$

where only the leading terms in β are retained. Equating the derivative to zero, the saddle point in β is given by

$$\beta_0 = \left(\frac{\alpha C(s)}{sE}\right)^{s/(1+s)} = \kappa_s E^{-(s/(1+s))}, \tag{20}$$

where

$$\kappa_s = \left(\frac{\alpha C(s)}{s}\right)^{s/(1+s)}. \tag{21}$$

Substituting this in the saddle-point expression for the density of states in eq. (4), the asymptotic density of states is given by

$$\bar{\rho}_k^{(s)}(E) \approx \kappa_s\sqrt{s} \times \frac{\exp[\kappa_s(s+1)E^{1/(1+s)}]}{\sqrt{2\pi(s+1)E^{(3s+1)/(s+1)}[1-\exp(-(k+1)\kappa_s E^{-s/(s+1)})]}}, \tag{22}$$

where the subscript k in $\bar{\rho}$ indicates the main property of the statistical system under consideration. We have kept the exponential term under the square-root sign in the denominator to indicate the interpolation property of the density of states between the Bose ($k \to \infty$) and the Fermi ($k = 1$) limits even though keeping the exponential may not be consistent with the order of the expansion. The correct expression is given below.

For finite k there always exists an energy E which is large enough such that the following approximation is useful:

$$\bar{\rho}_k^{(s)}(E) \approx \sqrt{s\kappa_s}\frac{\exp[\kappa_s(s+1)E^{1/(1+s)}]}{\sqrt{2\pi(s+1)(k+1)E^{(2s+1)/(s+1)}}}. \tag{23}$$

Note that the above expression is identical to eq. (23) for $d_s(n)$ in [4] corresponding to the special case of $k = 1$. The degeneracy or k dependence in the above equation is also hidden in the parameter κ_s. An asymptotic formula derived in ref. [5] based on Gentile statistics for the special case of $s = 1$ is the same as the one given in eq. (23). However, to the best of our knowledge, no general formula for arbitrary s and k exists.

In particular for $s = 1, k = 1$ the above equation reduces to the well-known formula [3,12] for distinct partitions of an integer into a set of integers

$$\bar{\rho}(E) \approx d(n = E) = \frac{\exp\left[\pi\sqrt{E/3}\right]}{4 \times 3^{1/4}E^{3/4}} . \tag{24}$$

The bosonic limit may be obtained from eq. (22) by taking the limit $k \to \infty$. However, we also note that this limit should be taken in such a way that the β independent factor in eq. (16) is also included. With this proviso we obtain

$$\bar{\rho}_k^{(s)}(E) \approx \kappa_s\sqrt{s}\frac{\exp[\kappa_s(s+1)E^{1/(1+s)}]}{(2\pi)^{(s+1)/2}\sqrt{(s+1)E^{(3s+1)/(s+1)}}}, \tag{25}$$

which is the celebrated Hardy–Ramanujan asymptotic formula for integer partitions. In particular for $s = 1$, we have the well-known result (see for example ref. [12]).

$$\bar{\rho}(E) \approx \ p(n = E) = \frac{\exp\left[\pi\sqrt{2E/3}\right]}{4\sqrt{3}E} . \tag{26}$$

In figure 1 we show a comparison between the exact $p_k(n)$ (continuous line), and the asymptotic density of states, $\bar{\rho}_k^{(1)}(n)$ (dashed line) obtained from eq. (23) for various values of k when the power $s = 1$. The asymptotics is reached earlier for smaller k than for larger k as can be seen easily from the figures. In fact for $k = 1$ the formula is almost exact even for n small. We note that the numerical estimates for the density based on eq. (22) are usually higher than the exact values and asymptotically reach the exact value later than the density calculated based on eq. (23) which is shown in the figure. The saddle point evaluated by keeping

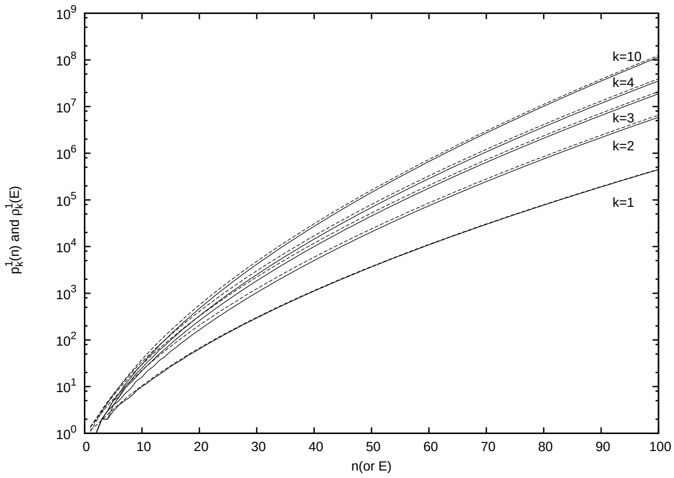

Figure 1. Comparison of the exact $p_k^1(n)$ (solid line) and the asymptotic $\bar{\rho}_k^1(n)$ (dashed line) for $s = 1$

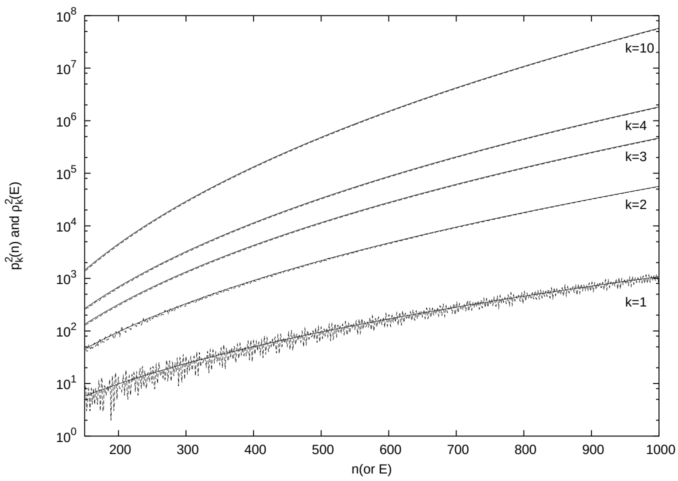

Figure 2. Comparison of the exact $p_2^k(n)$ (solid line) and the asymptotic $\bar{\rho}_k^2(n)$ (dashed line) for $s = 2$.

only the two leading terms in β ensures that eq. (23) is more accurate for small values of k though asymptotically both equations yield the same result.

Similarly, in figure 2, the computed $p_k^2(n)$ is compared with $\bar{\rho}_k^2(n)$. We note that in this case the fluctuation about the mean is larger for small values of k. In general the fluctuations decrease with increasing k for a given s but increase with increasing s for a given k as shown in figure 3 where we have plotted the asymptotic density of states for $s = 3$ and various values of k.

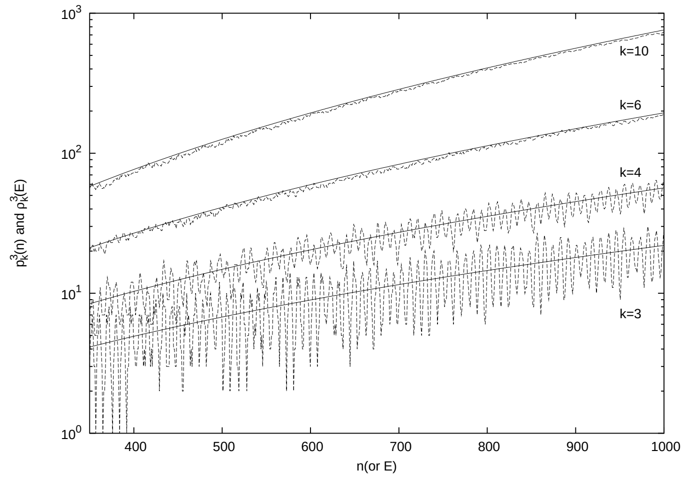

Figure 3. Comparison of the exact $p_3^k(n)$ (solid line) and the asymptotic $\bar{\rho}_k^2(n)$ (dashed line) for $s = 3$.

4. Summary

To summarise, this paper extends the results of ref. [4] to the case of restricted or coloured partitions. We identify the relevant generating function for coloured partitions as the partition function of Gentile statistics. The resulting asymptotic or smooth density states interpolates between the Bose (arbitrary partitions) and Fermi (distinct partitions) systems as a function of the maximal occupancy of the quantum state.

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